QUASI-NORMAL MATRICES AND PRODUCTS

N. A. WIEGMANN

(Received 17 March 1969)

Communicated by G. B. Preston

1. Introduction

A normal matrix $A = (a_{ij})$ with complex elements is a matrix such that $AA^{CT} = A^{CT}A$ where A^{CT} denotes the (complex) conjugate transpose of A. In an article by K. Morita [2] a quasi-normal matrix is defined to be a complex matrix A which is such that $AA^{CT} = A^TA^C$, where T denotes the transpose of A and A^C the matrix in which each element is replaced by its conjugate, and certain basic properties of such a matrix are developed there. (Some doubt might exist concerning the use of 'quasi' since this class of matrices does not contain normal matrices as a sub-class; however, in deference to the original paper and the normal canonical form of Theorem 1 below, the terminology in [2] is used.)

Here further properties of quasi-normal matrices are developed, their relation, in a sense, to normal matrices is considered, and further results concerning normal products are obtained including an analog (Theorem 4) for quasi-normal matrices.

2. Properties of quasi-normal matrices

The basic theorem developed in [2] is the following, for which an alternate proof is supplied here for brevity and easy reference.

THEOREM 1. *A matrix A is quasi-normal if and only if there exists a unitary matrix U such that UAU^T is a direct sum of non-negative real numbers and of 2×2 matrices of the form*:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where a and b are non-negative real numbers.

Let A be quasi-normal where $A = S+T$ where $S = S^T$ and $T = -T^T$. Then $AA^{CT} = A^TA^C$ gives $(S+T)(S^{CT}+T^{CT}) = (S^T+T^T)(S^C+T^C)$ or $(S+T)(S^C-T^C) = (S-T)(S^C+T^C)$ and so: $SS^C+TS^C-ST^C-TT^C = SS^C-TS^C+ST^C-TT^C$ or $TS^C = ST^C$. There exists a unitary matrix U (see [3] or [5]) such that $USU^T = D$ is a diagonal matrix with real, non-negative elements. Therefore $UTU^TU^CS^CU^{CT} =$

$USU^TU^CT^CU^{CT}$ or $WD = DW^C$ where $W = -W^T$. Let U be chosen so that D is such that $d_i \geqq d_j \geqq 0$ for $i < j$ where d_i is the i^{th} diagonal element of D. If $W = (t_{ij})$, where $t_{ji} = -t_{ij}$, then $t_{ij}d_j = d_i\bar{t}_{ij}$, for $j > i$, and 3 possibilities may occur: if $d_j = d_i \neq 0$, then t_{ij} is real; if $d_j = d_i = 0$, t_{ij} is arbitrary (though $W = -W^T$ still holds); and if $d_j \neq d_i$, then $t_{ij} = 0$ for if $t_{ij} = a+ib$, then $(a+ib)d_j = d_i(a-ib)$ and $a(d_j-d_i) = 0$ implies $a = 0$ and $b(d_i+d_j) = 0$ implies $d_i = -d_j$ (which is not possible since the d_i are real and non-negative and $d_j \neq d_i$) or $b = 0$ so $t_{ij} = 0$. So if $USU^T = d_1I_1 \dotplus d_2I_2 \dotplus \cdots \dotplus d_kI_k$ where $\dotplus$ denotes direct sum, then $UTU^T = T_1 \dotplus T_2 \dotplus \cdots \dotplus T_k$ where $T_i = -T_i^T$ is real and $T_k = -T_k^T$ is complex if and only if $d_k = 0$. For each real T_i there exists a real orthogonal matrix V_i so that $V_iT_iV_i^T$ is a direct sum of zero matrices and matrices of the form

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

where b is real (see [1] page 65 for example). If $T_k = -T_k^T$ is complex, there exists a complex unitary matrix V_k such that $V_kT_kV_kT$ is a direct sum of matrices of the same form (see [4]), so that if $V = V_1 \dotplus V_2 \dotplus \cdots \dotplus V_k$, then $VUSU^TV^T = D$ and $VUTU^TV^T = F =$ the direct sum described. Therefore $VUAU^TV^T = D+F$ which is the desired form.

Among properties of quasi-normal matrices obtained in [2] are the following: If A and B are two quasi-normal matrices such that $AB^C = BA^C$, then A and B can be simultaneously brought into the above normal form under the same U (with a generalization to a finite number) but not conversely; if A is quasi-normal, AA^C is normal in the usual sense, but not conversely; and if A is quasi-normal and AA^C is real, there is a real orthogonal matrix which gives the above form.

Among properties of quasi-normal matrices not obtained in [2] but of subsequent use are the following:

(a) A is quasi-normal if and only if $A = HU = UH^T$ where H is hermitian and U is unitary.

