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On a Conjecture of Livingston

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Abstract. In an attempt to resolve a folklore conjecture of Erdös regarding the non-vanishing at $s = 1$ of the L-series attached to a periodic arithmetical function with period q and values in $\{-1,1\}$, Livingston conjectured the $\overline{\mathbb{Q}}$-linear independence of logarithms of certain algebraic numbers. In this paper, we disprove Livingston's conjecture for composite $q \geq 4$, highlighting that a new approach is required to settle Erdös conjecture. We also prove that the conjecture is true for prime $q \geq 3$, and indicate that more ingredients will be needed to settle Erdös conjecture for prime q.

1 Introduction

In a written correspondence with Livingston, Erdös [5] conjectured the following:

Conjecture 1.1 (Erdös) *Let q be a positive integer and let f be an arithmetical function, periodic with period q. If $f(n) \in \{-1,1\}$ when $q \nmid n$ and $f(n) = 0$ otherwise, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

whenever the series is convergent.

In 1965, Livingston [5] attempted to resolve the above conjecture. He predicted that to settle Conjecture 1.1, one would first have to prove the following conjecture.

Conjecture 1.2 (Livingston) *Let $q \geq 3$ be a positive integer. The numbers*

$$\Big\{ \log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2} \Big\} \quad \textit{and} \quad \pi$$

when q is odd, and

$$\Big\{ \log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2} \Big\}, \quad \pi, \quad \textit{and} \quad \log 2$$

when q is even, are linearly independent over the field of algebraic numbers.

The above statement does not depend on the branch of logarithm considered, as the values would only differ by an integer multiple of $2\pi i$. In this paper, we disprove Livingston's conjecture in the case when q is not prime and show that the conjecture is true when q is prime. More precisely, we prove the following theorems.

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Theorem 1.3 *Conjecture 1.2 does not hold for $q \geq 4$ and q not prime. In fact, for a composite positive integer $q \geq 6$, the numbers*

$$\left\{ \log\left(2 \sin \frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2} \right\}$$

are $\mathbb{Q}$-linearly dependent.

Theorem 1.4 *Let p be an odd prime. The numbers*

$$\left\{ \log\left(2 \sin \frac{a\pi}{p}\right) : 1 \leq a \leq \frac{p-1}{2} \right\} \quad \text{and} \quad \pi$$

are $\overline{\mathbb{Q}}$-linearly independent. Thus, Conjecture 1.2 is true when the modulus p is an odd prime.

In both of these theorems, log denotes the principal branch. We have the following as a corollary of Theorem 1.4.

Corollary 1.5 *Let p be an odd prime and let f be an arithmetical function, periodic with period p such that $f(n) \in \{-1, 1\}$ when $p \nmid n$ and $f(n) = 0$ otherwise. Assume that $\sum_{a=1}^{p} f(a) = 0$. Then only one of the following is true, either*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

or

$$\sum_{a=1}^{p-1} f(a) \cot\left(\frac{a\pi}{p}\right) = \sum_{a=1}^{p-1} f(a) \cos\left(\frac{2\pi ab}{p}\right) = 0,$$

for $1 \leq b \leq (p-1)/2$.

2 Preliminaries

This section introduces some results that are fundamental to the proofs.

2.1 Baker's Theorem on Linear Forms in Logarithm of Algebraic Numbers

We will use an important theorem of A. Baker concerning linear forms in logarithms of algebraic numbers.

Theorem 2.1 ([1, Theorem 2.1, p. 10]) *If $\alpha_1, \alpha_2, \dots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \log \alpha_2, \dots, \log \alpha_n$ are linearly independent over the rationals, then $1, \log \alpha_1, \log \alpha_2, \dots, \log \alpha_n$ are linearly independent over the field of all algebraic numbers.*

2.2 Matrices of the Dedekind Type

Let $\mathfrak{M}$ be an $n \times n$ matrix with complex entries. Let $m_{i,j}$ denote the (i, j)-th entry of $\mathfrak{M}$. Then $\mathfrak{M}$ is said to be of Dedekind type if there exists a finite abelian group $G = \{x_1, x_2, \dots, x_n\}$ and a complex valued function f on G such that $m_{i,j} = f(x_i^{-1} x_j)$

for all $1 \leq i, j \leq n$. We will use the following widely known theorem regarding matrices of the Dedekind type.

