Ergod. Th. & Dynam. Sys. (1986), **6**, 415–448
Printed in Great Britain

Chains, entropy, coding

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(*Received* 10 *January* 1985)

Abstract. Various definitions of the entropy for countable-state topological Markov chains are considered. Concrete examples show that these quantities do not coincide in general and can behave badly under nice maps. Certain restricted random walks which arise in a problem in magnetic recording provide interesting examples of chains. Factors of some of these chains have entropy equal to the growth rate of the number of periodic orbits, even though they contain no subshifts of finite type with positive entropy; others are *almost sofic* – they contain subshifts of finite type with entropy arbitrarily close to their own. Attempting to find the entropies of such subshifts of finite type motivates the method of entropy computation by loop analysis, in which it is not necessary to write down any matrices or evaluate any determinants. A method for variable-length encoding into these systems is proposed, and some of the smaller subshifts of finite type inside these systems are displayed.

1. *Introduction*

Among the most familiar systems in symbolic dynamics are the subshifts of finite type, or SFT's for short. Their popularity arises from the relative ease with which they can be analyzed as well as their importance for coding and for the classification of certain differentiable dynamical systems. A larger class, which has the desirable property of being closed under factors, is that of the sofic systems – continuous images of SFT's under shift-commuting maps. All of these systems are intrinsically ergodic – they have unique (shift-invariant Borel probability) measures of entropy equal to the topological entropy. For a survey of the useful properties of these kinds of systems as well as an explanation of how they come up in problems of magnetic recording, see [**2**] and [**17**].

The generalization of the idea of a subshift of finite type to the case of a countable alphabet, called a countable-state topological Markov chain, or, more briefly, simply a *chain*, is a natural one to make and comes up in various contexts, including again in a very specific problem in magnetic recording (which we will discuss below). The basic results on chains were obtained by D. Vere-Jones [**27**], [**28**]; an up-to-date account is in [**24**]. These systems, which can be considerably more complicated than the finite-alphabet SFT's (for example, they need no longer be intrinsically ergodic), have been studied from the dynamical point of view by Dinaburg [**10**], Gurevič [**14**], [**15**], Weiss [**30**], Blanchard [**3**], Blanchard and Hansel [**4**], Salama

[**22**], Wagoner [**29**] and others. Some immediate problems are caused by the facts that various natural ways to define the entropy of a chain do not all coincide, as they do in the finite-alphabet case, and that certain kinds of entropy can increase under factor maps (one-block maps, or finite labellings). We give several such examples below.

Our interest in chains arises from their usefulness in analyzing a class of restricted random walks found in a problem of encoding data for recording on a magnetic medium which was first considered by P. Siegel [**25**]. When encoding signals by the two symbols + and −, in order to guarantee that the power spectrum of the encoded messages vanish at a certain frequency α (so that this frequency might be available, for example, for feedback, control, or checking operations), one might require that the accumulated sums

$$S_n(x, \alpha) = \sum_{k=0}^{n-1} x_k e^{ik\alpha}$$

stay bounded by a certain constant c, for then the power being transmitted at frequency α is

$$\lim_{n\to\infty} \frac{1}{n}|S_n|^2 = P(x, \alpha) = 0.$$

(Here x is a doubly infinite sequence on the symbols 1 and −1, representing an input message.) This leads us to consider three systems: the *subshift* of all those sequences x in $\{1, -1\}^Z$ for which

$$\left|\sum_m^n x_k e^{ik\alpha}\right| < c \qquad \text{for all } m, n;$$

the *restricted random walk* on the set of all points in the disk of radius c in the complex plane reached by starting at 0, taking one step forward or back, turning through an angle α, and repeating; and the *chain* of all sequences, on the countable alphabet consisting of the attainable points in the disk, which are consistent with the transitions allowed by the random walk. Siegel [**25**] noted that when $\alpha = 2\pi/m$ and $m = 1, 2, 3, 4$, or 6, the resulting subshift is sofic; but for other values of m the set of states has accumulation points in the disk, and the dynamics of the subshift can be complicated.

We have two main results. First, in the case $\omega = e^{i\alpha}$ is transcendental over $\{1, -1\}$ (i.e. satisfies no equation of the form $\sum_{k=0}^{n-1} \xi_k \omega^k = 0$, where each $\xi_k = \pm 1$) and c is large enough, the subshifts generated in this manner are most definitely *not* almost sofic, since they contain no SFT's whatsoever with positive entropy; yet they still have one desirable property of SFT's and sofic systems: each has topological entropy equal to the growth rate of the number of periodic orbits. This latter property of a dynamical system we call *periodic saturated.* Second, in the case where α is rational, we can prove that most of the subshifts generated in this way are almost sofic. A practical consequence is that an arbitrarily large amount of the entropy of any such system is available for machine-implementable and finitely understandable coding. (Recall that the sofic systems are exactly the ones whose elements can be recognized by finite-state machines.) We suggest ways to construct variable-length or constant-length codes into these disk systems. We also give some sufficient conditions for the subshifts determined by restricted random walks in groups to be almost sofic.

For small values of c and particular choices of α the disk systems can be described explicitly: they are SFT's which increase in size, complexity, and entropy as c increases. For each α there is a critical value of c beyond which the corresponding disk system is no longer an SFT. We describe several of the smaller SFT's and compute their entropies (using the 'loop method' which allows one, having counted first returns to a fixed vertex, to write down an equation for e^h immediately, without having to examine any matrices or evaluate any determinants). Completely to describe the variation of these subshifts and their entropies as functions of c and α is an intriguing but extremely difficult problem.

I would like to thank the many people who contributed ideas to this work, especially F. Blanchard, I. Salama, S. Williams, and most of all B. Marcus, who first conjectured that the disk systems might be almost sofic.

This research was supported in part by NSF grants MCS-8001590 and DMS-8400730.

2. *Various kinds of entropy for chains*

Let Γ be a strongly connected directed graph on a countable set of vertices $A=\{a_1, a_2, \ldots\}$. The elements of the alphabet A are also called states, letters, or symbols. Give A the discrete topology, form the product space $A^{\mathbb{Z}}$, and consider the set $U(\Gamma)$ consisting of all those doubly infinite sequences with entries from A which are consistent with the graph Γ:

$$U(\Gamma)=\{x\in A^{\mathbb{Z}}\colon \text{for all } i, \text{ there is an edge in } \Gamma \text{ from } x_i \text{ to } x_{i+1}\}.$$

Then $U(\Gamma)$, together with the shift transformation σ defined by $(\sigma x)_i = x_{i+1}$ for all i, is a (non-compact) dynamical system, called the *chain* determined by the directed graph Γ. Fix a vertex v in Γ and define

$B_n = B_n(v)$ = number of paths of length n in Γ from v to v;

$f_n = f_n(v)$ = number of paths of length n in Γ from v to v with no other occurrences of v in between;

R = radius of convergence of $\sum_{n=1}^{\infty} B_n t^n$.

D. Vere-Jones [**27**], [**28**] classified the graph Γ, or its associated 0, 1 transition matrix M, as *transient*, *null recurrent*, or *positive recurrent* according to the following table:

	transient	null recurrent	positive recurrent
$\sum f_n R^n$	<1	$=1$	$=1$
$\sum nf_n R^n$	$=\infty$	$=\infty$	$<\infty$
$\sum B_n R^n$	$<\infty$	$=\infty$	$=\infty$
$\lim_{n\to\infty} B_n R^n$	$=0$	$=0$	>0.

This classification and the number R, called the *parameter of convergence* of Γ or of M, are independent of the choice of the base vertex v, so long as Γ is connected; indeed, we arrive at the same R and the same classification if we consider paths between any pair of fixed vertices in Γ instead of from v to itself. We assume henceforth that $R>0$. The number $1/R$ is an analogue of the Perron–Frobenius maximum positive eigenvalue of a finite non-negative matrix. This is especially so in the positive recurrent case, when $1/R$ is actually an eigenvalue of M and has positive left and right eigenvectors. Then $-\log R$ is a candidate for the topological entropy of $U(\Gamma)$.

Gurevič [**14**] considered a compactification $U(\Gamma)$ obtained as follows. Give A the totally bounded metric compatible with the discrete topology obtained by identifying a_i with the point $1/i \in [0, 1]$, and let $\bar{A} = A \cup \{0\}$. Then we let $\bar{U}(\Gamma)$ be the closure of $U(\Gamma)$ in the compact space $\bar{A}^{\mathbf{Z}}$. The shift transformation σ on the compact metric space $\bar{U}(\Gamma)$ now has a well-defined topological entropy, which we denote by $h_G(\Gamma)$. Gurevič showed that $h_G(\Gamma)$ coincides with the supremum of the topological entropies of the SFT's $U(F)$ determined by finite connected subgraphs F of Γ, and that $h_G(\Gamma) = -\log R$. Thus the topological entropy of this compactification of $U(\Gamma)$ coincides with the growth rate of the number $B_n(v)$ of loops from any fixed vertex to itself.

Gurevič [**15**] also discussed the question of the intrinsic ergodicity of $\bar{U}(\Gamma)$. Continue to assume that $R>0$ and Γ is connected. Gurevič showed that $\bar{U}(\Gamma)$ has a measure of maximal entropy if and only if Γ is positive recurrent, and in this case the measure of maximal entropy is unique. Indeed, the maximal measure is (countable-state) Markov and is given by formulae similar to those for the Shannon–Parry measure on an SFT (see [**20**]).

There are other possible definitions of the topological entropy of $U(\Gamma)$. First we mention the entropy $h^*(\Gamma)$ considered by Salama [**22**], which is defined to be the growth rate of the number of all paths in Γ which begin at a fixed vertex v: if $T_n(v)$ is the number of all allowed paths $vx_2 \ldots x_n$ of length n in Γ, then

$$h^*(\Gamma) = \limsup_{n\to\infty} \frac{1}{n} \log T_n(v).$$

Clearly $h^*(\Gamma) \geq h_G(\Gamma)$. If A is given the totally bounded metric mentioned above, then $U(\Gamma)$ also becomes a metric space, and $h_G(\Gamma)$ coincides with the topological entropy of the uniformly continuous map σ as defined by Bowen [**6**]; on the other hand, if A is given the metric according to which any pair of points are distance 1 apart, then the Bowen entropy coincides with $h^*(\Gamma)$. The latter observation has also been made by Wagoner [**29**]. In general $h_G(\Gamma)$ and $h^*(\Gamma)$ can be different: Salama has shown how to construct, given $0 \leq \alpha \leq \beta$, a connected, locally finite (with even a bounded number of edges into and out of each vertex) graph Γ for which $h_G(\Gamma) = \alpha$ and $h^*(\Gamma) = \beta$.

The idea of Gurevič was to refer the entropy of a chain to that of SFT's by considering finite-alphabet subsystems of the given dynamical system. There is a dual method of reducing to the finite-alphabet case: instead of taking finite-alphabet subsystems, we may consider *labellings* of the (vertices or edges of the) graph Γ by a finite alphabet. Most simply, let us consider a map $\pi: A \to B$, where B is a finite set. (We could also deal with a *finite code* $\pi: A^r \to B$ for some r.) Then π determines a map $\pi: A^{\mathbf{Z}} \to B^{\mathbf{Z}}$ by $(\pi x)_n = \pi(x_n)$ for all n. The image of $U(\Gamma)$ under π is clearly shift-invariant, but it is not in general a subshift, since it may not be closed. Define $D_\pi(\Gamma)$ to be the closure in the compact space $B^{\mathbf{Z}}$ of $\pi(U(\Gamma))$, so that $D_\pi(\Gamma)$ is a subshift; it consists of all those sequences in $B^{\mathbf{Z}}$ whose finite sub-blocks appear along paths in $U(\Gamma)$.

We will see in the next section that there are even very nice chains and labellings (one-block maps on positive recurrent chains) for which

$$h_{\text{top}}(D_\pi(\Gamma) > h_G(\Gamma).$$

This possibility complicates the proof that the disk systems with a spectral notch at a rational frequency are almost sofic. Remember that in this case we have a graph in a disk of radius c in the complex plane whose edges are labelled by +'s and −'s according to the selections made by a (restricted) random walker. Serious confusion could result in this case if the topological entropy of the subshift in $\{1, -1\}^{\mathbf{Z}}$ were larger than the Gurevič entropy of the corresponding chain. We are able to show that in this case, luckily, this does not happen.

There are still more possible definitions of the topological entropy of $U(\Gamma)$: for example, each choice of a metric compatible with the discrete topology on the alphabet A will provide, via Bowen's definition, a version of the topological entropy for $U(\Gamma)$. One could also find the growth rate of the number of periodic orbits passing through a fixed vertex at time 0 or estimate the algorithmic complexity of the allowed sequences as in [7]. It is an interesting problem to find necessary or sufficient conditions for some of these versions of topological entropy to agree.

3. *Some bad examples*

Let us make more precise the definition of *almost sofic*, a concept which we have already mentioned above. For the moment we will deal with a finite alphabet B and the shift transformation $\sigma: B^{\mathbf{Z}} \to B^{\mathbf{Z}}$. By a *subshift* we mean any closed shift-invariant subset $X \subset B^{\mathbf{Z}}$. Recall that a subshift is called *sofic* when it is the continuous image under a shift-commuting map of an SFT. Every sofic system is in fact the boundedly finite-to-one homomorphic image of an SFT (in fact we can have the map be one-to-one a.e. with respect to every ergodic measure which has full support). This implies in particular that sofic systems are intrinsically ergodic. Useful characterizations of sofic sysems can be given in terms of follower sets, semigroups, finite-state automata, and formal languages; see [3] and [4] for a discussion and further references to the literature.