For if $A = HU$ is a polar form of A, then $U^{CT}HU = K$ is such that $A = HU = UK$ and if $AA^{CT} = A^TA^C$, then $H^2 = (K^T)^2$ and since this is a hermitian matrix with non-negative roots, $H = K^T$ and $A = HU = UH^T$. The converse is immediate. This same result may be seen as follows. If $UAU^T = F$ is the normal form in Theorem 1, $F = D_rV = VD_r$ where D_r is real diagonal and V is a direct sum of 1's or blocks of the form

$$(a^2+b^2)^{-\frac{1}{2}} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

which are unitary. Therefore $A = U^{CT}D_rUU^{CT}VU^C = U^{CT}VU^CU^TD_rU^C$ which exhibits the polar form in another guise.

(b) A is both normal and quasi-normal if and only if $A = HU = UH = UH^T$ so $H = H^T = H^{CT}$ so that H is real.

https://doi.org/10.1017/S1446788700006716 Published online by Cambridge University Press

(c) If $A = HU = UH^T$ is quasi-normal, then UH is quasi-normal if and only if $HU^2 = U^2H$, i.e. if and only if HU^2 is normal. For if UH is quasi-normal, $UH = H^TU$ so that $HU^2 = UH^TU = U^2H$; and if $HU^2 = U^2H$, then $HUU = UH^TU = UUH$ or $H^TU = UH$.

(d) A matrix A is quasi-normal if and only if A can be written $A = SW = W^CS$ where $S = S^T$ and W is unitary. If A is quasi-normal, from the above $A = U^{CT}FU^C = U^{CT}D_rU^CU^TVU^C = SW = U^{CT}VUU^{CT}D_rU^C = W^CS$ where $S = U^{CT}D_rU^C$ is symmetric and $W = U^TVU^C$ is unitary. Conversely, if $A = SW = W^CS$, $AA^{CT} = SWW^{CT}S^{CT} = A^TA^C = S^TW^{CT}WS^C$.

Note that if B is quasi-normal and if $B = SU$ where $S = S^T$ and U is unitary, it does not necessarily follow that $B = U^CS$; but it is possible to find an S_1 and U_1 such that $B = S_1U_1 = U_1^CS_1$ holds. This may be seen as follows. If $B = SU$ is quasi-normal, let V be unitary such that $VSV^T = D$ is diagonal, real, and non-negative, so that $VBV^T = VSV^TV^CUV^T = DW$ is quasi-normal from which $DWW^{CT}D^C = W^TD^TD^CW^C$ or, since D is real, $WD^2 = D^2W$ and $WD = DW$ since D is non-negative. Then $B = (V^{CT}DV^C)(V^TWV^C) = SU = (V^{CT}WV)(V^{CT}DV^C)$ which is not necessarily $=$ to $U^CS = (V^{CT}W^CV)(V^{CT}DV^C)$. However, if $D = r_1I_1 \dotplus r_2I_2 \dotplus \cdots \dotplus r_kI_k, r_i > r_j$ for $i > j$, then $W = W_1 \dotplus W_2 \dotplus \cdots \dotplus W_k$. Since each W_i is unitary, it is quasi-normal and there exist unitary X_i so that $X_iW_iX_i^T = F_i$ is in the real normal form of Theorem 1. If $X = X_1 \dotplus X_2 \dotplus \cdots \dotplus X_k$, then $XVBV^TX^T = XDWX^T = DXWX^T = DF = FD$ where $F = F_1 \dotplus F_2 \dotplus \cdots \dotplus F_k$. So

$$\begin{aligned} B &= (V^{CT}X^{CT}DX^CV^C)(V^TX^TFX^CV^C) \\ &= (V^{CT}X^{CT}FXV)(V^{CT}X^{CT}DX^CV^C) = S_1U_1 = U_1^CS_1 \quad \text{and} \\ S_1 &= V^{CT}X^{CT}DX^CV^C \neq V^{CT}DV^C = S \quad \text{and} \\ U_1 &= V^TX^TFX^CV^C \neq V^TWV^C = U. \end{aligned}$$

3. Normal products of matrices

It was shown in [6] that the following are true: if A, B, and AB are normal matrices, the BA is normal; a necessary and sufficient condition that the product, AB, of two normal matrices A and B be normal is that each commute with the hermitian polar matrix of the other. First a generalization of this theorem is obtained here and then an analog for the quasi-normal case is developed.

THEOREM 2. *Let A be a normal matrix. Then AB and BA are normal if and only if $(A^{CT}A)B = B(AA^{CT})$ and $(B^{CT}B) = A(BB^{CT})$.*

(In a sense, the latter conditions might be described as stating that each matrix is 'normal relative to the other'.)