Theorem 2.2 *Let $\mathfrak{M}$ be an $n \times n$ matrix of the Dedekind type. For a character χ on G (a homomorphism of G into $\mathbb{C}^*$), define*

$$S_\chi := \sum_{s \in G} f(s)\chi(s).$$

Then the determinant of $\mathfrak{M}$ is equal to $\prod_\chi S_\chi$, where the product runs over all characters of G. Thus, $\mathfrak{M}$ is invertible if and only if $S_\chi \neq 0$, for all characters χ of G.

For a proof of Theorem 2.2 and an exposition on properties of matrices of the Dedekind type, we refer the reader to [8]. The determinant of a matrix of the Dedekind type is often referred to as *Dedekind determinant.*

2.3 Linear Forms in Logarithm of Algebraic Numbers with Dirichlet Coefficients

A Dirichlet character χ modulo q is a group homomorphism,

$$\chi \colon (\mathbb{Z}/q\mathbb{Z})^* \longrightarrow \mathbb{C}^*,$$

which can be extended to a periodic function on all of integers by setting

$$\chi(n) = \begin{cases} \chi(n \bmod q) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The trivial Dirichlet character, χ_0 is given by

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet L-function associated with a Dirichlet character χ is defined as

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which converges absolutely for $\Re(s) > 1$. The series $L(s,\chi)$ can be analytically continued to the entire complex plane except when $\chi = \chi_0$, in which case the series has a simple pole at $s = 1$. Since χ is a periodic arithmetical function, the proof of analytic continuation of $L(s,\chi)$ follows from the analytic continuation of the series $L(s, f)$ for a periodic arithmetical function f, proved in the next section, and the fact that $\sum_{a=1}^{q} \chi(a) = 0$ for a non-trivial Dirichlet character χ modulo q. We will make use of the following well-known lemma towards proving Theorem 1.4.

Lemma 2.3 *Let χ be a non-trivial even Dirichlet character modulo an odd prime p,* i.e., $\chi(-1) = 1$. *Then*

$$\sum_{a=1}^{p-1} \overline{\chi}(a) \log\left|1 - \zeta_p^a\right| = -\frac{p}{\tau(\chi)} L(1,\chi),$$

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where

$$\tau(\chi) = \sum_{a=1}^{p} \chi(a)\zeta_p^a,$$

is the Gauss sum associated with χ *and* $\zeta_p = e^{2\pi i/p}$.

In the interest of completeness, we include a proof of this lemma.

Proof Let χ be a non-trivial even Dirichlet character modulo an odd prime p. Let $\widehat{\chi}$ denote the discrete Fourier transform of χ, given by

$$\widehat{\chi}(k) := \frac{1}{p} \sum_{a=1}^{p} \chi(a)\zeta_p^{-ak}.$$

This can be inverted using the identity

$$\chi(n) = \sum_{k=1}^{p} \widehat{\chi}(k)\zeta_p^{kn}. \tag{2.1}$$

Substituting expression (2.1) in the definition of the Dirichlet L-function associated with χ and noting that $\widehat{\chi}(p) = \sum_{a=1}^{p} \chi(a) = 0$ for a non-trivial Dirichlet character χ, we get

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{p-1} \widehat{\chi}(k)\zeta_p^{kn}. = \sum_{k=1}^{p-1} \widehat{\chi}(k) \sum_{n=1}^{\infty} \frac{\zeta_p^{kn}}{n^s}. \tag{2.2}$$

The inner sum converges for $s = 1$. To see this, recall the partial summation formula.