We define a subshift X to be *almost sofic* if given $\varepsilon > 0$ there is an SFT $\Sigma \subset X$ with $h_{\text{top}}(\Sigma) > h_{\text{top}}(X) - \varepsilon$. Marcus [17] has pointed out that sofic systems are almost sofic in this sense. Typical systems are not almost sofic; minimal subshifts with positive entropy are ready examples. The almost sofic systems may be considered to be among the most useful ones for the purposes of encoding data. Frequently the sequences which result from coding have to satisfy certain restrictions, for example to make them suitable for magnetic recording (see [2] and [17].) These restrictions might be summarized by requiring the image of the coding to be an SFT, a sofic system, or another subshift. If the subshift is almost sofic, then we are assured that as much of its topological entropy as we please is available for machine-implementable coding.

It would be nice if the class of almost sofic systems possessed good dynamical properties, such as intrinsic ergodicity, being closed under factors, being closely related to SFT's or chains, etc. In this section we present several examples to indicate some of the bad things that can happen. A useful class of almost sofic, intrinsically ergodic systems which contains the sofic systems and is closed under the usual dynamical operations such as passing to factors has not yet been identified. Perhaps such a class could be defined by some combination of the following properties:

(1) The family of all follower sets of left rays is countable.

(2) It is a factor of a chain by a finite labelling.

(3) It is a factor of a chain by a finite labelling, with countable fibres.

(4) The semigroup of all finite sub-blocks is countably presented.

(5) The allowable sub-blocks are determined by a countable semigroup (in the same way that for a sofic system they are determined by a finite semigroup).

Example 3.1. *An almost sofic system which is not intrinsically ergodic.*
This example was suggested by Blanchard. The idea is that the system should be almost sofic in two ways, in that there are two distinct ways to approximate it from the inside by SFT's.

For the specific example, let X be the subshift of $\{a_1, a_2, b\}^{\mathbf{Z}}$ of all those sequences whose finite sub-blocks are sub-blocks of concatenations of blocks of the form $\{a_1, a_2\}^n b^n$, $n \geq 1$. Then it is not hard to see that $h_{\text{top}}(X) = \log 2$. Since X contains the 2-shift $\{a_1, a_2\}^{\mathbf{Z}}$, the Bernoulli measure $B(\frac{1}{2}, \frac{1}{2})$ on this 2-shift will be an ergodic measure of maximal entropy. On the other hand, assigning equal probabilities to a_1 and a_2 and probability 2^{-k-1} to each cylinder set B_k determined by a block of the form $ba^k b^{k-1}$ will also define a measure with entropy $\log 2$. This is so because the cylinder set $[b] = \{x \in X: x_0 = b\}$ has measure $\frac{1}{2}$, and the first return map σ_b to this set is Bernoulli with independent generator $\{B_k: k = 1, 2, \dots\}$, so that

$$h_{\mu_b}(\sigma_b) = -\sum_{k=1}^{\infty} \mu_b(B_k) \log \mu_b(B_k)$$
$$= -\sum_{k=1}^{\infty} 2^{-k} \log (2^{-k}) = 2 \log 2,$$

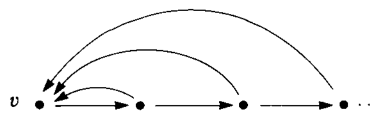
and

$$h_\mu(\sigma) = \mu[b] h_{\mu_b}(\sigma_b) = \log 2.$$

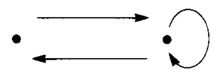
Here one way to approximate X from the inside in entropy by SFT's is with the 2-shift $\{a_1, a_2)^{\mathbf{Z}}$. Another way is with the SFT's Σ_m consisting of all sequences whose sub-blocks are sub-blocks of concatenations of the blocks $\{a_1, a_2\}^n b^n$, $1 \leq n \leq m$.

Example 3.2. *A positive recurrent chain with a finite labelling which is not almost sofic and is not intrinsically ergodic.*
This example is a variation on an idea of Salama **[22]**. Start with the chain $U(\Gamma)$ on countably many states (all represented below by dots) determined by the following

graph Γ:

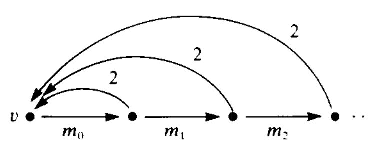


The number of loops beginning with v which first return to v at time n is 1 for each $n \geq 2$. This graph has the same loop structure based at v as the notorious golden mean SFT, which has graph



Therefore these two graphs both have $R = 1/\phi$, where ϕ is the golden mean, and hence topological entropy $\log \phi$; and since every SFT is positive recurrent, Γ is positive recurrent.

Now choose a minimal set $M \subset \{0, 1\}^{\mathbb{Z}}$ which is not intrinsically ergodic and for which $h_{\text{top}}(M) > h_G(U(\Gamma))$ (if necessary, see [**8**, p. 157]). Then M is certainly not almost sofic, since a minimal set cannot contain any SFT's whatsoever with positive entropy. Let $m_0 m_1 m_2 \ldots$ be the right half of an element of M, and use this to label the horizontal arrows in Γ:



The remaining arrows may be labelled by 2. This determines a 2-block map π: $U(\Gamma) \to \{0, 1, 2\}^{\mathbb{Z}}$; let X denote the closure of $\pi(U(\Gamma))$. Then $X \supset M$, so

$$h_{\text{top}}(X) \geq h_{\text{top}}(M) > h_G(U(\Gamma)).$$

We claim that actually

$$h_{\text{top}}(X) = h_{\text{top}}(M).$$

For, fix an $n \geq 1$ and consider a typical n-block

$$x_1 x_2 \ldots x_k 2 x_{k+2} \ldots 2 x_l \ldots x_n$$

in X, where the displayed 2's are the first and last ones which appear. There are of the order of $\exp(k h_{\text{top}}(M))$ initial blocks $x_1 \ldots x_k$, of the order of $\exp((l-k-1)\log\phi) \leq \exp((l-k-1)h_{\text{top}}(M))$ middle blocks $2x_{k+2}\ldots 2$, and of the order of $\exp((n-l+1)h_{\text{top}}(M))$ terminal blocks $x_l \ldots x_n$. Thus there are at most $\exp(n h_{\text{top}}(M))$ blocks of this form; summing over the $n(n-1)/2$ possible choices of k and l, we see that the number of n-blocks in X has growth rate $\exp(n h_{\text{top}}(M))$.

These considerations are also enough to show that X *is not almost sofic.* For let Y be a closed invariant transitive subset of X which is disjoint from M. Then every element of Y must contain infinitely many 2's, so the growth rate of the number of blocks in Y, which equals the growth rate of the number of blocks in Y that begin and end with 2, is no greater than $\exp(n \log \phi)$. Thus in order to find subsystems

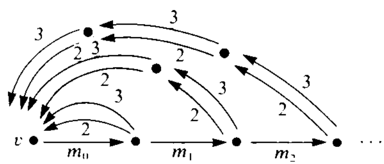
https://doi.org/10.1017/S014338570000359X Published online by Cambridge University Press

of X with larger entropy than this, we would have to consider Y's which intersected M. However, any such Y would have to contain M. Suppose now that Y were an SFT with $M \subset Y \subset X$. By what we have shown, $h_{\text{top}}(M) = h_{\text{top}}(Y) = h_{\text{top}}(X)$. Since Y is an SFT and M is minimal, there are many blocks which occur in Y which do not occur in M. Choose any such block and let Y_0 be the SFT obtained from Y by excluding this block. Then $M \subset Y_0 \subset Y \subset X$, yet $h_{\text{top}}(Y_0) < h_{\text{top}}(Y)$, and this is impossible.

It follows also that X *is not intrinsically ergodic.* For since M has at least two measures with entropy equal to $h_{\text{top}}(M) = h_{\text{top}}(X)$, so does X.

If we replace the symbol 2 by 0 in X, then the resulting factor will still be neither almost sofic nor intrinsically ergodic. Further, by passing to 2-block representations (in which new alphabets consist of 2-blocks on the original alphabets), we can arrange for such an example in which the labelling π is a one-block map.

Example 3.3. *An almost sofic system with a factor which is not almost sofic.* Consider a chain $U(\Gamma)$ with labelling π by 0, 1, 2, 3 as shown:



Here the m_i are as in example 3.2. Again let X denote the closure of $\pi(U(\Gamma))$.

Note that $X \supset \{(2, 3)\}^{\mathbb{Z}}$, so that $h_{\text{top}}(X) \geq \log 2$. We claim that actually $h_{\text{top}}(X) = \log 2$. First note that the number of loops of length $2n$ based at v with no other v's in between is $f_{2n} = 2^n$. From this we can show by induction that (with $B_0 = 1$)

$$B_{2n} = 2^n + \sum_{j=1}^{n-1} 2^j B_{2(n-j)} = 2^{2n-1} \qquad \text{for } n \geq 1.$$

It follows that $R = \frac{1}{2}$ and Γ is positive recurrent. Now a typical n-block in X has the form PQR, where P and R contain only the symbols 0 and 1. There are of the order of at most $B_{l(Q)} = 2^{l(Q)-1}$ choices for the block Q, and at most

$$\exp\,(l(P)h_{\text{top}}(M))\,\exp\,(l(R)h_{\text{top}}(M)) < 2^{(l(P)+l(R))}$$

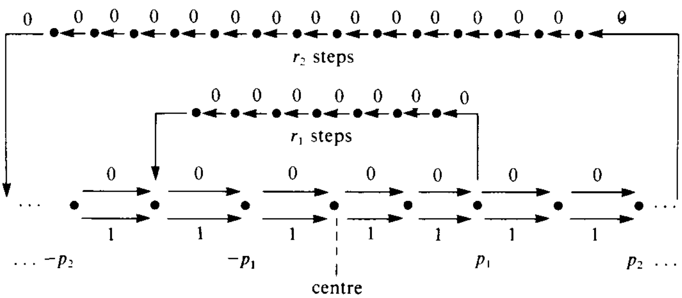
choices of P and R, since $h_{\text{top}}(M) < \log 2$; thus there are of the order of at most 2^n n-blocks in X.

It follows immediately that X is almost sofic, since the 2-shift $\Sigma_2 \subset X$ has full entropy. Now consider the factor Y that results from X by replacing the symbol 3 everywhere by 2. This factor is very similar to the one in example 3.2, and it fails to be almost sofic for pretty much the same reason. We have lengthened the consecutive strings of 2's, so this just drives the entropy of this part of Y down. Again $h_{\text{top}}(Y) = h_{\text{top}}(M)$, and Y cannot be almost sofic.

These examples might lead to the suspicion that the extra entropy (along with the non-sofic and non-intrinsically-ergodic behaviour) is picked up by the factors when we take the closure of the image, and perhaps such phenomena might not occur if the finite labelling π produced a map *onto* the downstairs subshift. With a little more effort, though, we can show that the entropy can still go up in this case as well.

Example 3.4. *A graph of entropy less than* $\log 2$ *with a finite labelling which contains the* 2*-shift.*

Choose fairly rapidly growing sequences of integers $0<p_1<p_2<\dots$ and $0<r_1<r_2<\dots$ and form the graph Γ with labelling of paths by 0's and 1's as shown:



The k'th returning (leftward) path consists of r_k edges which are all labelled 0. Then this is a connected graph whose labelling X contains the 2-shift Σ_2 and hence equals Σ_2.

The number of paths of length $2p_k+r_k+1$ from the centre c to itself which do not hit c in between is 2^{2p_k}, so the loop function for this graph is

$$\sum_{n=1}^{\infty} f_n x^n = \sum_{n=1}^{\infty} 2^{2p_k} x^{2p_k+r_k+1}.$$

If we consider a finite connected subgraph of Γ, it will have loop function

$$L_N(x) = \sum_{n=1}^{N} 2^{2p_k} x^{2p_k+r_k+1}.$$

Since an SFT is always recurrent (or by the loop analysis of § 7), the SFT determined by such a finite connected subgraph has entropy $-\log(t_N)$, where t_N is the solution of the equation $L_N(x)=1$; by [**14**], $\log(t_N)$ converges to $h_G(\Gamma)$. It is clear that given any sequence $\{p_k\}$, we can choose the r_k increasing rapidly enough that all the t_N will stay below some level $2-\varepsilon$.

4. *The disk systems*

Before data is stored as a string of 1's and -1's on a magnetic disk or tape, frequently it is recoded to assure that the transitions between the blocks of constant sign appear neither too far apart nor too close together, or that the accumulated charge starting at any time (i.e. the forward sum of the sequence entries) stays bounded. Such restrictions are due to, for example, the signal-detecting capabilities of electronic devices and the limited accuracy of motors and clocks. Thus one demands that $\Sigma_2=\{0,1\}^{\mathbf{Z}}$, representing arbitrary input data, be encoded into the subshift (which one hopes is of finite type or sofic, or, failing that, at least almost sofic) consisting of all sequences in $\{1,-1)^{\mathbf{Z}}$ which meet the given restrictions. Of course one seeks codes that are efficient, have limited error propagation, and can be easily implemented – or, more realistically, ones with a compromise among these competing attributes. See [2], [17], [11], [12], [16], [21], [26], [19] for some of the previous work on this sort of problem.