If AB and BA are normal, let U be a unitary matrix such that $UAU^{CT} = D$ is diagonal, $d_i\bar{d}_i \geqq d_j\bar{d}_j \geqq 0$ for $i < j$, and let $UBU^{CT} = B_1 = (b_{ij})$. From

https://doi.org/10.1017/S1446788700006716 Published online by Cambridge University Press

$ABB^{CT}A^{CT} = B^{CT}A^{CT}AB$ it follows that $DB_1B_1^{CT}D^C = B_1^{CT}D^CDB_1$; by equating diagonal elements it follows that $\sum_{j=1}^n d_i\bar{d}_ib_{ij}\bar{b}_{ij} = \sum_{j=1}^n d_j\bar{d}_jb_{ji}\bar{b}_{ji}$ for $i = 1, 2, \cdots, n$. Similarly from $BAA^{CT}B^{CT} = A^{CT}B^{CT}BA$ follows $B_1DD^CB_1^{CT} = D^CB_1^{CT}B_1D$ and $\sum_{j=1}^n d_j\bar{d}_jb_{ij}\bar{b}_{ij} = \sum_{j=1}^n \bar{d}_id_i\bar{b}_{ji}b_{ji}$. Let $i = 1$ in each of these equations so that $\sum_{j=1}^n d_1\bar{d}_1b_{1j}\bar{b}_{1j} = \sum_{j=1}^n d_j\bar{d}_jb_{j1}\bar{b}_{j1}$ and $\sum_{j=1}^n d_j\bar{d}_jb_{1j}\bar{b}_{1j} = \sum_{j=1}^n \bar{d}_1d_1\bar{b}_{j1}b_{j1}$ from which follows $\sum_{j=1}^n(d_1\bar{d}_1-d_j\bar{d}_j)b_{1j}\bar{b}_{1j} = \sum_{j=1}^n(d_j\bar{d}_j-d_1\bar{d}_1)b_{j1}\bar{b}_{j1}$ so that $\sum_{j=1}^n(d_1\bar{d}_1-d_j\bar{d}_j)(b_{1j}\bar{b}_{1j}+b_{j1}\bar{b}_{j1}) = 0$. Let $d_1\bar{d}_1 = d_2\bar{d}_2 = \cdots = d_l\bar{d}_l > d_{l+1}d_{l+1}$; then $b_{1j}\bar{b}_{1j}+b_{j1}\bar{b}_{j1} = 0$ for $j = l+1, l+2, \cdots, n$ since $d_1\bar{d}_1-d_j\bar{d}_j$ is zero or positive and is the latter for $j > l$. So $b_{1j} = 0$ and $b_{j1} = 0$ for $j = l+1, l+2, \cdots, n$. For $i = 2, \cdots, l$ in turn it follows that $b_{ij} = 0$ and $b_{ji} = 0$ for $i = 1, 2, \cdots, l$ and for $j = l+1, l+2, \cdots, n$. Let $UAU^{CT} = D = r_1D_1\dotplus r_2D_2\dotplus \cdots \dotplus r_sD_s$ where the r_i are real, $r_i > r_j$ for $i < j$ and the D_i are unitary. Then by repeating the above process it follows that $UBU^{CT} = B_1 = C_1\dotplus C_2\dotplus \cdots \dotplus C_s$ is conformable to D.

It follows from the given conditions that $r_iD_iC_iC_i^{CT}D_i^Cr_i = C_i^{CT}(r_iD_i^C)(D_ir_i)C_i$ and $C_ir_iD_iD_i^Cr_iC_i^{CT} = r_iD_i^CC_i^{CT}C_iD_ir_i$ or that $D_iC_iC_i^{CT} = C_i^{CT}C_iD_i$ and $D_iC_iC_i^{CT} = C_i^{CT}C_iD_i$ if $r_i > 0$. If $r_s = 0$, D_s is arbitrary insofar as D is concerned and so may be chosen so that $D_sC_sC_s^{CT} = C_s^{CT}C_sD_s$ in which case D_s may not be diagonal. But whether or not this is done, it follows that $DB_1B_1^{CT} = B_1^{CT}B_1D$ and that $B_1DD^{CT} = D^{CT}DB_1$ so that $A(BB^{CT}) = (B^{CT}B)A$ and $B(AA^{CT}) = (A^{CT}A)B$.

The converse is immediate. It may be noted that if the roots of A are all distinct in absolute value, B must be normal. The following further clarifies the situation.

THEOREM 3. *Let $A = LW = WL$ be the polar form of the normal matrix A. Then AB and BA are normal if and only if $B = NW^{CT}$ where N is normal and $LN = NL$.*

In the above proof let $C_i = H_iU_i = U_iK_i$ be polar forms of the C_i. Then $U_i^{CT}H_iU_i = K_i$ so that $U_i^{CT}C_iC_i^{CT}U_i = C_i^{CT}C_i$ or $U_i^{CT}C_iC_i^{CT} = C_i^{CT}C_iU_i^{CT}$. Also, from the above $D_iC_iC_i^{CT} = C_i^{CT}C_iD_i$. Let $R_i = D_i^CU_i^{CT}$; then

$$R_iC_iC_i^{CT} = D_i^CU_i^{CT}C_iC_i^{CT} = D_i^CC_i^{CT}C_iU_i^{CT} = C_iC_i^{CT}D_i^CU_i^{CT} = C_iC_i^{CT}R_i$$

where R_i is unitary. (If $r_s = 0$, D_s may be chosen $= U_s^{CT}$ as described above). So $R_iH_i^2 = H_i^2R_i$ and since H_i has positive or zero roots, $R_iH_i = H_iR_i$ and so $H_iR_i^{CT} = R_i^{CT}H_i$. Then $A = U^{CT}DU = U^{CT}D_rUU^{CT}D_uU = LW = WL$ and