Theorem *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of complex numbers and let* f *be a* C^1 *function on* $\mathbb{R}_{>0}$. *For* $x > 0$, *if* $A(x) := \sum_{n\le x} a_n$, *then*

$$\sum_{1\le n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

For $1 \le k \le p-1$, let $a_n = \zeta_p^{kn}$ and $f(x) = 1/x$. Thus, $A(x) = \sum_{n\le x} \zeta_p^{kn}$ and the partial summation formula gives us that

$$\sum_{1\le n\le x} \frac{\zeta_p^{kn}}{n} = \frac{A(x)}{x} + \int_1^x \frac{A(t)}{t^2} dt. \tag{2.3}$$

Now, note that for $1 \le k \le p-1$, $\sum_{n=1}^{p} \zeta_p^{kn} = 0$. Hence, the partial sums $A(x)$ are bounded above by p for all $x > 0$. Therefore, the integral in (2.3) is absolutely convergent as x tends to infinity. Thus, taking the limit as x goes to infinity in (2.3), we get the convergence of the inner sum in (2.2) and can conclude that

$$L(1,\chi) = -\sum_{k=1}^{p-1} \widehat{\chi}(k) \log(1-\zeta_p^k), \tag{2.4}$$

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where log is the principal branch. Since χ is an even character, equation (2.4) can be rewritten as

$$L(1,\chi) = -\sum_{k=1}^{p-1}\widehat{\chi}(k)\log(1-\zeta_p^k) = -\sum_{k=1}^{\lfloor (p-1)/2\rfloor}\widehat{\chi}(k)\big[\log(1-\zeta_p^k)+\log(1-\zeta_p^{-k})\big]$$
$$= -\sum_{k=1}^{\lfloor (p-1)/2\rfloor}\widehat{\chi}(k)\log\big|1-\zeta_p^k\big|^2 = -\sum_{k=1}^{p-1}\widehat{\chi}(k)\log\big|1-\zeta_p^k\big|,$$

where $\widehat{\chi}$ denotes the Fourier transform of χ. Now, note that the Fourier transform of χ can be evaluated in terms of the Gauss sum $\tau(\chi)$ as follows. For every $(k,p)=1$,

$$\widehat{\chi}(k) = \frac{1}{p}\sum_{a=1}^{p-1}\chi(a)\zeta_p^{-ak} = \frac{1}{p}\sum_{t=1}^{p-1}\chi(-tk^{-1})\zeta_p^t = \frac{\overline{\chi(-k)}}{p}\sum_{t=1}^{p-1}\chi(t)\zeta_p^t = \frac{\overline{\chi(-k)}}{p}\tau(\chi).$$

Thus, the $L(s,\chi)$ for a non-trivial Dirichlet character χ has the value

$$L(1,\chi) = -\frac{\tau(\chi)}{p}\sum_{k=1}^{p}\overline{\chi}(k)\log|1-\zeta_p^k|$$

at $s=1$. Another elementary but important fact about the Gauss sum is that when χ is a non-trivial Dirichlet character modulo p, $\tau(\chi)\neq 0$. For a proof of this fact, we refer the reader to [6, Theorem 5.3.3, p. 76]. This proves Lemma 2.3. ■

3 Livingston's Approach

We first review general theory of L-series attached to a periodic arithmetical function following [7]. Let q be a positive integer and let f be an arithmetical function that is periodic with period q. We define

$$L(s,f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}.$$

Let us observe that $L(s,f)$ converges absolutely for $\Re(s)>1$. Since f is periodic,

$$L(s,f) = \sum_{a=1}^{q}f(a)\sum_{k=0}^{\infty}\frac{1}{(a+kq)^s} = \frac{1}{q^s}\sum_{a=1}^{q}f(a)\zeta(s,a/q),$$

where $\zeta(s,x)$ is the Hurwitz zeta function. For $\Re(s)>1$ and $0<x\le 1$, recall that the Hurwitz zeta function is defined as

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s}.$$

In 1882, Hurwitz [4] proved that $\zeta(s,x)$ has an analytic continuation to the entire complex plane except for a simple pole at $s=1$ with residue 1. In particular,

$$\zeta(s,x) = \frac{1}{s-1}-\psi(x)+O(s-1), \tag{3.1}$$

where ψ is the digamma function, which is defined as the logarithmic derivative of the gamma function. This can be used to conclude that $L(s,f)$ can be extended analytically to the entire complex plane except for a simple pole at $s=1$ with residue

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$\frac{1}{q}\sum_{a=1}^{q} f(a)$. Thus, $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ exists if and only if $\sum_{a=1}^{q} f(a) = 0$, which we will assume henceforth.

