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WEIGHTED NORMAL NUMBERS

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We show that if $\{a_k\}_k$ is bounded then $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} a_k(-1)^{d_k} = 0$ for almost every $0 \leqslant x \leqslant 1$ where $x = \sum_{k=1}^{\infty} d_k 2^{-k}$ is the dyadic expansion of x. It is also shown that $(1/n)\sum_{k=1}^{n} a_k \exp\left(2\pi i \cdot p^k x\right) \to 0$ almost everywhere where $p > 1$ is any fixed integer.

Let (X, μ) be a probability measure space. A measurable transformation $T : X \to X$ is said to be *measure preserving* if $\mu\left(T^{-1}E\right) = \mu(E)$ for every measurable subset E. A measure preserving transformation T on X is called *ergodic* if $f(Tx) = f(x)$, $f \in L^1(X, \mu)$, holds only for constant functions. Let 1_E be the indicator function of a measurable set E and consider the behaviour of the sequence $\sum_{k=0}^{n-1} 1_E\left(T^k x\right)$ which equals the number of times that the points $T^k x$ visit E. The Birkhoff Ergodic Theorem implies that the relative frequency of the visits equals $\mu(E)$, that is,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1_E(T^n x) = \mu(E).$$

Consider the ergodic transformation $T : x \mapsto \{2x\}$ on $[0, 1)$, where $\{t\}$ is the fractional part of t. If $x = \sum_{k=1}^{\infty} d_k\, 2^{-k}$ is the dyadic expansion of x, then $d_k = 1_{[(1/2),1)}\left(T^{k-1}x\right)$. The same theorem applied to $T : x \mapsto \{2x\}$ on $[0, 1)$ gives the classical Borel's Theorem on normal numbers:

$$\lim_{n} \frac{1}{n}\sum_{k=0}^{n-1} d_k = \frac{1}{2} \quad \text{almost everywhere,}$$

hence almost every x is *normal*, that is, the relative frequency of the digit 1 in the binary expansion of x is $1/2$. Equivalently we may rephrase it as $\lim_{n}(1/n)\sum_{k=0}^{n-1}(-1)^{d_k} = 0$

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almost everywhere, with respect to the Lebesgue measure. For general references, see [**6, 7**]. For recent results on spectral properties of uniform distribution, see [**2**].

In this paper we obtain weighted ergodic theorems, in other words, we show that for some T and a bounded sequence $\{a_k\}_k$ of complex numbers the limit of $(1/n)\sum_{k=1}^{n} a_k f(T^k x)$ exists almost everywhere, if f satisfies certain orthogonality conditions. Similar problems were studied by Nair [**5**] for the Gauss transformation $x \mapsto \{1/x\}$ and sequences satisfying $a_k \in \{0,1\}$.

We need the following lemma. For the proof see Proposition 1.9 in [**1**].

LEMMA. *Let $\{u_j\}_{j=1}^{\infty}$ be a bounded sequence of complex numbers and let $\{v_j\}_{j=1}^{\infty}$ be a sequence of complex numbers for which there exists a constant $M > 0$ such that $(1/n)\sum_{j=0}^{n-1} |v_j|^2 \leqslant M$ for every n. Suppose that an increasing sequence of positive integers $\{N_k\}_{k=1}^{\infty}$ satisfies*

(i) $\lim_{k\to\infty} (N_{k+1})/(N_k) = 1$,

(ii) $(1/N_k)\sum_{j=0}^{N_k-1} u_j v_j$ *converges to A as $k \to \infty$.*

Then $(1/n)\sum_{j=0}^{n-1} u_j v_j$ also converges to A.

For a unitary operator U in a Hilbert space $\mathcal{H}$ with the inner product $(\ ,\ )$ there exists a spectral measure P such that $U = \int_{|z|=1} z\,dP(z)$ where $P(E)$ is an orthogonal projection in $\mathcal{H}$ for every measurable subset E. For $h \in \mathcal{H}$ we have a positive finite measure λ_h such that $\lambda_h(E) = (P(E)h,h)$ and $(U^n h, h) = \int_{|z|=1} z^n\,d\lambda_h(z)$. But for a noninvertible measure preserving transformation T acting on a probability space (X,μ) the induced linear operator U_T in $L^2(X,\mu)$ defined by $(U_T f)(x) = f(Tx)$ is not unitary, hence the spectral measure does not exist and the spectral theorem is not applicable.

Here is one of the ways to overcome this difficulty: Let U be the isometry in $\mathcal{H}$ which is not necessarily invertible. Put $c_n = (U^n h, h)$ for $n \geqslant 0$ and $c_n = \left((U^*)^{|n|}h, h\right) = \left(h, U^{|n|}h\right)$ for $n < 0$, where U^* is the adjoint of U. Then $c_{-n} = \overline{c_n}$ and the sequence $\{c_n\}_{n\in\mathbb{Z}}$ is positive definite. Hence by Bochner's theorem there exists a positive finite measure λ_h such that $c_n = \int_{|z|=1} z^n\,d\lambda_h(z)$ for $n \in \mathbb{Z}$. Note that $\left(U^k h, U^j h\right) = \left(U^{k-j}h, h\right) = c_{k-j}$ for $k \geqslant j$ and $\left(U^k h, U^j h\right) = \left(h, U^{j-k}h\right) = c_{k-j}$ for $k < j$. If there is an element $h \in \mathcal{H}$ such that $(U^n h, h) = 0$ for every $n > 0$, then $d\lambda_h$ and the normalised Lebesgue measure on the circle dz have the same Fourier-Stieltjes coefficients, hence we see that $d\lambda_h = C \cdot dz$ for $C = \|h\|^2$. For details on Bochner's theorem, see [**3, 4**].