$$\begin{aligned} B &= U^{CT}B_1U = U^{CT}(C_1\dotplus C_2\dotplus \ldots \dotplus C_s)U \\ &= U^{CT}(H_1U_1\dotplus H_2U_2\dotplus \cdots \dotplus H_sU_s)U \\ &= U^{CT}(H_1R_1^{CT}D_1^C\dotplus H_2R_2^{CT}D_2^C\dotplus \cdots \dotplus H_sR_s^{CT}D_s^C)U = NW^{CT} \end{aligned}$$

where $N = U^{CT}(H_1R_1^{CT}\dotplus H_2R_2^{CT}\dotplus \cdots \dotplus H_sR_s^{CT})U$ (which is normal since the hermitian H_i and unitary R_i^{CT} commute) and $W^{CT} = U^{CT}(D_1^C\dotplus D_2^C\dotplus \cdots \dotplus D_s^C)U$. It is evident that $LN = NL$.

Conversely, if $A = LW = WL$ and $B = NW^{CT}$ as described, then $AB = WLNW^{CT}$ which is obviously normal as is $BA = NW^{CT}WL = NL$.

It is easily seen that $B = NW^{CT}$ is normal if and only if $NW^{CT} = W^{CT}N$. If $B = NW^{CT} = (HR)W^{CT}$ is quasi-normal, then $B = H(RW^{CT}) = (RW^{CT})H^T = RHW^{CT}$ (from property a), section 2) so $W^{CT}H^T = HW^{CT}$ or $WH = H^TW$ and $W(BB^{CT}) = (B^{CT}B)W$.

If A is normal, if B is quasi-normal, and if AB is normal, it does not necessarily follow that BA is normal though it can occur. For example, if $B = HU = UH^T$ is quasi-normal and if $A = U^{CT}$, then $AB = U^{CT}UH^T = H^T$ and $BA = HUU^{CT} = H$ are both normal. But the following is an example in which AB is normal but not BA. Let $B = HU = UH^T$ be quasi-normal but not normal (i.e., H is not real by property b) section 2) and let H be non-singular. Let $A = H^{-1}$ which is hermitian (so normal) and not quasi-normal (since H^{-1} is not real). Then $AB = H^{-1}HU = U$ is normal. If BA were also normal, then by the above theorem $(A^{CT}A)B = B(AA^{CT})$ and $(B^{CT}B)A = A(BB^{CT})$. But $(B^{CT}B)A = (H^T)^2H^{-1}$ and $A(BB^{CT}) = (H^{-1})(H^2)$ and if these were equal, $(H^T)^2 = H^2$ would follow which means that $H^2 = (H^T)^2 = (H^{CT})^2$ so that H^2 is real. But this is not possible for if $H = VDV^{CT}$ where D is diagonal with positive real elements (since H is non-singular), then $H^2 = VD^2V^{CT} = V^CD^2V^T$ if H^2 is real so that $V^TVD^2 = D^2V^TV$ so $V^TVD = DV^TV$ so $VDV^{CT} = V^CDV^T = H$ is real which contradicts the above assumption.

But the following theorems result when A and B are both quasi-normal.

THEOREM 4. *If A and B are quasi-normal and if AB is normal, then BA is normal.*

Let U be a unitary matrix such that $UAU^T = F$ is the normal form described in Theorem 1 and where $FF^{CT} = FF^T = r_1^2I_1 + r_2^2I_2 + r_3^2I_3 + \cdots + r_k^2I_k$ which is real diagonal with $r_1^2 > r_2^2 > \cdots > r_k^2 \geqq 0$. These r_i^2 may be either the squares of diagonal elements of F or they may arise when matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

are squared. Assume that any of the latter whose r_i^2 are equal are arranged first in a given block followed by any diagonal elements whose square is the same r_i^2.

Let $U^CBU^{CT} = B_1$ which is quasi-normal and then $UAU^TU^CBU^{CT} = FB_1$ is normal. Let V be the unitary matrix

$$\sqrt{2}^{-1}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Then the following matrix relation holds, independent of a and b:

$$V\begin{bmatrix} a & b \\ -b & a \end{bmatrix}V^{CT} = \begin{bmatrix} a-bi & 0 \\ 0 & a+bi \end{bmatrix}$$

https://doi.org/10.1017/S1446788700006716 Published online by Cambridge University Press