Let us also note that (3.1) helps us to express $L(1, f)$ as a linear combination of values of the digamma function. Therefore,

$$L(1,f) = -\frac{1}{q}\sum_{a=1}^{q} f(a)\psi\Big(\frac{a}{q}\Big). \tag{3.2}$$

Let f be an Erdös function, *i.e.*, $f(n) = \pm 1$ when $q \nmid n$ and $f(n) = 0$ whenever $q|n$. The condition for the existence of $L(1, f)$ implies that

$$\sum_{a=1}^{q} f(a) = \sum_{a=1}^{q-1} f(a) = 0. \tag{3.3}$$

As seen earlier, $L(1, f)$ can be written as a linear combination of the values of the digamma function. Gauss ([3, pp. 35–36]) proved the following formula for $1 \le a < q$:

$$\psi\Big(\frac{a}{q}\Big) = -\gamma - \log q - \frac{\pi}{2}\cot\Big(\frac{a\pi}{q}\Big) + \sum_{b=1}^{r}\Big\{\cos\Big(\frac{2\pi ab}{q}\Big)\log\Big(4\sin^2\frac{\pi b}{q}\Big)\Big\} + (-1)^a \log 2\frac{1+(-1)^q}{2}, \tag{3.4}$$

where $r := \lfloor (q-1)/2 \rfloor$.

Substituting (3.4) in (3.2), we have

$$L(1,f) = \frac{-1}{q}\Big[\sum_{a=1}^{q-1} f(a)\Big\{\gamma + \log q + \frac{\pi}{2}\cot\Big(\frac{a\pi}{q}\Big) - \sum_{b=1}^{r}\Big\{\cos\Big(\frac{2\pi ab}{q}\Big)\log\Big(4\sin^2\frac{\pi b}{q}\Big)\Big\} + (-1)^a\log 2\,\frac{1+(-1)^q}{2}\Big\}\Big].$$

Simplifying this expression using (3.3), we get

$$L(1,f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\Big(\frac{a\pi}{q}\Big) + \frac{2}{q}\sum_{b=1}^{r}\Big\{\Big[\sum_{a=1}^{q-1} f(a)\cos\Big(\frac{2\pi ab}{q}\Big)\Big]\log\Big(2\sin\frac{\pi b}{q}\Big)\Big\} - T_q, \tag{3.5}$$

where

$$T_q = \begin{cases} \frac{\log 2}{q}\Big(\sum_{k=1}^{q-1}(-1)^k f(k)\Big) & \text{if } q \text{ is even,} \\ 0 & \text{otherwise.}\end{cases}$$

Let us note that the numbers

$$\cot\Big(\frac{a\pi}{q}\Big) \quad \text{and} \quad \cos\Big(\frac{2\pi ab}{q}\Big)$$

are algebraic for $1 \le a < q$ and $1 \le b < q$. Since $f(a) \in \overline{\mathbb{Q}}$ and $f(q) = 0$, we are led to deduce that $L(1, f)$ is an algebraic linear combination of

$$\pi, \log\Big(2\sin\frac{\pi}{q}\Big), \log\Big(2\sin\frac{2\pi}{q}\Big), \dots, \log\Big(2\sin\frac{(q-1)\pi}{2q}\Big)$$

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together with $\log(2)$ when q is even. This led Livingston to predict that if Conjecture 1.1 were to be true, the above numbers should be linearly independent over $\overline{\mathbb{Q}}$. At this point, we make the following key observation: to conclude Conjecture 1.1 as an implication of Conjecture 1.2, one is still required to prove that the resulting relation is non-trivial. That is, if f is an Erdös function, not identically zero, then at least one of

$$\sum_{a=1}^{q-1} f(a)\cot\Big(\frac{a\pi}{q}\Big), \tag{3.6}$$

and

$$\sum_{a=1}^{q-1} f(a)\cos\Big(\frac{2\pi ab}{q}\Big), \quad 1 \le b \le r \tag{3.7}$$

or T_q is not zero. This question is not addressed by Conjecture 1.2, and hence, Livingston's conjecture alone is not sufficient to settle the conjecture of Erdös.