A further restriction arises from attempts to control the power spectrum of the encoded signal. If the incoming signals $x = \ldots x_{-1}x_0x_1 \ldots \in \{1, -1\}^{\mathbb{Z}}$ are governed by a stationary (i.e. shift-invariant) measure μ on the subshift X, then the *autocorrelation* of the signals is given by

$$A(k) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} x_i x_{i+k} = \int_X x_0 x_k \, d\mu.$$

This positive-definite function on $\mathbb{Z}$ is, by Bochner's theorem, the Fourier transform of a positive measure λ, which is taken as describing the distribution of various frequencies within the typical signal x. In the case where λ is absolutely continuous with respect to Lebesgue measure m on the circle,

$$\frac{d\lambda}{dm}(\theta) = \lim_{n\to\infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} x_k e^{ik\theta} \right|^2 = P(\theta),$$

which is supposed to measure the *power* which the signal is transmitting at frequency θ. It may be desirable occasionally to encode the data so that the resulting signal will have zero power at a certain frequency θ, so that, for example, this frequency will be available for feedback, control, or other system functions. A particularly direct way to assure such a 'spectral notch' is by requiring that

$$S_{0,n-1}(x, \theta) = \sum_{k=0}^{n-1} x_k e^{ik\theta}$$

remain bounded. Thus we define, for $0 \le \theta < 2\pi$ and $c \ge 0$,

$$\Sigma(c, \theta) = \left\{ x \in \{1, -1\}^{\mathbb{Z}} : \left| \sum_{k=m}^{n} x_k e^{ik\theta} \right| < c \quad \text{for all } m \le n \right\}.$$

(To obtain a shift-invariant set, we start the sum at an arbitrary point.) At each time we take a step forward or back and then turn through an angle θ, always refusing to step across the boundary of the disk of radius c. Such systems were first studied by Siegel [25].

For $x \in \{1, -1\}^{\mathbb{Z}}$, $0 \le \theta \le 2\pi$, and integers $m \le n$, let

$$S_{m,n}(x, \theta) = \sum_{m}^{n} x_k e^{ik\theta}.$$

Let $S = \{S_{m,n}(x, \theta) : x \in \{1, -1\}^{\mathbb{Z}}, \ m \le n\}$ be the set of all these sums. $S \times \{k\theta \bmod 1 : k \in \mathbb{Z}\}$ will form the set of vertices of a countable directed graph defined as follows. We draw an edge from one of these points (s_1, t_1) to another (s_2, t_2), and label this edge with $\xi = \pm 1$, if there are m, n such that $s_1 = S_{m,n}(x, \theta)$, $s_2 = s_1 + \xi e^{i(n+1)\theta}$, and $t_2 = t_1 + \theta \bmod 1$. In this way we obtain a countable graph $\Gamma(\theta)$; the vertices are all the possible sums that can be accumulated by starting at any place in any sequence x, together with the possible directions of motion away from

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these vertices, and the transitions between sums and directions are defined in the natural way.

Restrict now to the set of all those states (vertices) (s, t) for which s is in the open disk of radius c centred at the origin, and denote the resulting graph by $\Gamma(c, \theta)$. We can now define several closely related dynamical systems:

(1) $U(c, \theta)$ is the chain on the countable graph $\Gamma(c, \theta)$.

(2) $D(c, \theta)$ is the subshift of $\{1, -1\}^{\mathbb{Z}}$ composed of all those sequences whose sub-blocks are found as labellings of paths in $U(c, \theta)$. Reading the labels on the paths in $U(c, \theta)$ produces a map π: $U(c, \theta) \to \{1, -1\}^{\mathbb{Z}}$; then $D(c, \theta)$ is the closure of the image of π.

(3) $U_r(c, \theta)$ is the chain which results from restricting $\Gamma(c, \theta)$ to only those vertices $z = (s, t)$ in the disk from which one can return arbitrarily close to 0: given $\delta > 0$, there is a sequence $\ldots z_{-1}z_0z_1 \ldots \in U(c, \theta)$ such that $z_0 = z$ and $|s_k| < \delta$ for some $k > 0$.

(4) $D_r(c, \theta)$ is the subshift 'below' $U_r(c, \theta)$, that is, the closure of the image of $U_r(c, \theta)$ under the restriction of the labelling π. This is the system which will draw most of our attention.

(5) $\Delta(c, \theta)$ is the 'chain above' the subshift $\Sigma(c, \theta)$, as in [3]. The countably many states are pairs (F, ξ), where $\xi = \pm 1$ and F is a *follower set* of a block in $\Sigma(c, \theta)$: there is a block B which appears in a sequence in $\Sigma(c, \theta)$ for which

$$F = \{\text{blocks } f\text{: } Bf \text{ appears in a sequence in } \Sigma(c, \theta)\}.$$

The allowed transitions between states are specified as follows: we draw an edge from (F, ξ) to (F', ξ') if and only if

$$F' = \{\text{blocks } f'\text{: } \xi' f' \in F\}.$$

One may think of (F, ξ) as the state in which one is looking at the symbol ξ in position 0 (the present or central spot in a message or sequence) and in which one has the ability to append any blocks in F starting at position 1. The transition to symbol ξ' is legal exactly when this moves one to the state in which one can append blocks f' which would have been legal to append in the previous state if ξ' had been appended first. Then $\Delta(c, \theta)$ consists of all the doubly-infinite sequences on these countably many symbols which are consistent with these allowed transitions.

(6) $\bar{\Delta}(c, \theta)$ is the closure of $\Delta(c, \theta)$ in the compact space $\bar{A}^{\mathbb{Z}}$, where A is the countable alphabet of states described in (5), a_i is identified with $1/i \in [0, 1]$, and $\bar{A} = A \cup \{0\}$ with the usual topology.

(7) In the case where $e^{i\theta}$ is algebraic over $\{1, -1\}$ (e.g. $\theta = 2\pi k/m$ for some integers m and k), restrict the graph $\Gamma(c, \theta)$ to the set of states z in the open disk of radius c from which one can return to 0 *exactly*: there is a sequence $\ldots z_{-1}z_0z_1 \ldots \in U(c, \theta)$ such that $z_0 = z$ and $z_k = 0$ for some $k > 0$. The chain determined by this graph is denoted by $U_0(c, \theta)$.

(8) $D_0(c, \theta)$ is the subshift 'below' $U_0(c, \theta)$. It is the closure of the image of $U_0(c, \theta)$ under the restriction of the labelling π.

(9) The chains $U(c, \theta)$, $U_r(c, \theta)$, and $U_0(c, \theta)$ have closures $\bar{U}(c, \theta)$, $\bar{U}_r(c, \theta)$, and $\bar{U}_0(c, \theta)$, respectively, when their alphabets are compactified as in (6).

Remarks 4.1. (1) Among the algebraic conjugates of some of the T-numbers of Salem [**23**] there are numbers $\omega = e^{i\alpha}$, with α an irrational multiple of π, which are roots of a polynomial with coefficients ± 1. (K. Schmidt has shown that $1 - x - x^2 - x^3 + x^4$ has such a root.) For such α, $D_0(c, \alpha)$ is still periodic saturated, but we don't know whether it is almost sofic.

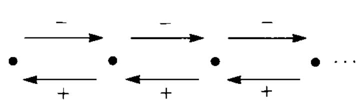
(2) The systems $D_r(c, \theta)$ and $D_0(c, \theta)$, with $\theta = 2\pi/m$, will draw most of our attention.

(3) If $\theta = 2\pi/p$ and p is prime, then $S_{0,n} = 0$ implies that $n+1$ divides p. Thus in this case, in studying D_0, there is no need to keep track of the time mod p, since it is determined by knowing the point reached in the disk. The same is true if θ is transcendental over $\{-1, 1\}$.

(4) The follower set of a block $B = b_0 \ldots b_n$ in $\Sigma(c, \theta)$ is determined by the set of all backward sums $\sum_{j=0}^{k} b_{n-j} e^{-ij\theta}$ using $e^{-i\theta}$, because in order to know whether or not a block BF is allowable, where $F = f_0 \ldots f_p$, we need to check all the sums over its sub-blocks; and

$$\sum_{j=n-k}^{n} e^{ij\theta} b_j + e^{in\theta} \sum_{j=1}^{r} e^{ij\theta} f_j = e^{in\theta} \left[\sum_{j=0}^{k} b_{n-j} e^{-ij\theta} + \sum_{j=1}^{r} e^{ij\theta} f_j \right].$$

(5) The labellings π are not onto in general and need not extend to continuous maps when closures are taken, as in (6) and (9). The following graph and labelling provide a simple example:



(6) If $\theta = 2\pi/m$, then $\Sigma(2c - \varepsilon, \theta) \subset D_0(c, \theta)$ for every $\varepsilon > 0$. Consequently the labelling π of $U_0(c, \theta)$ maps *onto* $D_0(c - \varepsilon, \theta)$ for each $\varepsilon > 0$. This is because given $x \in \Sigma(2c - \varepsilon, \theta)$, by a compactness argument we can select a state w in the disk to assign to x as its position at time 0 in such a way that beginning from w and moving forwards or backwards in time according to the steps specified by the entries of x, we will never leave the disk of radius c and we will always be able to return to 0. We omit the details of the argument, which are similar to the technical considerations of the following section.

(7) For $c = \infty$ and $\theta = 2\pi/m$ with m prime, the system $U(c, \theta)$ is *transient*. This is because if for each $k = 1, 2, \ldots$ we denote by B_{mk} the number of blocks B of length mk with $\mathrm{Sum}(B) = 0$, then B_{mk} equals the number of those blocks of 1's and -1's which have an equal number of positive and negative entries in each congruence class of places mod m. This number is easily seen to be

$$B_{mk} = \sum_{j=0}^{k} \binom{k}{j}^m \sim \frac{2^{mk}}{m^{\frac{1}{2}}} \left(\frac{2}{\pi k} \right)^{(m-1)/2}$$

[**13**]. It follows that $\limsup\,(B_{mk})^{1/mk} = 2$, $R = \frac{1}{2}$, and $h_G(\Gamma) = \log 2$. But $B_{mk} R^{mk} \to 0$, so the full plane system is transient.

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(8) $h_{\text{top}}(D_0(c, \theta))$ is clearly an increasing function of c, and therefore has at most countably many discontinuities. We conjecture that it is in fact continuous from the left, but that this would not be so if the systems were defined by requiring that the sums stay in the closed, rather than open, disk of radius c centred at the origin.

5. *A periodic saturated subshift which is not almost sofic*

Subshifts of finite type are examples of dynamical systems which contain an abundance of periodic points: any loop (closed path) in the graph can be repeated infinitely many times, and the graph always contains many loops (as long as the SFT has positive entropy). Let us agree to call a dynamical system (X, ϕ) (where X is a metric space and ϕ is a homeomorphism) *periodic saturated* if it has lots of periodic points, in the following sense: if P_k is the number of points $x \in X$ such that $\phi^k x = x$, then

$$\rho(X, \phi) = \limsup_{k\to\infty} \frac{1}{k} \log P_k = h_{\text{top}}(X, \phi).$$

It is easy to see that $\rho(X, \phi) \leq h_{\text{top}}(X, \phi)$ for every expansive dynamical system (X, ϕ) - see [**8**, p. 110] - and that SFT's are periodic saturated. It follows that sofic systems and also almost sofic systems are periodic saturated. For if X is almost sofic and if X_n are SFT's with $X_n \subset X$ for $n = 1, 2, \ldots$ and $h_{\text{top}}(X_n) \to h_{\text{top}}(X)$, then

$$\limsup_{k\to\infty} \frac{1}{k} \log P_k(X) \geq \limsup_{k\to\infty} \frac{1}{k} \log P_k(X_n) = h_{\text{top}}(X_n)$$

for each n, so that

$$\rho(X, \phi) \geq \sup_n h_{\text{top}}(X_n) = h_{\text{top}}(X).$$

In this section we will show that for $e^{i\theta}$ transcendental over $\{1, -1\}$ and c large enough, the disk systems $D_r(c, \theta)$ provide natural examples of periodic saturated subshifts which are certainly not almost sofic, since they contain no SFT's whatsoever with positive entropy. Thus in these systems most blocks can be repeated over and over to produce allowable sequences, but no *pair* of blocks can be concatenated freely.

Note first of all that for large enough c, $D_r(c, \theta) \neq \emptyset$, since, for example, if $x_n = 1$ for all n, then

$$|S_{m,n}(x, \theta)| = \left| \sum_{k=m}^{n} x_k e^{ik\theta} \right| = \frac{|e^{i(n-m+1)\theta} - 1|}{|(e^{i\theta} - 1|} \leq \frac{2}{|e^{i\theta} - 1|}.$$

We will see later that in fact $D_r(c, \theta)$ has positive entropy for large enough c. Before we get to that, though, we need a few technical results which will provide us with enough elbow room to be able to manufacture *infinitely repeatable* blocks - blocks B for which $B^\infty = \ldots BBBBB \ldots \in D_r(c, \theta)$.

If $B = b_1 \ldots b_n$ is a block on the symbols 1 and -1, let

$$\text{Sum}(B) = \sum_{k=1}^{n} b_k e^{ik\theta}.$$

PROPOSITION 5.1. *Let θ be irrational. Let $\delta > 0$. For large enough c, the closure of the set of states of $D_r(c, \theta)$ (all sums $S(B)$ which are in the disk of radius c) contains a δ-neighbourhood of* 0.