Let $F = F_1 \dotplus F_2 \dotplus \cdots \dotplus F_k$ where the direct sum is conformable to that of FF^{CT} given above (i.e., $F_i F_i^{CT} = r_i^2 I_i$) and consider $F_1 = G_1 \dotplus G_2 \dotplus \cdots \dotplus G_l \dotplus r_1 I$ where each G_i is 2×2 as described above and I is an identity matrix of proper size. Let $W_1 = V \dotplus V \dotplus \cdots \dotplus V \dotplus I$ be conformable to F_1; define W_i for each F_i in like manner and let $W = W_1 \dotplus W_2 \dotplus \cdots \dotplus W_k$. If $r_k = 0$, $W_k = I$. Then $WFW^{CT} = D$ is complex diagonal, where if d_i is the i^{th} diagonal element $d_i \bar{d}_i \geqq d_{i+1} \bar{d}_{i+1}$. Then $W(UAU^T)W^{CT} W(U^C BU^{CT})W^{CT} = (WFW^{CT})(WB_1 W^{CT}) = DB_2$ is normal for $B_2 = WB_1 W^{CT}$ (or $B_1 = W^{CT} B_2 W$). Since B_1 is quasi-normal, $B_1 B_1^{CT} = B_1^T B_1^C$ so that $W^{CT} B_2 W W^{CT} B_2^{CT} W = W^T B_2^T W^C W^T B_2^C W^C$ or that $B_2 B_2^{CT} WW^T = WW^T B_2^T B_2^C$. Now VV^T is a matrix of the form

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

so that WW^T is a direct sum of matrices of this form and 1's.

Let $B_2 = (b_{ij})$ and consider $(WW^T)^{CT} B_2 B_2^{CT} (WW^T) = B_2^T B_2^C$. Let $B_2 B_2^{CT} = (c_{ij})$, $B_2^T B_2^C = (f_{ij})$. c_{ij} and f_{ij} are identifiable with the b_{ij}, both matrices being hermitian. Consider two cases:

a) If $d_1 \bar{d}_1 = d_j \bar{d}_j$ for all j (where d_j is the j^{th} diagonal element of D), then $D = kD_u$ where D_u is unitary diagonal. Since $WFB_1 W^{CT} = DB_2 = kD_u B_2 = D_u(kB_2)$ is normal, then $D_u^C(D_u B_2 k)D_u = B_2 D = WB_1 FW^{CT}$ is normal as is $B_1 F = U^C BU^{CT} UAU^T$ so BA is normal.

b) If $d_1 \bar{d}_1 \neq d_j \bar{d}_j$ for some j, let $d_1 \bar{d}_1 = d_2 \bar{d}_2 = \cdots = d_l \bar{d}_l$ for $1 \leqq l < n$ (so that $d_l \bar{d}_l > d_{l+1} \bar{d}_{l+1}$).

Suppose $F_1 = G_1 \dotplus G_2 \dotplus r_1 I_1$ where I_1 is the 2×2 identity matrix. (The general case will be seen to follow from this example.) From $(WW^T)^{CT} B_2 B_2^{CT} (WW^T) = B_2^T B_2^C$ and the fact that $W_1 = V \dotplus V \dotplus I_1$, it follows that $c_{11} = f_{22}$, $c_{22} = f_{11}$, $c_{33} = f_{44}$, $c_{44} = f_{33}$, $c_{55} = f_{55}$, $c_{66} = f_{66}$ (and $\bar{c}_{12} = f_{12}$, $\bar{c}_{34} = f_{34}$, etc.). These equalities supply the following relations (where the summations is over $i = 1$ to n):

$$c_{11} = \Sigma b_{1i} \bar{b}_{1i} = \Sigma b_{i2} \bar{b}_{i2} = f_{22}; \qquad c_{22} = \Sigma b_{2i} \bar{b}_{2i} = \Sigma b_{i1} \bar{b}_{i1} = f_{11}$$
$$c_{33} = \Sigma b_{3i} \bar{b}_{3i} = \Sigma b_{i4} \bar{b}_{i4} = f_{44}; \qquad c_{44} = \Sigma b_{4i} \bar{b}_{4i} = \Sigma b_{i3} \bar{b}_{i3} = f_{33}$$
$$c_{55} = \Sigma b_{5i} \bar{b}_{5i} = \Sigma b_{i5} \bar{b}_{i5} = f_{55}; \qquad c_{66} = \Sigma b_{6i} \bar{b}_{6i} = \Sigma b_{i6} \bar{b}_{i6} = f_{66}.$$

DB_2 is normal so that the following relations also hold:

$$d_1 \bar{d}_1 \Sigma b_{1i} \bar{b}_{1i} = \Sigma d_i \bar{d}_i b_{i1} \bar{b}_{i1}; \qquad d_2 \bar{d}_2 \Sigma b_{2i} \bar{b}_{2i} = \Sigma d_i \bar{d}_i b_{i2} \bar{b}_{i2}$$
$$d_3 \bar{d}_3 \Sigma b_{3i} \bar{b}_{3i} = \Sigma d_i \bar{d}_i b_{i3} \bar{b}_{i3}; \qquad d_4 \bar{d}_4 \Sigma b_{4i} \bar{b}_{4i} = \Sigma d_i \bar{d}_i b_{i4} \bar{b}_{i4}$$
$$d_5 \bar{d}_5 \Sigma b_{5i} \bar{b}_{5i} = \Sigma d_i \bar{d}_i b_{i5} \bar{b}_{i5}; \qquad d_6 \bar{d}_6 \Sigma b_{6i} \bar{b}_{6i} = \Sigma d_i \bar{d}_i b_{i6} \bar{b}_{i6}.$$