Remark If f is allowed to take values in $\overline{\mathbb{Q}}$ and q is odd, then there exist a plethora of examples of functions f that are not identically zero but for which (3.6) and (3.7) are both zero for all $1 \le b \le r$. These are given by the following theorem from [2].

Theorem *Let $q \ge 3$ be a natural number. Then all odd, algebraically-valued functions f, periodic mod q, for which $L(1, f) = 0$ are given by the totality of linear combinations with algebraic coefficients of the following $\lfloor \frac{1}{2}(q-3) \rfloor$ functions:*

$$f_l(n) = (-1)^{n-1}\Big(\frac{\sin n\pi/q}{\sin \pi/q}\Big)^l, \quad \textit{for } l = 3, 5, \dots, (q-2)$$

when q is odd and

$$f_l(n) = (-1)^{n-1}\Big(\frac{\cos n\pi/q}{\cos \pi/q}\Big)\Big(\frac{\sin n\pi/q}{\sin \pi/q}\Big)^l, \quad \textit{for } l = 3, 5, \dots, (q-1)$$

when q is even. The functions are linearly independent and take values in $\mathbb{Q}(\zeta_q)$, i.e., *the q-th cyclotomic field.*

Each f_l in the above theorem is an odd function. Since $\cos(2\pi ab/q)$ is an even function for $1 \le a < q$, (3.7) is zero for all $1 \le b \le r$. $T_q = 0$ as q is odd. Thus,

$$L(1, f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\Big(\frac{a\pi}{q}\Big),$$

which is zero by the above theorem from [2].

4 Proof of the Main Theorems

We make a useful observation before proceeding with the proofs. If q is a positive integer and $1 \le a < q/2$, then

$$2\sin\frac{a\pi}{q} = \frac{e^{ia\pi/q} - e^{-ia\pi/q}}{i} = ie^{-ia\pi/q}(1 - \zeta_q^a), \tag{4.1}$$

where $\zeta_q = e^{2\pi i/q}$. Since $\sin\frac{a\pi}{q} > 0$, for $1 \le a < q/2$ and log denotes the principal branch,

$$\text{(4.2)}\qquad \log\Big(2\sin\frac{a\pi}{q}\Big) = \log\Big(\big|1-\zeta_q^a\big|\Big) + i0 = \log\Big(\big|1-\zeta_q^a\big|\Big) = \log\Big(\big|1-\zeta_q^{-a}\big|\Big) = \log\Big(2\sin\frac{(q-a)\pi}{q}\Big).$$

4.1 Proof of Theorem 1.3

Conjecture 1.2 does not hold for $q = 4$, because the numbers in consideration, namely

$$\log\Big(2\sin\frac{\pi}{4}\Big) = \log\sqrt{2} = \frac{1}{2}\log 2, \quad \log 2, \quad \text{and} \quad \pi$$

are $\mathbb{Q}$-linearly dependent.

Henceforth, assume that $q \ge 6$. We prove the linear dependence of the numbers

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \le a < \frac{q}{2}\Big\}$$

by giving an explicit $\mathbb{Q}$-relation among them. Before proceeding, we note that by (4.2), it suffices to exhibit a relation among logarithms of cyclotomic numbers. Now, since q is not prime, there is a divisor d of q such that $d \ne 1, q$. For such a divisor d, we have the following polynomial identity in $\mathbb{C}[X, Y]$:

$$X^{q/d} - Y^{q/d} = \prod_{j=1}^{q/d}\big(X - \zeta_{q/d}^{j} Y\big),$$

where $\zeta_{q/d} = e^{2\pi i d/q}$. Substituting $X = 1$ and $Y = \zeta_q^a$ for $(a, q) = 1$, we have

$$1 - e^{2\pi i a/d} = \prod_{j=1}^{q/d}\big(1 - e^{2\pi i(dj/q + a/q)}\big) = \prod_{j=1}^{q/d}\big(1 - e^{2\pi i(a+dj)/q}\big).$$

Thus, taking absolute values of both sides of the above equation gives us

$$\Big(\big|1-\zeta_q^{aq/d}\big|\Big) = \prod_{j=1}^{q/d}\Big(\big|1-\zeta_q^{(a+dj)}\big|\Big).$$