Proof. Let $\eta > 0$. If L is large enough that the points $e^{ij\theta}, j = 0, 1, \ldots, L$, are very nearly uniformly distributed on the unit circle, then the points $e^{ij\theta} + e^{ik\theta}$, $0 \le j, k \le L$, will be η-dense in a δ-neighbourhood of 0. Suppose also that L is chosen so that $|\mathrm{Sum}\,(1^{L+1})| < \eta$. If the block B_{jk} is obtained from the block $B = 1^{L+1}$ by changing the j and k entries to -1, then the states $\mathrm{Sum}\,(B_{jk})$ will be, say, 3η-dense in a δ-neighbourhood of 0. Note also that

$$|\mathrm{Sum}\,(1^m)| \le \frac{2}{|e^{i\theta} - 1|} \qquad \text{for all } m = 1, 2, \ldots,$$

and since $\mathrm{Sum}\,(B_{jk}) = \mathrm{Sum}\,(B) - 2(b_j\, e^{ij\theta} + b_k\, e^{ik\theta})$, for each initial sub-block C of each B_{jk} we have

$$|\mathrm{Sum}\,(C)| \le |\mathrm{Sum}\,(B)| + 4 \le \frac{2}{|e^{i\theta} - 1|} + 4.$$

It follows that all the states arrived at by moving along these blocks, for all η and L, are contained in some fixed disk. Moreover, these blocks can all be continued forward and backward in such a way as to cluster at 0, and thus they are allowable blocks in $D_r(c, \theta)$. For starting with any of the blocks B_{jk}, we may add 1's until we arrive at a block P which has a length L such that $|e^{iL\theta} - 1| \ge \frac{1}{2}$. Then the periodic point $\omega = P^\infty$ (with an occurrence of P beginning at the central entry) is in $D_r(c, \theta)$ (for some c depending only on θ), since it can be found as the sequence ω of labels along an allowable path in $U_r(c, \theta)$: if $C = P^{r+1} p_0 \ldots p_k$ is a typical initial block of this sequence, then

$$\begin{aligned} |\mathrm{Sum}\,(C)| &= |\mathrm{Sum}\,(P) + e^{iL}\,\mathrm{Sum}\,(P) + \cdots + e^{irL}\,\mathrm{Sum}\,(P) + \mathrm{Sum}\,(p_0 \ldots p_k)| \\ &\le |\mathrm{Sum}\,(P)| \frac{|e^{iL(r+1)\theta} - 1|}{|e^{iL\theta} - 1|} + \frac{2}{|e^{i\theta} - 1|} + 4, \end{aligned}$$

which is bounded by a constant depending only on θ; and similarly each state z from which one would have arrived at 0 at time 0 has modulus

$$\sum_{k=-1}^{-n} \omega_k\, e^{ik\theta} = -\mathrm{Sum}\,(p_s \ldots p_L P^r),$$

which is bounded by a constant depending only on θ. Thus the states reached by beginning at 0 at time 0 and moving either forward or backward in time according to the steps specified by the entries of ω are all in a certain fixed disk. Finally, we recur arbitrarily near 0 because

$$|\mathrm{Sum}\,(P^k)| \le |\mathrm{Sum}\,(P)| \frac{|e^{ikL\theta} - 1|}{\frac{1}{2}},$$

which is very close to 0 for appropriate choices of k.

COROLLARY 5.2. *If c is big enough and δ is small enough, then for each $\eta > 0$ and each $z_0 \in B_\delta(0)$ there is a point $x \in D_r(c, \theta)$ such that*

$$|z_0 + \mathrm{Sum}\,(x_0 \ldots x_L)| < \eta \quad \textit{for some } L.$$

Proof. Using the preceding proposition, select a state of $D_r(c, \theta)$ which is within a distance η of $-z_0$.

PROPOSITION 5.3. *Suppose that θ is an irrational multiple of π. Fix $\varepsilon > 0$. Then for each $\delta > 0$ there is an $n = n(\delta)$ such that for each allowable block B in $D_r(c-\varepsilon, \theta)$ there is a block C such that*

(1) $l(C) \leq n$;

(2) *BC is allowable in $D_r(c-\varepsilon/2, \theta)$;*

(3) $|\mathrm{Sum}\,(BC)| < \delta$.

Proof. On the disk G of radius $c-\varepsilon$ centred at 0 we consider the *normalized transitions*

$$T_\xi z = e^{-i\theta}(z+\xi) \qquad \text{for } z \in G \text{ and } \xi = \pm 1.$$

These normalized transitions were used by Siegel [**25**]; they are useful because $T_\xi z$ depends only on the current state z and the selected step ξ, and not on the path which brought us to the state z (e.g. through its length), while if $B = b_0 \ldots b_n$, then

$$\begin{aligned}\mathrm{Sum}\,(B) &= b_0 + b_1 e^{i\theta} + \cdots + b_n e^{in\theta} \\ &= e^{in\theta}[b_n + e^{-i\theta}(b_{n-1} + e^{-i\theta}(\ldots(e^{-i\theta}(b_0+0))\ldots)] \\ &= e^{i(n+1)\theta} T_{b_n} \ldots T_{b_1} T_{b_0} \\ &= e^{i(n+1)\theta} T_B z,\end{aligned}$$

so that the restricted random walks and the corresponding normalized transitions bring one to states having equal moduli.

Denote by E the closure of the set of all those states z which can be reached starting from 0 with normalized transitions which never leave the closed disk of radius $c-\varepsilon$ centred at 0:

$$E = \mathrm{cl}\,\{T_B 0 \colon |T_A 0| \leq c-\varepsilon \quad \text{for each initial block } A \text{ of } B\}.$$

For each $n = 1, 2, \ldots$ let U_n be the set of all those points $z \in E$ for which there is a block $B = b_0 \ldots b_s$ such that

(i) $l(B) \leq n$;

(ii) $|T_A z| < c - \varepsilon/2$ for each initial block A of B;

(iii) $|T_B z| < \delta$.

If $z = T_A 0 \in U_n$ is the endpoint of a path of allowable normalized transitions in the disk of radius $c-\varepsilon$, if c is large enough and $\delta = \delta(\varepsilon)$ is small enough, and if B is as in (iii), then, by the proof of proposition 5.1, AB can be extended infinitely to produce an allowable sequence of normalized transitions and hence an allowable sequence in $D_r(c-\varepsilon/2, \theta)$. We may assume that in fact δ is small enough so that this is possible.

Each set U_n is relatively open in E: if a normalized path B starting from z never hits or exceeds distance $c-\varepsilon/2$ from 0 and lands in a δ-neighbourhood of 0, then the identical sequence of normalized steps followed from any starting point sufficiently close to z will have these same properties. Also, the U_n cover the set E. For given any $z_0 \in E$, choose a very small δ, a point $z = T_D 0$ within a distance $\delta/2$ of z_0, and a path B in the disk of radius $c-\varepsilon$ which starts at z and ends within $\delta/2$ of 0. If δ is small enough, this same path will bring z_0 to within δ of 0 without leaving the disk of radius $c - \varepsilon/2$. Again we may assume that δ is in fact small enough (depending on ε) that this is possible.

If we take a finite subcover of E from among $\{U_n\}$, we are done.

Remark 5.4. For large enough c, $h_{\text{top}}(D_r(c, \theta)) > 0$.

Proof. Choose a large c and δ small enough that the two preceding propositions are operative. Choose N different blocks $P_1, \ldots, P_N$ for which

$$|\text{Sum}\,(P_k)| < \delta, \qquad k = 1, \ldots, N,$$

while $\{\arg(\text{Sum}\,(P_k)) : k = 1, \ldots, N\}$ is δ-dense in $[-\pi, \pi]$. If necessary enlarge c so that all of these paths stay in the disk of radius c and so that each $P_k^\infty \in D_r(c, \theta)$. (Note the estimates in the proof of proposition 5.1, where it was shown that $|\text{Sum}\,(B)|$ is bounded by a constant depending on P and θ for each sub-block B of P_k^∞.) We claim that it is possible to concatenate these blocks, if not in a completely arbitrary way, at least with *some* choice at each stage, in order to build up allowable sequences in $D_r(c, \theta)$. For if $P = P_{i_1} \ldots P_{i_k}$ has $|\text{Sum}\,(P)| < \delta$, then by corollary 5.2 P can be found as a sub-block of a point of $D_r(c, \theta)$. Moreover, we claim that there are at least two choices for a block $P_{i_{k+1}}$ which can be appended to P to form another such block Q, with $|\text{Sum}\,(Q)| < \delta$: we merely need to choose a P_r whose argument is approximately opposite to that of Sum (P).

Thus, if L is the maximum of the lengths of the P_k, then for each $k = 1, 2, \ldots$ the number of kL-blocks in $D_r(c, \theta)$ is at least 2^k, so that $h_{\text{top}}(D_r(c, \theta)) \geq \log 2/L$.

We will need a result related to proposition 5.3 which will allow us to choose the length of a path from 0 back to a small neighbourhood of 0 arbitrarily.

PROPOSITION 5.5. *For any θ, if c is sufficiently large, then for every $\delta > 0$ there is a $K(\delta)$ such that given any $k \geq K(\delta)$ there is an allowable block B in $D_r(c, \theta)$ of length k such that $|\text{Sum}\,(B)| < \delta$.*

Proof. Let $a = 2/|e^{i\theta} - 1|$. Fix $K(\delta)$ large enough that the endpoints of paths of length less than or equal to $K(\delta)$ which are allowed in $D_r(a, \theta)$ are $\delta/2$-dense in the set of all states of $D_r(a, \theta)$. Let $k \geq K(\delta)$, and notice that $v = \sum_0^{k-1} e^{ij\theta}$ is a state of $D_r(a, \theta)$. Choose an allowable block B with $l(B) \leq K(\delta)$ and $|\text{Sum}\,(B) - v| < \delta/2$. Then by changing fewer than $K(\delta)$ +'s to −'s in the block 1^k we can form a block A of length k for which Sum (A) is within $\delta/2$ of 0. Moreover, A will be allowable in $D_r(c(\delta), \theta)$ for some $c(\delta)$.

THEOREM 5.6. *For θ an irrational multiple of π and c large enough and a point of left continuity of $h_{\text{top}}(D_r(c, \theta))$, $D_r(c, \theta)$ is periodic saturated.*

Proof. For each $k = 1, 2, \ldots$ let ρ_k denote the number of blocks P in $D_r(c, \theta)$ of length k for whcih $P^\infty \in D_r(c, \theta)$. Let $\varepsilon > 0$, and let $\delta < \varepsilon/10$. Using proposition 5.3, choose n so that each allowable block in $D_r(c - \varepsilon, \theta)$ can be extended by a block of length less than or equal to n to produce an allowable block in $D_r(c - \varepsilon/2, \theta)$ with sum having modulus less than δ. Choose a large L and let B be any L-block in $D_r(c - \varepsilon, \theta)$, find C as in proposition 5.3, with $l(C) \leq n$, C allowable in $D_r(c - \varepsilon/2, \theta)$, and $|\text{Sum}\,(BC)| < \delta$. Let D be a block of length less than or equal to n such that $P=BCD$ is allowable in $D_r(c-\varepsilon,\theta)$, $|e^{il(BCD)\theta}-1|\geq\frac{1}{2}$, and $|\mathrm{Sum}\,(BCD)|<2\delta$. (We may assume that n is also sufficiently large to permit this.) We claim that $P^\infty\in D_r(c,\theta)$.

It is enough to check that $|\mathrm{Sum}\,(P^kp_0\ldots p_r)|<c$ and $|\mathrm{Sum}\,(p_r\ldots p_lP^k)|<c$ for each k, r, and l, where $P=p_0\ldots p_l$, and that given $\eta>0$ there is k such that $|\mathrm{Sum}\,(P^k)|<\eta$. Now

$$|\mathrm{Sum}\,(P^kp_0\ldots p_r)|\leq|\mathrm{Sum}\,(P)|\frac{|e^{ikl\theta}-1|}{|e^{il\theta}-1|}+|\mathrm{Sum}\,(p_0\ldots p_r)|$$

$$\leq 2\delta\frac{2}{\frac{1}{2}}+(c-\varepsilon)<c.$$

A similar calculation applies to $\mathrm{Sum}\,(p_r\ldots p_lP^k)$. Also,

$$\mathrm{Sum}\,(P^k)=\mathrm{Sum}\,(P)\frac{e^{ikl\theta}-1}{e^{il\theta}-1},$$

which can be made arbitrarily small by choosing k correctly.

Thus if N_L denotes the number of allowable L-blocks in $D_r(c-\varepsilon,\theta)$, then $N_L\leq\rho_{L+t}$ for some $t\leq 2n$, with n depending only on θ and ε. This implies that

$$\limsup\frac{1}{s}\log\rho_s\geq\limsup\frac{1}{s}\log N_s=h_{\mathrm{top}}(D_r(c-\varepsilon,\theta)).$$

Since this is true for each $\varepsilon>0$, we have $\limsup\log\rho_s/s\geq h_{\mathrm{top}}(D_r(c,\theta))$.

THEOREM 5.7. *If $e^{i\theta}$ is transcendental over $\{1,-1\}$, then $D_r(c,\theta)$ contains no* SFT *with positive entropy.*

Proof. If X were an SFT of positive entropy inside $D_r(c,\theta)$, then there would be two different paths A and B from some vertex of the graph of X back to itself. Now A and B can be concatenated arbitrarily inside X, but $\mathrm{Sum}\,(A)$ and $\mathrm{Sum}\,(B)$ are two different non-zero complex numbers. Clearly we can choose words C on the two symbols A and B to make $|\mathrm{Sum}\,(C)|$ as large as we please. (For example, if we begin with A and always extend the block at hand by appending the one of A, B which least decreases the sum accumulated up to that time, we will always be increasing the modulus of the accumulated sum by an amount which is bounded below.)

6. *Sufficient conditions for restricted random walks to be almost sofic*

PROPOSITION 6.1. *Let U be a chain on a countable connected graph Γ, $\pi: U\to\{0,1,\ldots,n\}^{\mathbb{Z}}$ a labelling of the paths of U by a finite alphabet, and D the closure of $\pi(U)$. Suppose that π is countable-to-one (e.g. right resolving: no two paths emanating from any vertex of Γ have the same labels). Then $h_{\mathrm{top}}(D)\geq h_G(U)$.*

Proof. For each finite connected subgraph F of Γ, the restriction of π to the SFT $U(F)$ is a finite-to-one continuous factor map onto a sofic subshift $D_F\subset D$, since a finite code on an SFT must be either finite-to-one or else uncountable-to-one (see [1] or [20]). Then $h_{\mathrm{top}}(U(F))=h_{\mathrm{top}}(D_F)$, so by [14]

$$h_G(U)=\sup_F \mathrm{h}_{\mathrm{top}}(U(F))=\sup_F h_{\mathrm{top}}(D_F)\leq h_{\mathrm{top}}(D).$$

It is a corollary of the proof of this proposition, (since the D_F are sofic) that *a finite labelling of a chain, with countable fibres, will be almost sofic as soon as the entropy downstairs does not exceed the entropy upstairs.* Whether or not this happen depends on how much branching takes place in the graph Γ before there is recurrence and what relationship the branching bears to the labelling. We will see that for (restricted) random walks in finite-dimensinal spaces, and in particular for our disk systems with $\theta = 2\pi/m$, the entropy does *not* increase under our labellings, and so these systems are almost sofic. For more general random walks, the Gurevič entropy of the upstairs chain U is just the supremum of the metric entropies of certain *recurrent* measures on the downstairs subshift D.