Since $d_1 \bar{d}_1 = d_2 \bar{d}_2$, on combining the first 2 relations in each of these sets, $d_1 \bar{d}_1 (\Sigma b_{1i} \bar{b}_{1i} + \Sigma b_{2i} \bar{b}_{2i}) = d_1 \bar{d}_1 (\Sigma b_{i1} \bar{b}_{i1} + \Sigma b_{i2} \bar{b}_{i2}) = \Sigma d_i \bar{d}_i (b_{i1} \bar{b}_{i1} + b_{i2} \bar{b}_2)$ so that $\Sigma(d_1 \bar{d}_1 - d_i \bar{d}_i)(b_{i1} \bar{b}_{i1} + b_{i2} \bar{b}_{i2}) = 0$. $d_1 \bar{d}_1 = d_j \bar{d}_j$ for $j = 1, 2, \cdots, 6$ but for j beyond 6, $d_1 \bar{d}_1 - d_j \bar{d}_j > 0$ so that $b_{i1} \bar{b}_{i1} + b_{i2} \bar{b}_{i2} = 0$ or $b_{i1} = 0$ and $b_{i2} = 0$

for $i = 7, 8, \cdots n$. Similarly, $b_{13} = 0$ and $b_{i4} = 0$ for $i > 6$. The third relations in each set give $b_{i5} = 0$ and $b_{i6} = 0$ for $i > 6$.

On adding all 6 relations in the first set,

$$\sum_{i,j=1}^{6} b_{ij}\bar{b}_{ij} + \sum_{i=1}^{6}\sum_{j=7}^{n} b_{ij}\bar{b}_{ij} = \sum_{i,j=1}^{6} b_{ij}\bar{b}_{ij} + \sum_{i=7}^{n}\sum_{j=1}^{6} b_{ij}\bar{b}_{ij}$$

and on cancelling the first summations on each side,

$$\sum_{i=1}^{6}\sum_{j=7}^{n} b_{ij}\bar{b}_{ij} = \sum_{i=7}^{n}\sum_{j=1}^{6} b_{ij}\bar{b}_{ij}.$$

But the right side is 0 from the above, so the left side is 0 and so $b_{ij} = 0$ for $i = 1, 2, \cdots, 6$ and $j > 6$.

From this it is evident that this procedure may be repeated, and that if

$$D = r_1 D_1 \dotplus r_2 D_2 \dotplus \cdots \dotplus r_k D_k$$

where the D_i are unitary and the r_i non-negative real, as above, then

$$B_2 = C_1 \dotplus C_2 \dotplus \cdots \dotplus C_k$$

conformable to D. Then $r_i D_i C_i$ is normal so $D_i^{CT}(D_i C_i r_i) D_i = C_i r_i D_i$ is normal so $B_2 D$ is normal, so $B_1 F$ and so $U^C B U^{CT} U A U^T$ and BA.

THEOREM 5. *If A and B are quasi-normal, then AB is normal if and only if $A^{CT}AB = BAA^{CT}$ and $ABB^{CT} = B^{CT}BA$ (i.e., if and only if each is 'normal relative to the other').*

If AB is normal, from the above, $D^{CT}DB_2 = B_2 DD^{CT}$ so that $F^{CT}FB_1 = B_1 FF^{CT}$ or $A^{CT}AB = BAA^{CT}$. Similarly, since DB_2 is normal, $DB_2 B_2^{CT} D^C = B_2^{CT} D^C D B_2$ so $DB_2 B_2^{CT} = B_2^{CT} B_2 D$ or $FB_1 B_1^{CT} = B_1^{CT} B_1 F$ or $ABB^{CT} = B^{CT}BA$. The converse is directly verifiable.

THEOREM 6. *Let A and B be quasi-normal. If AB is normal, then $A = LW = WL^T$ (with L hermitian and W unitary) and $B = NW^{CT}$ where N is normal and $L^T N = NL^T$; and conversely.*

As above, let $UAU^T = F = W^{CT}DW = W^{CT}D_r WW^{CT}D_u W$ (where D_r and D_u are the hermitian and unitary polar matrices of D) and $U^C B U^{CT} = B_1 = W^{CT} B_2 W = W^{CT}(C_1 \dotplus C_2 \dotplus \cdots \dotplus C_k)W$. As in the proof of Theorem 3 it follows that for all i, $D_i C_i C_i^{CT} = C_i^{CT} C_i D_i$ and $U_i^{CT} C_i C_i^{CT} = C_i^{CT} C_i U_i^{CT}$, with U_i as defined there, so that when $R_i = D_i^C U_i^{CT}$ (where D, here, $= r_1 D_1 \dotplus r_2 D_2 \dotplus \cdots \dotplus r_k D_k$, as earlier), then $C_i = H_i U_i = H_i R_i^{CT} D_i^C$ with $H_i R_i = R_i H_i$. Then, since

$$WD_r = D_r W,\ UAU^T = W^{CT}D_r WW^{CT}D_u W = D_r(W^{CT}D_u W) \quad \text{and}$$
$$A = (U^{CT}D_r U)(U^{CT}W^{CT}D_u WU^C) = LX$$
$$= (U^{CT}W^{CT}D_u WU^C)(U^T D_r U^C) = XL^T$$