Taking logarithms of both sides, we obtain the $\mathbb{Q}$-linear relation

$$\log\Big(\big|1-\zeta_q^{aq/d}\big|\Big) - \sum_{j=1}^{q/d}\log\Big(\big|1-\zeta_q^{(a+dj)}\big|\Big) = 0,$$

for all $1 \le a < q$ and $(a, q) = 1$ and $d|q$, $d \ne 1, q$. Hence, using (4.2), we have

$$\text{(4.3)}\qquad \log\Big(2\sin\Big(\frac{aq}{d}\,\frac{\pi}{q}\Big)\Big) - \sum_{j=1}^{q/d}\log\Big(2\sin\frac{(a+dj)\pi}{q}\Big) = 0.$$

Since we want a linear relation among

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \le a < \frac{q}{2}\Big\},$$

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we will replace $\log(2\sin(b\pi/q))$ by $\log(2\sin((q-b)\pi/q))$ whenever $b \ge q/2$. This is valid by (4.2). Now, we make the following observations. Suppose that there exists an integer k such that $1 \le k < q/2$ and $k \equiv a + dj \equiv a + dl \bmod q$, for some $1 \le j, l \le q/d$, and $j \ne l$. This implies that $q | d(j-l)$, which is impossible, since $(j-l) < q/d$. Thus,

$$a + dj \not\equiv a + dl \bmod q, \tag{4.4}$$

for $1 \le j, l \le q/d$ and $j \ne l$. Similarly,

$$-(a + dj) \not\equiv -(a + dl) \bmod q, \tag{4.5}$$

for $1 \le j, l \le q/d$ and $j \ne l$. Suppose there exists a k such that $1 \le k < q/2$ and

$$k \equiv a + dj \equiv -(a + dl) \bmod q,$$

for $1 \le j, l \le q/d$ and $j \ne l$. Thus, $q | (2a + d(j+l))$. Since $d|q$, we have $d | (2a + d(j-l))$, *i.e.*, $d|2a$. But $(a,q) = 1$. Hence, $(a,d) = 1$, which implies that $d|2$. We assumed that $d \ne 1, q$. Therefore, $d = 2$. As a result, we have

$$a + dj \not\equiv -(a + dl) \bmod q, \tag{4.6}$$

for $1 \le j, l \le q/d$, and $j \ne l$ unless $d = 2$.

Thus, for $(a,q) = 1$, $d|q$ and $2 < d < q$, (4.3) along with (4.4), (4.5), and (4.6) give us a non-trivial $\mathbb{Q}$-relation, namely,

$$\mathfrak{R}_{a,d} := \sum_{1 \le k < q/2} \alpha_k \log\Big(2\sin\frac{k\pi}{q}\Big) = 0,$$

where α_k is determined as follows:

$$\alpha_k = -1 \text{ if} \begin{cases} \text{either } (aq/d \bmod q) < q/2,\ k \not\equiv aq/d \bmod q \ \&\ k \equiv \pm(a + dj) \bmod q, \\ \text{or } (aq/d \bmod q) \ge q/2,\ k \not\equiv -(aq/d) \bmod q \ \&\ k \equiv \pm(a + dj) \bmod q, \end{cases}$$

for some $1 \le j \le q/d$,

$$\alpha_k = 1 \text{ if} \begin{cases} \text{either } (aq/d \bmod q) < q/2,\ k \equiv aq/d \bmod q \ \&\ k \not\equiv \pm(a + dj) \bmod q, \\ \text{or } (aq/d \bmod q) \ge q/2,\ k \equiv -(aq/d) \bmod q \ \&\ k \not\equiv \pm(a + dj) \bmod q, \end{cases}$$

for some $1 \le j \le q/d$ and $\alpha_k = 0$, otherwise.

To see that the above relation is non-trivial for q not prime and $q \ge 6$, note that at least one of the following scenarios happens: either $(aq/d \bmod q) < q/2$, in which case for $k \equiv aq/d \bmod q$, we have $\alpha_k = \pm 1$, or $(aq/d \bmod q) \ge q/2$, in which case for $k \equiv -(aq/d) \bmod q$, we have $\alpha_k = \pm 1$.

Hence, the numbers under consideration in Conjecture 1.2 are $\mathbb{Q}$-linearly dependent. As a result, Livingston's conjecture is false when q is a composite number greater than or equal to 4.