Let G be a separable locally compact group, G_0 a finite subset of G, and $f_1, f_2, \ldots$ a sequence of measurable functions from a probability space (X, B, μ) to G_0. For each $n \geq 1$, define

$$\sigma_n(x) = f_1(x)f_2(x) \ldots f_n(x).$$

We define a graph Γ whose states are $\{\sigma_n(x): n \geq 1, x \in X\}$, with transitions $v_1 \to v_2$ if there are n and x such that $v_1 = \sigma_n(x)$ and $v_2 = \sigma_{n+1}(x)$. The chain $U = U(\Gamma)$ associated with this graph will be called the *random walk chain* associated with the random walk $\{f_n\}$. Let E be a subset of G, and define Γ_E to be the subgraph of Γ which results from deleting all vertices of Γ not in E, along with their associated transitions. (To avoid triviality, we assume that E contains the identity of G.) The chain $U(\Gamma_E)$ is called the *restricted random walk chain* associated with $\{f_n\}$ and E. Each of these chains has a natural labelling π, in which the path from v_1 to v_2 is labelled by the element $v_1^{-1}v_2$ of G_0.

A random walk will be called *isotropic* if the sets of labels and paths at any two vertices are identical, and *symmetric* if $G_0^{-1} = G_0$. The restriction of an isotropic (or symmetric) random walk chain will be called an *isotropic* (or *symmetric*) *restricted random walk chain.* In an isotropic restricted random walk, from each state one has a fixed set of possible allowed moves, subject only to the condition that one is not allowed to leave the set E.

THEOREM 6.2. *Let $U(\Gamma_E)$ be an isotropic symmetric restricted random walk chain in $\mathbb{Z}^d$, and let $D_\pi(\Gamma_E)$ be the closure of the image of $U(\Gamma_E)$ under the natural labelling π, so that $D_\pi(\Gamma_E)$ is a subshift of $G_0^{\mathbb{Z}}$. Then $D_\pi(\Gamma_E)$ is almost sofic.*

Proof. Abbreviate $D = D_\pi(\Gamma_E)$ and $U = U(\Gamma_E)$. Because of the preceding proposition and comments, it is sufficient to prove that $h_{\text{top}}(D) \leq h_G(U)$. The idea of the proof is very simple: since $h_{\text{top}}(D)$ is the growth rate of the number of blocks in D, while $h_G(U)$ is the growth rate of the number of loops from a fixed vertex v back to itself (with possibly many repeat visits to v), and since the number of vertices that can be visited in k steps is bounded by a constant times k^d, while the number of blocks in D grows like $\exp(k\, h_{\text{top}}(D))$, for large k most k-blocks in D must visit some vertex v repeatedly.

In order to be more precise, let $h = h_{\text{top}}(D)$ and let $\varepsilon > 0$. Choose k so large that the number N_k of k-blocks in D is at least $\exp(k(h-\varepsilon))$. By volume considerations, a path in E starting from 0 can reach at most Ak^d different states, for some constant A. Thus there is a state v which is the endpoint of at least $\exp(k(h-\varepsilon))/(Ak^d)$ paths in E beginning at 0. Let p_1 and p_2 be any two such paths. Then p_1 followed by the time reverse $\bar{p}_2$ of p_2 (if $p=g_1\dots g_k$, then $\bar{p}=g_k^{-1}\dots g_1^{-1}$) will be an allowed path in E from 0 to 0. Consequently, if $B_{2k}(0)$ denotes the number of paths of length $2k$ from 0 to 0 in U, then

$$B_{2k}(0)\geq\left(\frac{e^{k(h-\varepsilon)}}{Ak^d}\right)^2,$$

and so

$$\frac{1}{2k}\log(B^{2k}(0))\geq\frac{2k(h-\varepsilon)-2d\log k-2\log A}{2k},$$

which has limit $h-\varepsilon$. Since ε is arbitrary, we must have $h_G(U)\geq h$.

THEOREM 6.3. *For $\theta=2\pi k/m$ and c a point of left continuity of $h_{\mathrm{top}}(D_0(c,\theta))$, the disk system $D_0(c,\theta)$ is almost sofic.*

Proof. If $B=\xi_1\xi_2\dots\xi_k$ is an allowed k-block in $D=D_0(c,\theta)\subset\{1,-1\}^{\mathbb{Z}}$, and

$$S_{0,n-1}(\xi,\theta)=\sum_{r=1}^{k}\xi_r e^{ir\theta}$$

is the endpoint of the corresponding path in the random walk in the disk of radius c (starting at 0), then $S_{0,n-1}(\xi,\theta)$ also has the representation

$$S_{0,n-1}(\xi,\theta)=a_0+a_1\omega+\cdots+a_{m-1}\omega^{m-1},$$

where $\omega=e^{i\theta}$ and $a_0,\dots,a_{m-1}\in\mathbb{Z}$. Thus the number of states visited in k steps by the restricted random walk in the disk is less than or equal to the number of states in $\mathbb{Z}^m$ visited in k steps by a certain restricted random walk with steps of length 1. This is not yet enough, though, to yield the result, since the walk is not isotropic, and so the reverse of an allowed path might not be allowed. Also, we need to be sure that we return with the time the same mod m as when we started. With a little more effort, however, an argument similar to the preceding one can be made to work.

Fix $\varepsilon>0$ and define F_m to be the closure of the set of those states in the restricted random walk associated with $D_0(c-\varepsilon,\theta)$ which can be reached from 0 by a path of length km, for some $k\in\mathbb{Z}$, and from which one can return to 0 by a path of length $k'm$, for some $k'\in\mathbb{Z}$. For each $x\in F_m$, let $T(x)$ be the length of the shortest path in the open disk of radius $c-\varepsilon/2$ from x to an $\varepsilon/4$-neighbourhood of 0 in km steps:

$$\begin{aligned}T(x)=\inf\{n\colon n=km &\text{ for some } k, \text{ and there are } \xi_0,\dots,\xi_{n-1}\in\{1,-1\}\\ &\text{such that } |x+S_{0,r-1}(\xi,\theta)|<c-\varepsilon/2 \text{ for all } r=1,2,\dots,n,\\ &\text{while } |x+S_{0,n-1}(\xi,\theta)|<\varepsilon/4\}.\end{aligned}$$

Then $T(x)$ is upper semicontinuous and hence has an absolute maximum value T on the compact set F_m. Thus, starting from any point in F_m, we can get within $\varepsilon/4$ of 0 with an allowed path in the disk of radius $c-\varepsilon/2$ in at most T steps.

Let $h(\varepsilon)=h_{\mathrm{top}}(D_0(c-2\varepsilon,\theta))$. Pick a large n which is divisible by m, and an allowed block of length n in $D_0(c-2\varepsilon,\theta)$. By moving the starting point and prefixing a block of bounded length L, we can find this block as the end of an allowable path in $D_0(c-\varepsilon, \theta)$ starting at 0. Some state $v \in F_m$ is the endpoint of at least $\exp(n(h(\varepsilon)-\varepsilon))/(An^m)$ of these paths. To any such path ending at v we may add an allowed path of length (divisible by m and) less than or equal to T to make an allowed path from 0 to a vertex w in an $\varepsilon/4$-neighbourhood of 0. Now any such path followed by the *dual* (+ is replaced by − and − by +) of another such path will be an allowed path in $D_0(c, \theta)$ fom 0 to 0:

$$(\xi_0+\xi_1\omega+\cdots+\xi_{n-1}\omega^{n-1})-(\eta_0\omega^n+\eta_1\omega^{n+1}+\cdots+\eta_{n-1}\omega^{2n-1})=0 \qquad \text{if } \omega^n=1.$$

It follows that for large n (divisible by m) the number of allowed paths in $U_0(c, \theta)$ from 0 to 0 of length $2n+t$, for some t between 0 and $2(T+L)$, is

$$B_{2n+t} \geq \left(\frac{e^{n(h(\varepsilon)-\varepsilon)}}{An^m}\right)^2,$$

so that

$$h_G(U_0(c, \theta)) \geq \limsup_{n\to\infty} \frac{B_{2n+t}}{2n+t} \geq h(\varepsilon)-\varepsilon.$$

Since $h(\varepsilon)-\varepsilon$ increases to $h_{\text{top}}(D_0(c, \theta))$ as $\varepsilon \to 0$, we have

$$h_G(U_0(c, \theta)) \geq h_{\text{top}}(D_0(c, \theta)),$$

and the result follows from our previous observations.

Let G be a separable locally compact group and Γ a connected countable graph whose vertices are elements of G. If there is an edge in Γ from g to h, then label that edge by $g^{-1}h$. We assume that the set of edge labels G_0 is *finite*. Form the chain $U = U(\Gamma)$ and the subshift $D = D_\pi(\Gamma) \subset G_0^{\mathbb{Z}}$. If μ is a shift-invariant Borel probability measure on U, then $f_n(x) = x_n$ determines a stationary random walk in G. If μ has full support, then the chain associated with the walk is U.

Let $B = b_1 \ldots b_n$ be a block of length n with entries from G_0 which is an allowable path in Γ beginning at $e \in G$. The *state* of B is the product $b_1 \ldots b_n$ regarded as an element of G; it will be denoted by $\text{st}(B)$. The *height* of B is

$$\text{ht}(B) = \inf\{k: \text{st}(B) = \text{st}(C) \quad \text{for some } k\text{-block } C\}.$$

The *range* of B is

$$R(B) = \text{card}\{\text{st}(A): A \text{ is an initial sub-block of } B\}.$$

The *wait* of B is

$$\text{wt}(B) = \inf\{l(C): \text{st}(BC) = e\}.$$

Notice that since the loops between repeated states can be collapsed, $\text{ht}(B) \leq R(B)$ for each allowable block B. We define

$$\{\text{no return}\} = \{x \in D: \text{st}(x_0 \ldots x_{n-1}) \neq e \text{ for } n = 1, 2, \ldots\}.$$

Let μ be a shift-invariant ergodic measure on D. We will call μ *recurrent* if $\mu\{\text{no return}\} = 0$. Recall that the cylinder set determined by a block $B = b_0 \ldots b_n$ is

$$[B] = \{x: x_0 = b_0, \ldots, x_n = b_n\}.$$

A shift-invariant ergodic measure μ on D will be called *elastic* if there are constants

K_μ and δ_μ such that

$$\mu\left(\bigcup\left\{[B]: B \text{ is an allowed } n\text{-block with wt}\,(B) \leq K_\mu \text{ ht}\,(B)\right\}\right) \geq \delta_\mu$$

for infinitely many n.

Recall that by the Shannon-McMillan-Breiman theorem, for each $\varepsilon > 0$ and each n the n-blocks can be divided into two classes, a class of 'good' n-blocks, each of measure between $\exp[-n(h(\mu)+\varepsilon)]$ and $\exp[-n(h(\mu)-\varepsilon)]$, and whose corresponding cylinder sets form a set of total measure at least $1-\varepsilon$, and the remaining 'bad' n-blocks.

THEOREM 6.4. *Let U be a random walk chain with labelling D as described above. Then*

$$h_G(U) = \sup\{h_\mu(D): \mu \textit{ is shift-invariant, ergodic, recurrent, and elastic}\}.$$

Proof. It follows from [9] that if μ is an invariant ergodic probability measure on D, then

$$\frac{1}{n} R(x_0 \ldots x_{n-1}) \to \mu\{\text{no return}\} \qquad \text{for a.e. } x \in D.$$

If $F \subset \Gamma$ is a finite connected subgraph and μ is a shift-invariant ergodic measure on $D_\pi(F)$, then automatically $\mu\{\text{no return}\} = 0$. Thus clearly, since again π is countable-to-one,

$$\begin{aligned} h_G(U) &= \sup\nolimits_F h_{\text{top}}(U(F)) \\ &= \sup\nolimits_F h_{\text{top}}(D_\pi(F)) \\ &\leq \sup\{h_\mu(D): \mu\{\text{no return}\} = 0\}. \end{aligned}$$

For the reverse inequality, fix an ergodic elastic measure μ on D with $\mu\{$no return$\} = 0$, so that $R(x_0 \ldots x_{n-1})/n \to 0$ a.e. $d\mu$. Fix $\varepsilon > 0$, and consider n's large enough that

$$\exp[-n(h(\mu)+\varepsilon)] \leq \text{card}\{\mu\text{-good } n\text{-blocks}\} \leq \exp[-n(h(\mu)-\varepsilon)],$$

$$\mu\{x: R(x_0 \ldots x_{n-1})/n \geq \varepsilon\} < \varepsilon,$$

and

$$\mu\left(\bigcup\left\{[B]: B \text{ is an allowed } n\text{-block with wt}\,(B) \leq K_\mu \text{ ht}\,(B)\right\}\right) \geq \delta_\mu.$$

Denote the set of good n-blocks by G, the set where $R(x_0 \ldots x_{n-1})/n < \varepsilon$ by R, and the union of the cylinder sets $[B]$ over n-blocks B with wt $(B) \leq K_\mu$ ht (B) by W. Then $\mu(G) \geq 1-\varepsilon$, $\mu(R) \geq 1-\varepsilon$, and $\mu(W) \geq \delta_\mu$ imply that $\mu(G \cap R \cap W) \geq \delta_\mu/3$ if ε is chosen small enough. Thus the number N of good n-blocks B for which $R(B)/n < \varepsilon$ and wt $(B) \leq K_\mu$ ht (B) - call these *very good* n-blocks - is at least $(\delta_\mu/3)\exp[n(h(\mu)-\varepsilon)]$.