with $L = U^{CT}D_r U$ hermitian and $X = U^{CT}W^{CT}D_u WU^C$ unitary. Also,

$$U^C BU^{CT} = W^{CT}(H_1 R_1^{CT} D_1^C + H_2 R_2^{CT} D_2^C + \cdots + H_k R_k^{CT} D_k^C)W = N_1 Y$$

where

$$N_1 = W^{CT}(H_1 R_1^{CT} + H_2 R_2^{CT} + \cdots + H_k R_k^{CT})W$$

is normal and

$$Y = W^{CT}(D_1^C + D_2^C + \cdots + D_k^C)W$$

is unitary; then

$$B = U^T N_1 YU = (U^T N_1 U^C)(U^T YU) = NX^{CT}$$

where $N = U^T N_1 U^C$ is normal and $X^{CT} = U^T YU = U^T W^{CT} D_u^C WU$. Also $L^T N = NL^T$ since $D_r N_1 = N_1 D_r$, $D_r^C N_1 = N_1 D_r^C$ so

$$(U^C L^C U^T)(U^C NU^T) = (U^C NU^T)(U^C L^C U^T)$$

so $L^T N = NL^T$. The converse is immediate.

4. Quasi-normal products of matrices

It is possible if A is normal and B quasi-normal that AB is quasi-normal. For example, any quasi-normal matrix $C = HU = UH^T$ is such a product with $A = H$ and $B = U$. Or if $C = HU = UH^T$ and $A = H$, then $AC = H^2 U = HUH^T = U(H^T)^2$ is quasi-normal. The following theorems clarify this matter.

THEOREM 7. *If A is normal and B is quasi-normal, then AB is quasi-normal if and only if $ABB^{CT} = BB^{CT}A$ and $B^C AA^{CT} = A^T A^C B^C$ (or $BA^C A^T = A^{CT}AB$).*

(If one were to define 'N is normal with respect to M'' to mean $NN^{CT}M = MN^{CT}N$ and 'Q is quasi-normal with respect to P' to mean $PQQ^{CT} = Q^T Q^C P$, the above theorem would say that if A is normal and B quasi-normal, then AB is quasi-normal if and only if (quasi-normal) B is normal with respect to A and (normal) A is quasi-normal with respect to B^C.)

If the latter conditions hold, then: $(AB)(AB)^{CT} = ABB^{CT}A^{CT} = BB^{CT}AA^{CT}$ and $(AB)^T(AB)^C = B^T A^T A^C B^C = B^T B^C AA^{CT}$ which are equal.

Conversely, let AB be quasi-normal and let $UAU^{CT} = D = d_1 I + d_2 I_2 + \cdots + d_k I_k$ where $d_i \bar{d}_i > d_j \bar{d}_j$, $i > j$. Let $UB^T U^T = B_1 = (b_{ij})$. If $(AB)(AB)^{CT} = ABB^{CT}A^{CT} = AB^T B^C A^{CT} = (AB)^T(AB)^C = B^T A^T A^C B^C = B^T A^C A^T B^C$, then

$$(UAU^{CT})(UB^T U^T U^C B^C U^{CT})(UA^{CT}U^{CT})$$
$$= (UB^T U^T)(U^C A^C U^T U^C A^T U^T)(U^C B^C U^{CT})$$

so that $DB_1 B_1^{CT} D^{CT} = B_1 D^C DB_1^{CT}$. Equating diagonal elements on each side of this relation, $\sum_{j=1}^n d_i \bar{d}_i b_{ij} \bar{b}_{ij} = \sum_{j=1}^n d_j \bar{d}_j b_{ij} \bar{b}_{ij}$, $i = 1, 2, \cdots, n$, or $\sum_{j=1}^n (d_i \bar{d}_i - d_j \bar{d}_j) b_{ij} \bar{b}_{ij} = 0$.

Let $d_1 \bar{d}_1 = d_2 \bar{d}_2 = \cdots d_l \bar{d}_l > d_{l+1} \bar{d}_{l+1}$. Then $b_{ij} = 0$ for $i = 1, 2, \cdots, l$ and

https://doi.org/10.1017/S1446788700006716 Published online by Cambridge University Press

$j = l+1, l+2, \cdots, n$. Since B_1 is quasi-normal, $\sum_{j=1}^{n} b_{ij}\bar{b}_{ij} = \sum_{j=1}^{n} b_{ji}\bar{b}_{ji}$ for $i = 1, 2, \cdots, n$. On adding the first l of these equations and cancelling, $b_{ij} = 0$ for $i = l+1, l+2, \cdots, n$ and $j = 1,2, \cdots, l$. In this manner if $D = r_1 D_1 \dotplus r_2 D_2 \dotplus \cdots \dotplus r_t D_t$ with $r_i > r_{i+1}$ and D_i unitary, then $B_1 = C_1 \dotplus C_2 \dotplus \cdots \dotplus C_t$ conformable to D. Since $r_i D_i D_i^{CT} r_i C_i^T = r_i^2 C_i^T = C_i^T r_i^2 = C_i^T r_i D_i D_i^{CT} r_i$, all i, $DD^{CT}B_1^T = B_1^T DD^{CT}$ and so $U^{CT}DD^{CT}UU^{CT}B_1^T U^C = U^{CT}B_1^T U^C U^T DD^{CT} U^C$ or $AA^{CT}B = BA^TA^C$ or $A^{CT}AB = BA^TA^C$ or $A^TA^CB^C = B^CAA^{CT}$.