4.2 Proof of Theorem 1.4

We use the theory of Dedekind determinants developed in [8] and our knowledge of Dirichlet L-functions to prove that Conjecture 1.2 is true when the modulus q is prime. Consequently, let p be an odd prime. Our aim is to prove that the numbers

$$\Big\{ \log\Big(2\sin\frac{a\pi}{p}\Big) : 1 \le a \le \frac{p-1}{2} \Big\} \quad \text{and} \quad \pi$$

are $\overline{\mathbb{Q}}$-linearly independent.

Suppose, to the contrary, that the above numbers have a $\overline{\mathbb{Q}}$-linear relation among them. Thus, there exist algebraic numbers $\beta_0, \beta_1, \dots, \beta_r$, not all zero, such that

$$\beta_0 \pi + \sum_{a=1}^{r} \beta_a \log\Big(2 \sin \frac{a\pi}{p}\Big) = 0, \tag{4.7}$$

where $r = (p-1)/2$. If $\beta_0 \neq 0$, then (4.7) does not hold by the following lemma.

Lemma 4.1 ([7]) *If $c_0, c_1, \dots, c_n$ are algebraic numbers and $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive algebraic numbers with $c_0 \neq 0$, then $c_0\pi + \sum_{j=1}^{n} c_j \log \alpha_j \neq 0$.*

Thus, β_0 must be zero. Now, if the numbers

$$\Big\{ \log\Big(2 \sin \frac{a\pi}{p}\Big) : 1 \le a \le \frac{p-1}{2} \Big\}$$

are $\mathbb{Q}$-linearly independent, then by Theorem 2.1, the above numbers are also $\overline{\mathbb{Q}}$-linearly independent. This contradicts our assumption, and hence, the above numbers must satisfy a $\mathbb{Q}$-linear relation. Thus, there exist $b_1, b_2, \dots, b_r$ such that

$$\sum_{a=1}^{r} b_a \log\Big(2 \sin \frac{a\pi}{p}\Big) = 0. \tag{4.8}$$

On clearing denominators, we can assume that

$$b_a \in \mathbb{Z}, 1 \le a \le \frac{(p-1)}{2}.$$

Since log denotes the principal branch and $\sin a\pi/p \in \mathbb{R}_{>0}$, (4.8) gives us the multiplicative relation

$$\prod_{a=1}^{r} \Big(2 \sin \frac{a\pi}{p}\Big)^{b_a} = 1.$$

Using (4.1), this relation can be interpreted as a relation among roots of unity and cyclotomic numbers, *i.e.,*

$$\prod_{a=1}^{r} \Big(i e^{-ia\pi/p} (1 - \zeta_p^a) \Big)^{b_a} = 1.$$

The above relation can be further simplified by raising both sides of the equation to the $4p$-th power. Since $(ie^{-ia\pi/p})^{4p} = 1$, we are now left with the simpler multiplicative relation,

$$\prod_{a=1}^{r} \big(1 - \zeta_p^a\big)^{B_a} = 1, \tag{4.9}$$

where $B_a := 4pb_a$ and each factor in the product belongs to the cyclotomic field $\mathbb{Q}(\zeta_p)$.

Let G be the group $(\mathbb{Z}/p\mathbb{Z})^* / \{\pm 1\}$. Let $c \in G$ and σ_c be the unique automorphism of $\mathbb{Q}(\zeta_p)$ such that $\sigma_c(\zeta_p) = \zeta_p^c$.

The action of $\sigma_{c^{-1}}$ on (4.9) gives us

$$\prod_{a=1}^{r}\left(1-\zeta_p^{ac^{-1}}\right)^{B_a}=1.$$

On taking logarithm of the above equation, we obtain the relation

$$\sum_{a=1}^{r} B_a \log\left(2\sin\frac{ac^{-1}\pi}{p}\right)=0, \tag{4.10}$$

for all $1\le a\le r$ and $1\le c\le r$.