Proposition 1. *Let $\{a_j\}_j$ be a sequence of complex numbers such that*

$$\frac{1}{n}\sum_{j=0}^{n-1}|a_j|^2 \leqslant M$$

for every n. For almost every $0 \leqslant x \leqslant 1$ the limit of

$$\frac{1}{n}\sum_{j=1}^{n} a_j\,(-1)^{d_j}$$

exists and equals 0 where $x = \sum_{j=1}^{\infty} d_j\,2^{-j}$, $d_j \in \{0,1\}$, is the dyadic expansion of x.

Proof: Let $T:[0,1)\to[0,1)$ be the Lebesgue measure preserving transformation given by $Tx = \{2x\}$. Let $h(x) = 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x) = 1 - 2\cdot 1_{[1/2,1)}(x)$. Then $h(T^j x)$ is the jth Rademacher function and $\{h(T^j x)\}_{j=0}^{\infty}$ is an orthonormal family in $L^2(0,1)$.

Hence the isometry $U_T f(x) = f(Tx)$ satisfies for $j,k \geqslant 0$,

$$c_{j-k} = \left({U_T}^j h, {U_T}^k h\right) = \int_0^1 h(T^j x)h(T^k x)\,dx = \delta_{jk}$$

and

$$\left({U_T}^j h, h\right) = 0 \text{ for } j > 0.$$

Since

$$\begin{aligned}\left\|\sum_{j=0}^{n-1} a_j {U_T}^j h\right\|^2 &= \sum_{0\leqslant j,k\leqslant n-1} a_j\overline{a_k}c_{j-k}\\ &= \sum_{j=0}^{n-1}|a_j|^2 \leqslant n\cdot M,\end{aligned}$$

the Monotone Convergence Theorem implies that

$$\begin{aligned}\int_0^1 \sum_{n=1}^{\infty}\left|\frac{1}{n^2}\sum_{j=0}^{n^2-1} a_j h(T^j x)\right|^2 dx &= \sum_{n=1}^{\infty}\int_0^1\left|\frac{1}{n^2}\sum_{j=0}^{n^2-1} a_j h(T^j x)\right|^2 dx\\ &= \sum_{n=1}^{\infty}\left\|\frac{1}{n^2}\sum_{j=0}^{n^2-1} a_j {U_T}^j h\right\|^2\\ &\leqslant \sum_{n=1}^{\infty}\frac{1}{n^4}\cdot n^2\cdot M < \infty,\end{aligned}$$

hence

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^2}\sum_{j=0}^{n^2-1} a_j h(T^j x)\right|^2 < \infty \quad \text{almost everywhere}$$

and
$$\frac{1}{n^2}\sum_{j=0}^{n^2-1} a_j h(T^j x) \to 0 \quad \text{almost everywhere.}$$

Putting $u_j = h(T^j x)$, $v_j = a_j$ and $N_k = k^2$ we apply the Lemma. Then

$$\frac{1}{n}\sum_{j=0}^{n} a_j h(T^j x) \to 0 \quad \text{almost everywhere.}$$

Let $x = \sum_j d_j\, 2^{-j}$ be the dyadic expansion of x, and note that $d_j = 1_{[(1/2),1)}(T^{j-1}x)$ and use $h(T^{j-1}x) = 1 - 2\cdot 1_{[1/2,1)}(T^{j-1}x) = 1 - 2\cdot d_j(x) = (-1)^{d_j(x)}$. ∎

REMARK. Let $p > 1$ be a fixed integer. Using the Lebesgue measure preserving transformations $Tx = \{px\}$, $0 \leqslant x \leqslant 1$ and the corresponding function h defined by $h(x) = \exp((2\pi i(j-1)/p)\, x)$, $(j-1)/p \leqslant x < j/p$, $j = 1, \dots, p$, we can easily see that for a bounded sequence $\{a_k\}_k$ of complex numbers the limit of

$$\frac{1}{n}\sum_{k=1}^{n} a_k\, \lambda^{d_k} \qquad \text{where } \lambda = \exp(2\pi i/p)$$

is equal to 0 almost everywhere, where $x = \sum_{k=1}^{\infty} d_k\, p^{-k}$, $d_k \in \{0, 1, \dots, p-1\}$ is the p-adic expansion of x.

PROPOSITION 2. *Let $\{a_k\}_k$ be a bounded sequence of complex numbers. For almost every $0 \leqslant x \leqslant 1$ we have*

$$\frac{1}{n}\sum_{k=1}^{n} a_k \sin(2\pi i\cdot p^k x) \to 0,$$

$$\frac{1}{n}\sum_{k=1}^{n} a_k \cos(2\pi i\cdot p^k x) \to 0,$$

and
$$\frac{1}{n}\sum_{k=1}^{n} a_k \exp(2\pi i\cdot p^k x) \to 0$$

where $p > 1$ is a fixed integer.

PROOF: Define $Tx = \{px\}$, $0 \leqslant x \leqslant 1$. Note that the function $\exp(2\pi i x)$ satisfies the condition $(U_T{}^j h, h) = 0$ for $j > 0$. Proceed as in Proposition 1 and take real and imaginary parts. ∎

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