If B is a very good n-block, then wt $(B) \leq K_\mu$ ht $(B) \leq K_\mu R(B) \leq K_\mu n\varepsilon$, so that B can be extended to an $(n+nK_\mu\varepsilon)$-block which determines an allowable path in Γ from e to e. Therefore, if L_k denotes the number of allowable paths in Γ of length k from e to e,

$$(\delta_\mu/3)\, e^{n(h(\mu)-\varepsilon)} \leq L_{n(1+\varepsilon K_\mu)},$$

and hence

$$h_G(U) = \limsup \frac{1}{k} \log L_k \geq \lim_{n\to\infty} \frac{\log(\delta_\mu/3) + n(h(\mu)-\varepsilon)}{n(1+\varepsilon K_\mu)} = \frac{h(\mu)-\varepsilon}{1+\varepsilon K_\mu}.$$

Letting $\varepsilon \to 0$ shows that $h_G(U) \geq h(\mu)$ for all recurrent elastic measures μ.

COROLLARY 6.5. *If U and D are as above and for each* $\varepsilon > 0$

$$\frac{\text{card}\,\{n\text{-blocks } B\colon \text{wt}\,(B) \leq \varepsilon n\}}{\text{card}\,\{\text{allowable } n\text{-blocks } B\}}$$

does not converge to 0, *then D is almost sofic.*

Proof. As before, since the labelling π is countable-to-one, $h_{\text{top}}(D) \leq h_G(U)$. If N of the N_n allowed n-blocks B have wt $(B) \leq \varepsilon n$, then, since each of these N blocks can be continued to a loop of length $n(1+\varepsilon)$ in Γ from e to e, we have

$$L_{n(1+\varepsilon)} \geq N,$$

and

$$N \geq \delta N_n \qquad \text{for infinitely many } n,$$

for some fixed $\delta > 0$. Then again

$$\limsup \frac{1}{k} \log L_k \geq \lim \frac{\log \delta + \log N_n}{n(1+\varepsilon)} = \frac{h_{\text{top}}(D)}{1+\varepsilon}.$$

Then $h_G(U) \geq h_{\text{top}}(D)$, and D is almost sofic by the remarks at the beginning of this section.

Remark 6.6. It may not be difficult to check whether a given measure μ is recurrent. For example, in the case where $G = \mathbb{Z}$ and $G_0 = \{1, -1\}$, it follows from [**18**] that μ is recurrent if and only if $\mu[1] = \mu[-1]$; this is also true if μ is not ergodic. This condition can also be adapted to more complicated random walks.

7. *The loop method for computing the entropy of an SFT*

When analyzing the disk systems $D_r(c, \theta)$, it becomes necessary to attempt to compute the entropies of the SFT's that they contain. These SFT's can be so complicated (see figures 5–8) that entropy computation by the usual eigenvalue calculation method is impractical. However, the dynamical viewpoint that we are taking here, involving approximation from within by simpler systems and loop analysis, also provides a method for entropy computation which can be carried out easily even for some very large SFT's.

In this section we will show that the topological entropy of an irreducible subshift of finite type Σ is the logarithm of the positive root of the 'loop equation'

$$\sum_{i=0}^{\infty} \frac{N_i}{x^{i+1}} = 1, \tag{7.1}$$

where N_i is the number of blocks of (symbol) length $i+2$ that begin and end at a particular state and do not visit it in between. Although this equation is familiar in the theory of countable-state topological Markov chains (see [**10**], [**14**], [**15**], [**24**], [**27**], [**28**], and [**30**]), it seems not to be generally known that it also provides a nice

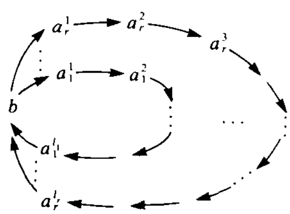
https://doi.org/10.1017/S014338570000359X Published online by Cambridge University Press

method for computing entropies in the finite alphabet case. In the particular case when every loop in the SFT Σ hits the selected base vertex b, the loop method coincides with the 'rome' technique of [**5**].

We derive the loop method here from a dynamical rather than matrix-theoretic or combinatorial viewpoint. We fix a vertex, b, in the graph of Σ and analyze the SFT in terms of the loops that begin and end at b, without hitting b in between. Restricting consideration to loops not exceeding a certain length, s, defines a subshift, Σ_s, of Σ whose characteristic equation is easily computed (proposition 7.1). The limit of these characteristic equations gives an equation for the maximum eigenvalue of Σ (theorem 7.5). Alternatively, one can obtain the maximum eigenvalue as the limit of those of the approximating subshifts Σ_s. The convergence is at exponential speed (theorem 7.4).

I thank T. Brylawski, W. Derrick, E. Coven, and D. Lind for helpful comments about the loop method.

Consider a finite *loop graph*, which consists of r disjoint loops of lengths $l_1, \ldots, l_r$ based at a vertex b



and let $L \subset \{b; a_1^1, \ldots, a_1^{l_1}; \ldots; a_r^1, \ldots, a_r^{l_r}\}^{\mathbb{Z}}$ be the associated SFT. The topological entropy of such an SFT is readily computed (and has in fact been computed many times before).

PROPOSITION 7.1. *The topological entropy of L is the logarithm of the positive root of the equation*

$$\sum_{i=1}^{r} \frac{1}{x^{l_i-1}} = 1.$$

Proof. For each vertex s of the graph of L and each $n \geq 1$, let $N_n(s)$ denote the number of n-blocks in L which end with the symbol s. Then

$$\begin{aligned} N_{n+1}(b) &= N_n(a_1^{l_1}) + \cdots + N_n(a_r^{l_r}), \\ N_{n+1}(a_i^1) &= N_n(b), \\ N_{n+1}(a_i^{k+1}) &= N_n(a_i^k), \qquad \text{for all } i = 1, \ldots, r \text{ and } k = 1, \ldots, l_i - 1. \end{aligned}$$

Combining these equations gives

$$N_{n+1}(b) = N_{n-l_1}(b) + \cdots + N_{n-l_r}(b).$$

If λ is the maximum positive eigenvalue of the transition matrix of L, then $N_n(b)$ is asymptotic to λ^n. Therefore

$$\lambda^{n+1} = \lambda^{n-l_1} + \cdots + \lambda^{n-l_r},$$

https://doi.org/10.1017/S014338570000359X Published online by Cambridge University Press

or

$$\sum_{i=1}^{r} \frac{1}{\lambda^{l_i-1}} = 1.$$

The measure of maximal entropy (Shannon–Parry measure) on a loop graph L also satisfies a type of loop equation.

PROPOSITION 7.2. *The maximal measure on L is the unique one-step Markov measure for which the transition probabilities $p_i = \nu(ba_i^1|b)$ satisfy the equation*

$$h_{\text{top}}(L)\left(1+\sum_{i=1}^{r} l_i p_i\right) = -\sum_{i=1}^{r} p_i \log p_i.$$

Proof. Any one-step Markov measure ν on L is completely determined by the probability vector $p = (p_i)$: the stochastic matrix P of transition probabilities of ν results from the transition 0, 1-matrix M of L by replacing the 1's in the first row of M by $p_1, \dots, p_r$. The fixed vector for ν is $(p_b; p_b p_1, \dots, p_b p_1; \dots; p_b p_r, \dots, p_b p_r)$ with

$$p_b = 1/\left(1+\sum_{i=1}^{r} l_i p_i\right).$$

Denote by σ_b the first-return map to $B = [b] = \{x \in L: x_0 = b\}$. Then σ_b is Bernoulli with respect to $\nu_b = \nu/\nu(B)$, so by Abramov's formula for the entropy of an induced transformation,

$$-\sum_{i=1}^{r} p_i \log p_i = H(p_1, \dots, p_r) = \frac{h(\nu)}{\nu(B)}.$$

There is a unique choice of $(p_1, \dots, p_r)$ for which the left-hand side of the latter equation can be $h_{\text{top}}(L)$.

We want now to approximate an arbitrary (irreducible) SFT from within by sub-SFT's which are topologically conjugate to loop-graph SFT's. Given an irreducible SFT Σ, a vertex b in the graph of Σ, and an $s = 0, 1, 2, \dots$, let

$$\Sigma_s = \{x \in \Sigma: x \text{ contains no block of length } s+1 \text{ which omits } b\}.$$

Then $\Sigma_s \subset \Sigma$ is also an SFT. We think of Σ_s as being generated by all loops from b to b of length no more than s. Unfortunately these loops need not all be disjoint as in a loop-graph SFT, but we can 'pull apart' these loops to create a loop-graph SFT which is conjugate to Σ_s.

Enumerate as $z_1, z_2, \dots, z_{r(s)}$ all the different blocks of the form $bc_1c_2\dots c_lb$, where $0 \le l \le s$ and no $c_k = b$, which appear in Σ. (The block bb could also be included if it is allowable in Σ.) Let $l_i = \text{length}(z_i) - 2$ for all i. Form the loop-graph as above with loop lengths $l_1, l_2, \dots, l_{r(s)}$, and denote by L_s the corresponding SFT. There is an obvious continuous onto factor map $\pi: L_s \to \Sigma_s$ defined by $\pi(ba_i^1 \dots a_i^{l_i} b) = z_i$.

PROPOSITION 7.3. *$\pi: L_s \approx \Sigma_s \subset \Sigma$; i.e. π is a conjugacy.*

Proof. π^{-1} is defined by $\pi^{-1}(z_i) = ba_i^l \dots a_i^{l_i} b$.

By propositions 7.1 and 7.3, $h_{\text{top}}(\Sigma_s)=\log \lambda_s$, where λ_s is the positive solution of

$$\sum_{i=1}^{r(s)} \frac{1}{x^{l_i-1}}=1.$$

We will show in a moment that $h_{\text{top}}(\Sigma)=\log \lambda$, where λ is the positive solution of

$$\sum_{i=1}^{\infty} \frac{1}{x^{l_i-1}}=1,$$

and that the λ_s increase to λ exponentially fast.

For each $k=0,1,2,\ldots$, denote by N_k the number of different blocks in Σ of the form bAb, where A is a block of length k which contains no b's. Define

$$f_s(x)=\sum_{i=1}^{r(s)} \frac{1}{x^{l_i}}=\sum_{k=0}^{s} \frac{N_k}{x^k} \qquad \text{for } s=0,1,2,\ldots.$$

We have $f_s(\lambda_s)=\lambda_s$ for all s, and clearly λ_s increases with s. Define

$$\lambda^*=\sup_s \lambda_s.$$

Since each f_s is decreasing,

$$f_s(\lambda^*)\le f_s(\lambda_s)=\lambda_s\le\lambda^* \qquad \text{for all } s,$$

so that

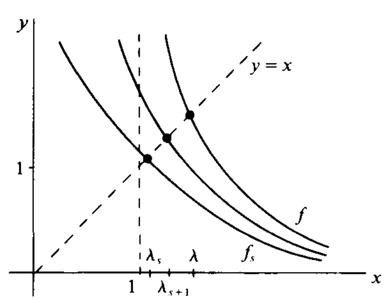
$$f(x)=\sum_{k=0}^{\infty} \frac{N_k}{x^k}$$

converges in a neighbourhood of λ^*.

THEOREM 7.4. $\lambda=\lambda^*$, *and there are constants* c *and* c' *and* $d<1$ *such that, for large* s,

$$\lambda-\lambda_s\le c\sum_{k=s+1}^{\infty} \frac{N_k}{\lambda^k}\le c'd^s.$$

Proof. The f_s increase with s and converge uniformly to f on an interval containing λ^*.



Then clearly the fixed point of f is the supremum of those of the f_s, i.e. $\lambda=\lambda^*$. By the Mean Value theorem, for each s there is a u_s between λ_s and λ such that

$$f_s'(u_s)(\lambda-\lambda_s)=f_s(\lambda)-f_s(\lambda_s)=f(\lambda)-\sum_{s+1}^{\infty} \frac{N_k}{\lambda^k}-\lambda_s,$$

so that

$$\lambda-\lambda_s=\frac{1}{1-f_s'(u_s)}\sum_{s+1}^{\infty}\frac{N_k}{\lambda^k}\le c\sum_{s+1}^{\infty}\frac{N_k}{\lambda^k},$$

since $-\infty<f_s'(u_s)\le f_{s_0}'(\lambda)<0$.

We claim now that there is $\eta<\lambda$ such that $N_k\le\eta^k$ for large k. For consider the graph that results from the graph of Σ when the vertex b and any edges attached to b are eliminated, and let Σ' be the corresponding SFT. Then $\Sigma'\subset\Sigma$ is a strictly smaller subsystem, so the Shannon–Parry measure on Σ' is different from that on Σ and hence $h_{\mathrm{top}}(\Sigma')<h_{\mathrm{top}}(\Sigma)$. Let $\lambda'=\exp(h_{\mathrm{top}}(\Sigma'))$. Then the number of blocks of the form bAb, with A having length k and containing no b's, is no more than the number of k-blocks in Σ', which is asymptotic to $(\lambda')^k$. Thus we may choose any $\eta\in(\lambda',\lambda)$.