Define an $r\times r$ matrix $\mathfrak{M}$ whose (a,c)-th entry is

$$\log\left(2\sin\frac{ac^{-1}\pi}{p}\right).$$

Thus, (4.10) can be rewritten as a matrix equation, *i.e.*, $\mathfrak{M}v=0$, where v is the $r\times 1$ column vector with the a-th entry being B_a. Since (4.8) was a non-trivial relation, $v\ne 0$. This is possible only if $\det\mathfrak{M}=0$.

Let $\mathfrak{M}^T$ denote the transpose of $\mathfrak{M}$. Notice that $\mathfrak{M}^T$ is a matrix of the Dedekind type with $\mathfrak{f}\colon G\to\mathbb{C}$ given by

$$\mathfrak{f}(a)=\log\left(2\sin\frac{a\pi}{p}\right),$$

where G is as defined above. As mentioned in Theorem 2.2, $\mathfrak{M}^T$ is invertible if and only if

$$S_\chi:=\sum_{a=1}^{r}\mathfrak{f}(a)\chi(a)\ne 0,$$

for all characters χ of the group G. Observe that all characters of the group G are precisely the even Dirichlet characters modulo p. Thus, for a non-trivial even Dirichlet character χ, we can use (4.2) to express S_χ as:

$$\begin{aligned}S_\chi&=\sum_{a=1}^{r}\chi(a)\log\left(2\sin\frac{a\pi}{p}\right)=\sum_{a=1}^{r}\chi(a)\log\left(\left|1-\zeta_p^a\right|\right)\\&=\frac{1}{2}\sum_{a=1}^{p-1}\chi(a)\log\left(\left|1-\zeta_p^a\right|\right)=-\frac{p}{2\tau(\overline{\chi})}L(1,\overline{\chi}),\end{aligned}$$

where the last equality follows from Lemma 2.3. By a famous theorem of Dirichlet [6, Sections 2.3 and 2.4],

$$L(1,\overline{\chi})\ne 0,$$

for non-trivial Dirichlet character χ. Therefore, $S_\chi\ne 0$ when χ is a non-trivial character on G.

Now, let χ_0 be the trivial character on G, *i.e.*, χ_0 is the trivial Dirichlet character modulo p. Then the factor S_{χ_0} is

$$\begin{aligned}S_{\chi_0}&=\sum_{a=1}^{r}\mathfrak{f}(a)=\sum_{a=1}^{r}\log\left(2\sin\frac{a\pi}{p}\right)=\sum_{a=1}^{r}\log\left(\left|1-\zeta_p^a\right|\right)\\&=\frac{1}{2}\log\left(\prod_{a=1}^{p-1}\left|1-\zeta_p^a\right|\right)=\frac{1}{2}\log p\ne 0,\end{aligned}$$

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where the last equality can be derived by noting that

$$\frac{1-X^p}{1-X} = \sum_{j=0}^{p-1} X^j = \prod_{a=1}^{p-1}(1-\zeta_p^a X),$$

substituting $X = 1$ and taking absolute values of both sides. Thus, $S_{\chi_0} \neq 0$.

Hence, $\mathfrak{M}^T$, and in turn, $\mathfrak{M}$ is invertible. Therefore, $\nu = 0$, which is a contradiction. This proves the theorem. ∎

4.3 Proof of Corollary 1.5

Suppose that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = L(1,f) = 0.$$

From Theorem 1.4, we see that Conjecture 1.2 is true when the period of f is an odd prime, *i.e.,* that the numbers

$$\Big\{ \log\Big(2\sin\frac{a\pi}{p}\Big) : 1 \le a \le \frac{p-1}{2}\Big\} \quad \text{and} \quad \pi$$

are $\overline{\mathbb{Q}}$ - linearly independent. Thus, the relation obtained from (3.5), namely,

$$0 = \frac{-\pi}{2p}\sum_{a=1}^{p-1} f(a)\cot\Big(\frac{a\pi}{p}\Big) + \frac{2}{p}\sum_{b=1}^{r}\Big\{\Big[\sum_{a=1}^{p-1} f(a)\cos\Big(\frac{2\pi ab}{p}\Big)\Big]\log\Big(2\sin\frac{\pi b}{p}\Big)\Big\}$$

is a trivial relation. Therefore, the co-efficients of π and $\log(2\sin(b\pi/p))$ must all be zero. This proves the corollary. ∎

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