We have then, for large s,

$$\lambda-\lambda_s\le c\sum_{s+1}^{\infty}\frac{N_k}{\lambda^k}\le c\sum_{s+1}^{\infty}\frac{\eta^k}{\lambda^k}=c'\left(\frac{\eta}{\lambda}\right)^s.$$

THEOREM 7.5. *Let λ equal the positive root of $\sum_{k=0}^{\infty}N_k/x^{k+1}=1$, which coincides with the limit of the positive roots λ_s of $\sum_{k=0}^{s}N_k/x^{k+1}=1$. Then $h_{\mathrm{top}}(\Sigma)=\log\lambda$.*

Proof. Let μ be the Shannon–Parry measure on Σ, let $B=\{x\in\Sigma\colon x_0=b\}$, let $\mu_B=\mu/\mu(B)$, again list all the different loops bAb with A containing no b as z_1, $z_2,\dots$, and let $p_i=\mu_B(z_i)$ for each i. Then $\sum p_i=1$, and

$$h_{\mu_B}(\sigma_b)=-\sum p_i\log p_i.$$

(The partition of B by the z_i's is finer than the first-return partition of B and is Bernoulli.) As in proposition 7.2, we have

$$\mu(B)=\frac{1}{1+\sum_{i=1}^{\infty}l_ip_i};$$

by Abramov's formula,

$$h_{\mathrm{top}}(\Sigma)=h_\mu(\Sigma)=\mu(B)h_{\mu_B}(\sigma_b)=-\frac{1}{1+\sum_{i=1}^{\infty}l_ip_i}\sum_{i=1}^{\infty}p_i\log p_i.$$

Consider now the SFT $\Sigma_s\subset\Sigma$ of entropy $\log\lambda_s$ and its corresponding loop-graph SFT L_s, as in proposition 7.3. For each $i=1,2,\dots,r(s)$, let

$$q_i=\frac{p_i}{\sum_{k=1}^{r(s)}p_k},$$

and use these weights to form a one-step Markov measure ν on L_s.

By the proof proposition 7.2,

$$\log\lambda_s=h_{\mathrm{top}}(L_s)\ge h_\nu(L_s)=-\frac{\sum_{i=1}^{r}q_i\log q_i}{1+\sum_{i=1}^{r}l_iq_i}.$$

Since $\log\lambda_s\le h_{\mathrm{top}}(\Sigma)$ for all s, it is enough to prove that

$$-\frac{\sum_{i=1}^{r}q_i\log q_i}{1+\sum_{i=1}^{r}l_iq_i}\to\frac{-\sum_{i=1}^{r}p_i\log p_i}{1+\sum_{i=1}^{r}l_ip_i}.$$

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The left side is

$$-\frac{\sum_{k=1}^{r} p_k}{\sum_{k=1}^{r} p_k+\sum_{i=1}^{r} l_i p_i}\,\frac{1}{\sum_{k=1}^{r} p_k}\left[\sum_{i=1}^{r} p_i \log p_i-\log\left(\sum_{k=1}^{r} p_k\right)\sum_{i=1}^{r} p_i\right],$$

so an application of the limit theorems of calculus yields the result.

Remarks 7.6. (1) Let A_k denote the number of *all* paths bBb in Σ with B having length k (and possibly containing some b's). Then

$$\frac{1}{1-\sum_{k=0}^{\infty} N_k/x^{k+1}}=1+\sum\frac{N_k}{x^{k+1}}+\left(\sum\frac{N_k}{x^{k+1}}\right)^2+\ldots=1+\sum_{k=0}^{\infty}\frac{A_k}{x^{k+1}}.$$

Since $\lim \log(A_k)/k=h_{\mathrm{top}}(\Sigma)$, the series on the right diverges for $x<\exp(h_{\mathrm{top}}(\Sigma))$. For any irreducible SFT on a finite alphabet, it diverges also when $x=\exp(h_{\mathrm{top}}(\Sigma))$, as may easily be seen by thinking in terms of the powers of the transition matrix of Σ. Therefore the left side tends to ∞ as x decreases to $\exp(h_{\mathrm{top}}(\Sigma))$. This combinatorial argument works to prove theorem 7.5 for all *recurrent* chains on countable alphabets (i.e. ones for which the right side diverges), but it does not provide as much dynamical information as the approach taken above.

(2) (W. Derrick (personal communication) and [**24**].) The loop equation is in fact a form of the characteristic equation of the transition matrix $M=(m_{ij})$ of the graph of Σ. If b is the j'th of n symbols, and $M_{jj}^{\#}$ denotes the cofactor of the j,j entry of $M-xI$, then the characteristic equation of M, $\det(M-xI)=0$, can be written in the form

$$(m_{jj}-x)M_{jj}^{\#}+P_j(x)=0,$$

or

$$\frac{m_{jj}M_{jj}^{\#}+P_j(x)}{xM_{jj}^{\#}}=1.$$

This equation is the same as (7.1).

(3) The *topological entropy of the first-return map* σ_b to a vertex b in a SFT Σ is

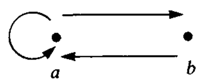
$$h_{\mathrm{top}}(\sigma_b)=\log\left(\sum_{i=0}^{\infty} N_i\right),$$

where N_i is the number of different blocks bAb in Σ with A containing no b's. For suppose that Σ contains at least r different loops bAb. Then the loop-graph SFT based at b with r different loops has first-return map to b with topological entropy $\log r$, since it is conjugate to the r-shift. Because this first-return map is a closed subsystem of (B, σ_b) (where $B=\{x\in\Sigma\colon x_0=b\}$), we have $h_{\mathrm{top}}(\sigma_b)\geq\log r$. Of course if there are exactly r loops, equality holds. Thus for the golden mean SFT (see example 7.7), the first-return map to one vertex has infinite topological entropy, while the first-return map to the other has topological entropy $\log 2$.

(4) Theorem 7.5 can be applied to the computation of the topological entropy of an SFT in two ways: one can try to determine in closed form the number N_k of loops of length k based at any selected vertex b and sum the left-hand side of (7.1) by geometric series to obtain a rational equation for e^h; or one can approximate e^h by the roots of the approximating equations $\sum_{k=0}^{s} N_k/x^{k+1}=1$. In the first case we always obtain a rational equation, by remark 7.2. In the second case, the

convergence of the roots λ_s to λ can be estimated by the rate of convergence to 0 of the tail of a power series, as in theorem 7.4.

Example 7.7. *The golden mean SFT.*



We will perform loop analysis using each vertex as a base in turn.

Using a as base vertex, we find $N_0 = 1$, $N_1 = 1$, $N_k = 0$ for $k \geq 2$. Thus (7.1) reads

$$\frac{1}{x} + \frac{1}{x^2} = 1,$$

the characteristic equation of the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Using b as base vertex, we have $N_0 = 0$ and $N_k = 1$ for all $k \geq 1$. Thus (7.1) reads

$$1 = \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} = \frac{1}{x^2} \frac{1}{1 - 1/x},$$

again a form of the characteristic equation.

Using theorem 7.4, we see that the roots λ_s of the approximating equations $\sum_{k=0}^{s} N_k / x^{k+1} = 1$ satisfy

$$\lambda - \lambda_s \leq c \sum_{k=s+1}^{\infty} \frac{1}{\lambda^k} \leq \frac{c'}{\lambda^k}.$$

Example 7.8. *A big SFT.*

Consider the SFT shown in figure 6, which is one of the smaller SFT's inside a disk system with turning angle $2\pi/5$. Since there are 61 states, the incidence matrix is rather cumbersome to work with, even though it has few non-zero entries. But we can find the topological entropy of this system by loop analysis, summing the left side of (7.1) by geometric series and applying Newton's method to the resulting polynomial.

We base the loop analysis at the vertex 0. Equation (7.1) is then

$$\lambda = \sum_{s=0}^{\infty} \frac{Q(s)}{\lambda^{9+5s}} = \lambda \xi^2 \sum_{s=0}^{\infty} Q(s) \xi^s,$$

where $\xi = \lambda^{-5}$ and $Q(s)$ is the number of loops $0A0$, where $l(A) = 9 + 5s$ and A contains no 0's. Consider only the unprimed half of the graph. There are two kinds of loops based at 0: those which do not visit state 4 and those which do. Loops of the first kind are distinguished by the number $(0, 1, \ldots, s)$ of times they visit state 1. (The number of times they visit state 2 is then determined, since s is fixed.) Thus there are $2(s+1)$ loops of the first kind, in the two halves of the graph, of length $9 + 5s$.

Loops of the second kind are distinguished by the number of times (p) they return to state 4 and the number of times (l) they return to state 1. For a fixed

$p=0,1,\ldots,[s/2]$, $t=s-2p$ can be divided (among the loops $1\leftrightarrow 3$ and $2'\leftrightarrow 2'$) $s-2p+1$ ways. This yields

$$2\sum_{p=0}^{s/2}(s-2p+1)=(s+2)^2/2 \qquad (s \text{ even})$$

$$\sum_{p=0}^{(s-1)/2}(s-2p+1)=(s+1)(s+3)/2 \qquad (s \text{ odd})$$

loops of the second kind, in the two halves of the graph, of length $9+5s$.

Equation (7.1) thus reads

$$\begin{aligned}1&=\xi^2\sum_{s=0}^{\infty}Q(s)\xi^s\\ &=2\xi^2\left[\sum_{s=0}^{\infty}(s+1)\xi^s+\sum_{k=0}^{\infty}\frac{(2k+2)^2}{2}\xi^{2k}+\sum_{k=1}^{\infty}\frac{2k(2k+2)}{2}\xi^{2k-1}\right]\\ &=2\frac{\xi^2}{(1-\xi)^2}+\frac{4\xi^4+4\xi^2}{(1-\xi^2)^3}+8\frac{\xi^3}{(1-\xi^2)^3},\end{aligned}$$

which reduces to

$$\xi^6+4\xi^5-\xi^4-12\xi^3-9\xi^2+1=0.$$

Newton's method gives $\xi=0.284079$ and

$$h_{\text{top}}(D_r(1.3282,2\pi/5))=-\log\xi/5=0.2517.$$

8. *Coding with restrictions*

The results of the preceding sections suggest some ways to code with restrictions. For example, perhaps the simplest way to code into a system $D_r(c,\theta)$, with θ a rational multiple of π, so as to produce signals with power bounded at frequency θ, is to select blocks $C_1,\ldots,C_n$ on 1 and -1 with $\text{Sum}(C_k)=0$ for each k, and then concatenate these blocks arbitrarily. This provides a map of the full n-shift into $D_r(c,\theta)$ for some c. By the theorem of Gurevič [**14**], we can arrange for the image Σ of the n-shift under this map to have entropy as close to log 2 as we please.

For this code to be efficient in practice, one should first code the source into the n-shift (Bernoulli scheme) $B(p_1,\ldots,p_n)$, where the p_k are chosen in such a way that the code described above will map this Bernoulli measure to the measure of maximal entropy on Σ. It is noted in proposition 7.2 that this is accomplished by choosing $p_1,\ldots,p_n$ to be the unique solution of

$$-\sum_{k=1}^{n}p_k\log p_k=h_{\text{top}}(\Sigma)\sum_{k=1}^{n}l_kp_k,$$

where $l_k=l(C_k)$.

In general, similar codes can be constructed between large subsets of any pair of almost sofic systems. Let X and Y be almost sofic, and select SFT's $\Sigma\subset X$ and $\Delta\subset Y$ with $h_{\text{top}}(\Sigma)$ as close as we please to $h_{\text{top}}(X)$ and $h_{\text{top}}(\Delta)$ as close as we please to $h_{\text{top}}(Y)$. Fix vertices a in Σ and b in Δ, and form the loop-graph resolutions of

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Σ and Δ based at these vertices, as in § 7. By selecting finitely many loops $ax_1 \ldots x_l a$, with no $x_k = a$, and considering only arbitrary concatenations of these loops, we obtain an SFT $\Sigma_0 \subset \Sigma$ whose entropy is as close to that of Σ as we please. We find a similar SFT Δ_0 inside Δ, which we may assume involves the same number of loops as does Σ_0. Now making the loops of Σ_0 correspond to those of Δ_0 provides a simple code between a large part of X and a large part of Y.

These are variable-length codes, but we can also produce *constant-length* codes of this kind. For suppose we are considering a system of arbitrary concatenations of n loops $C_1, \ldots, C_n$. Fix a large k, and consider all concatenations $B = B(i_1, \ldots, i_k) = C_{i_1} \ldots C_{i_k}$ of k of these loops. There are about n^k such concatenations B, all of length less than or equal to $k \max \{l(C_i)\}$, so for large k many of these blocks B must have the same length. Then forming concatenations of just these identical-length blocks $B(i_1, \ldots, i_k)$ will produce a subsystem of arbitrarily large entropy. If we deal with just these subsystems, we can have constant-length codes, although the lengths of the blocks involved will of course be larger.

9. *Examples of early SFT's in some disk systems*

For a fixed $\theta = 2\pi k/m$, $U(c, \theta)$ begins, for small c, as an SFT. As c increases, more and more states are added until a certain critical value $c_0(\theta)$ is reached, beyond which there are infinitely many states. The following figures give an idea of the pattern of the states in the disk and the graphs of the corresponding SFT's for some small values of c. In figures 1–4, we have plotted the attained points in the disk for the indicated value of c and θ. Figures 5–8 show the evolution of $U(c, 2\pi/5)$ for small values of c. The graphs do not show most states where a choice is not possible (and some of the choices are only illusory, since they lead immediately to states from which no legal move can be made). The states are labelled with their polar coordinates and congruence classes of the time mod 5, and the arrows according to the sequence of $\pm$ steps that they represent.

The numbers of choice states and of all states in $U(c, 2\pi/5)$ as a function c vary as follows:

c	Choice States	All States
1.24	5	23
1.3282	13	63
1.434	21	95
1.454	57	265
1.456	103	465
1.48	107	469
1.4865	111	481

Varying θ among the primitive fifth roots of unity also changes the number of choice states, the graph, and the entropy, although there are some obvious symmetries and isomorphisms among these systems.

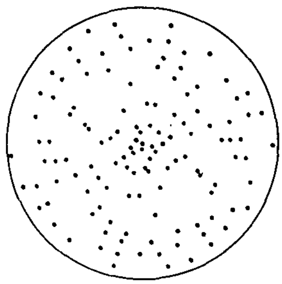

FIGURE 1. The states of $U(1.3, 2\pi/7)$.

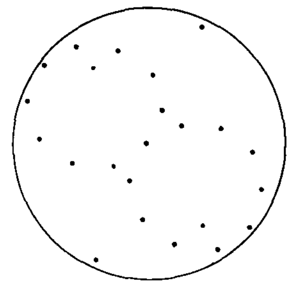

FIGURE 2. The states of $U(1.3, 2\pi/5)$.

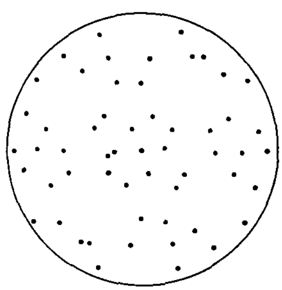

FIGURE 3. The states of $U(1.3282, 2\pi/5)$.

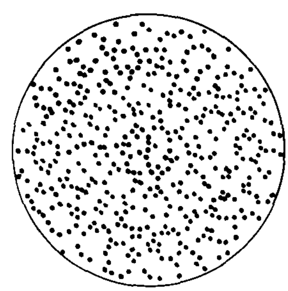

FIGURE 4. The states of $U(1.486, 2\pi/5)$.

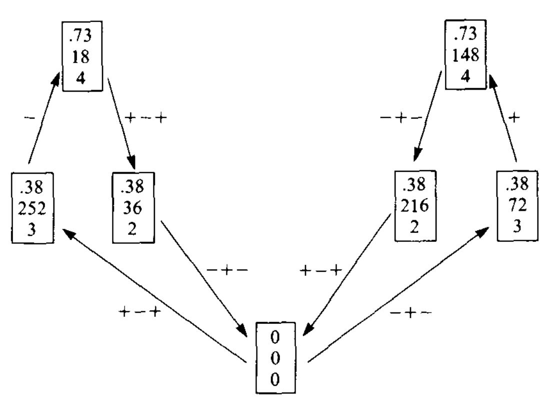

FIGURE 5. Transition diagram of $U(1.3, 2\pi/5)$. The entropy is $(\log 2)/10 = 0.069$.

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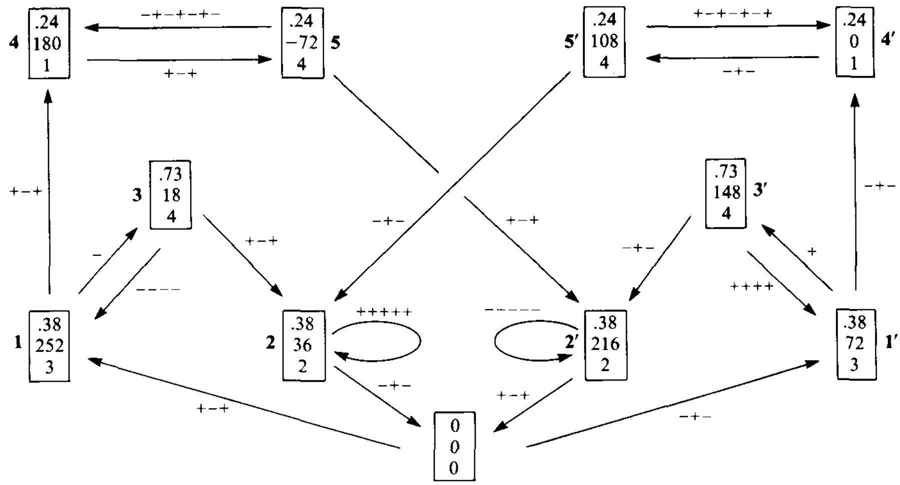

FIGURE 6. Transition diagram of $U(1.3282, 2\pi/5)$. The entropy is 0.2517 (see § 7).

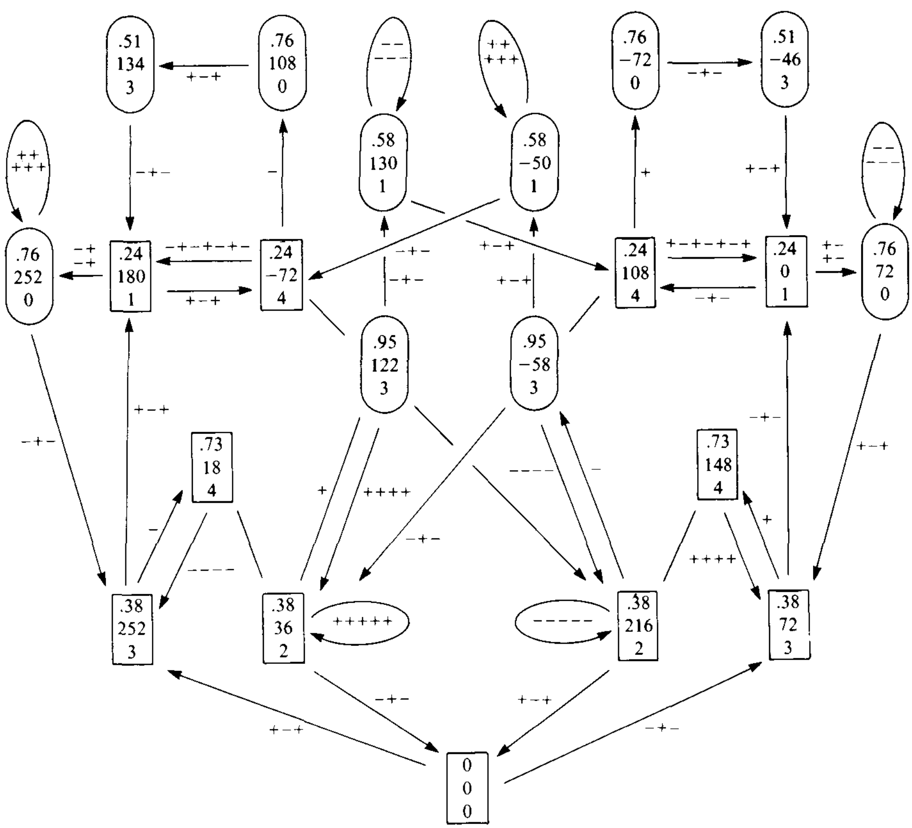

FIGURE 7. Transition diagram of $U(1.434, 2\pi/5)$.

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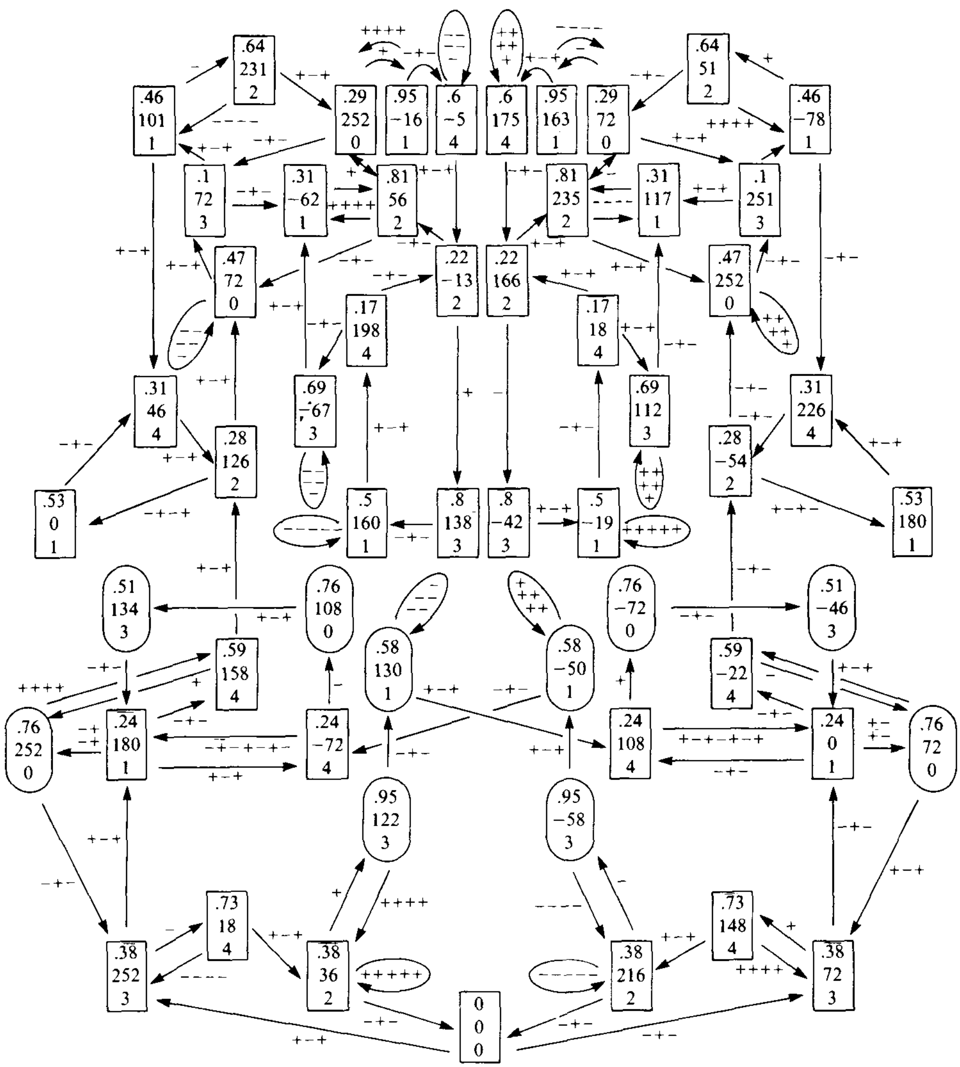

FIGURE 8. Transition diagram of $U(1.454, 2\pi/5)$. From some states it is impossible to return to 0, but when c is enlarged slightly 0 will again become accessible from these states.

REFERENCES

[1] R. L. Adler & B. Marcus. Topological entropy and equivalence of dynamical systems. *Mem. Amer. Math. Soc.* **219** (1979).

[2] R. Adler, D. Coppersmith & M. Hassner. Algorithms for sliding block codes - an application of symbolic dynamics to information theory, *IEEE Trans. Inf. Theory* **29** (1983), 5-22.

[3] F. Blanchard. Systèmes dynamiques topologiques associés à des automates récurrents. *Z. Wahr. verw. Geb.* **58** (1981), 549-564.

[4] F. Blanchard & G. Hansel. Systèmes codés. To appear.

[5] L. Block, J. Guckenheimer M. Misiurewicz & L.-S. Young. Periodic points and topological entropy for one dimensional maps. *Lecture Notes in Math.* **819**, Springer-Verlag, New York, 1980, 18-34.

[6] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** (1971), 401-414.

[7] A. A. Brudno. Entropy and the complexity of the trajectories of a dynamical system. *Trans. Moscow Math. Soc.* **44** (1982), 127-151.
[8] M. Denker, C. Grillenberger & K. Sigmund. *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. 527, Springer-Verlag, New York, 1976.
[9] Y. Derriennic. Quelques applications du théorème ergodique sous-additif. *Soc. Math. de France Astérisque* **74** (1980), 183-201.
[10] E. I. Dinaburg. An example of the computation of topological entropy. *Uspehi Mat. Nauk* **23** (1968), 249-250.
[11] P. A. Franaszek. A general method for channel coding. *I.B.M. J. Res. and Dev.* **24** (1980), 638-691.
[12] P. A. Franaszek. Construction of bounded delay codes for discrete channels. *I.B.M. J. Res. and Dev.* **26** (1982), 506-514.
[13] H. W. Gould. *Combinatorial Identities.* Morgantown, W. Va., 1972.
[14] B. M. Gurevič. Topological entropy of enumerable Markov chains. *Dokl. Akad. Nauk SSSR* **187** (1969). *Soviet Math. Dokl.* **10** (1969), 911-915.
[15] B. M. Gurevič. Shift entropy and Markov measures in the path space of a denumerable graph. *Dokl. Akad. Nauk SSSR* **192** (1970). *Soviet Math. Dokl.* **11** (1970), 744-747.
[16] A. Lempel & M. Cohn. Look ahead coding for input restricted channels. *IEEE Trans. Inf. Theory* **28** (1982), 933-937.
[17] B. Marcus. Sofic systems and encoding data. *IEEE Trans. Inf. Theory* **31** (1985), 366-377.
[18] B. Marcus & K. Petersen. Balancing ergodic averages. In *Ergodic Theory*, Lecture Notes in Math. 729, Springer-Verlag, New York, 1979, 126-143.
[19] K. Norris & D. S. Bloomberg. Channel capacity of charge-constrained run-length limited codes. *IEEE Trans. Magnetics* **17** (1981), 3452-3455.
[20] W. Parry & S. Tuncel. *Classification Problems in Ergodic Theory.* L.M.S. Lecture Note Series 67. Cambridge University Press: Cambridge, 1982.
[21] A. D. Patel. Zero modulation in magnetic recording. *I.B.M. J. Res. and Dev.* **19** (1975), 366-378.
[22] I. Salama. Topological entropy and classification of countable chains. Ph.D. Dissertation, University of North Carolina, 1984.
[23] R. Salem. Power series with integral coefficients. *Duke Math. J.* **12** (1945), 153-171.
[24] E. Seneta, *Non-negative Matrices and Markov Chains.* Springer-Verlag, New York, 1973 and 1981.
[25] P. H. Siegel. Modulation codes for magnetic recording. Preprint.
[26] D. Tang and L. Bahl. Block codes for a class of constrained noiseless channels. *Inf. and Control* **7** (1970), 436-461.
[27] D. Vere-Jones. Geometric ergodicity in denumerable Markov chains. *Quarterly J. Math.* **13** (1962), 7-28.
[28] D. Vere-Jones. Ergodic properties of nonnegative matrices, I. *Pacific J. Math.* **22** (1967), 361-386.
[29] J. B. Wagoner. Topological Markov chains, C^*-algebras, and K_2. To appear.
[30] B. Weiss. Intrinsically ergodic systems. *Bull. Amer. Math. Soc.* **76** (1970), 1266-1269.