Also, $D(B_1 B_1^{CT} D^{CT}) = B_1 D^C DB_1^{CT} = D^C DB_1^{CT} = D(D^C B_1 B_1^{CT})$ so that $C_i C_i^{CT}(r_i D_i^C) = (r_i D_i^C) C_i C_i^{CT}$ for $i = 1, 2, \cdots, t$. (If $r_t = 0$, this is still true and D_t may be chosen to be the identity matrix.) Therefore $B_1 B_1^{CT} D^{CT} = D^{CT} B_1 B_1^{CT}$ and $UB^TU^TU^CB^CU^{CT}UA^{CT}U^{CT} = UA^{CT}U^{CT}UB^TU^TU^CB_1^CU^{CT}$ so $B^TB^CA^{CT} = A^{CT}B^TB^C$ or $AB^TB^C = B^TB^CA$.

Corollary. *Let A be normal, B quasi-normal; if AB is quasi-normal, then BA^C is quasi-normal, and conversely.*

From the above, $UAU^{CT}UBU^T = DB_1^T$ is quasi-normal, and if $D = D_r D_u$, D_r real and D_u unitary, then since $D_u^C = D_u^{CT}$, $D_u^C(DB_1^T)D_u^C = D_r B_1^T D_u^C = B_1^T D_r D_u^C = B_1^T D^C$ is quasi-normal as are $UBU^TU^CA^CU^T$ and BA^C. Reversing the steps proves the converse.

If A is normal and B is quasi-normal, BA^C is quasi-normal if and only if AB is quasi-normal if and only if $(B^TB^C)A = A(BB^{CT})$ and $(A^TA^C)B^C = B^C(AA^{CT})$. Therefore, if A is normal and B quasi-normal, BA is quasi-normal if and only if $(B^TB^C)A^C = A^C(BB^{CT})$ and $(A^{CT}A)B^C = B^C(A^CA^T)$, i.e., replace A by A^C in the preceding, or $(B^{CT}B)A = A(B^CB^T) = A(B^{CT}B)$ and $(A^{CT}A)B^C = B^C(A^CA^T)$, thus exhibiting the fact that when AB is quasi-normal, BA is not necessarily so.

Theorem 8. *If $A = LW = WL$ is normal and $B = KV = VK^T$ is quasi-normal (where L and K are hermitian and W and V are unitary) then AB is quasi-normal if and only if $LK = KL$, $LV = VL^T$ and $WK = KW$.*

If the three relations hold, then $AB = LWKV = LKWV$ on one hand, and $AB = WLKV = WKLV = WKVL^T = WVK^TL^T = WV(LK)^T$ is quasi-normal since LK is hermitian and WV is unitary.

Conversely, let

$$A = U^{CT}DU = (U^{CT}D_r U)(U^{CT}D_u U) = LW \quad \text{and}$$
$$B = U^{CT}B_1^T U^C = (U^{CT}K_1 U)(U^{CT}V_1 U^C) = KV = VK^T$$

where K_1 and V_1 are hermitian and unitary and direct sums conformable to B_1^T and D. A direct check shows that $LK = KL$ and $LV = VL^T$; also $WK = U^{CT}D_u K_1 U = U^{CT}K_1 D_u U = KW$ since $D_u B_1 B_1^{CT} = B_1 B_1^{CT} D_u$ implies $D_u K_1 = K_1 D_u$.

A sufficient condition for the simultaneous reduction of A and B is given by the following:

THEOREM 9. *If A is normal, B quasi-normal, and $AB = BA^T$, then $WAW^{CT} = D$ and $WB^TW = F$, the normal form of Theorem* 1, *where W is a unitary matrix; also AB is quasi-normal.*

Let $UAU^{CT} = D$, diagonal, and $UBU^T = B_2$ which is quasi-normal. Then $AB = BA^T$ implies $DB_2 = UAU^{CT}UBU^T = UBU^TU^CA^TU^T = B_2D^T = B_2D$. Let $D = c_1I_1 \dotplus c_2I_2 \dotplus \cdots \dotplus c_kI_k$, where the c_i are complex and $c_i \neq c_l$ for $i \neq j$, and $B_2 = C_1 \dotplus C_2 \dotplus \cdots \dotplus C_k$. Let V_i be unitary such that $V_iC_iV_i^T = F_i =$ the real normal form of Theorem 1, and let $V = V_1 \dotplus V_2 \dotplus \cdots \dotplus V_k$. Then $VUAU^{CT}V^{CT} = D$, $VUBU^TV^T = F =$ a direct sum of the F_i.

Also, $AB = BA^T$ implies $B^TA^T = AB^T$ and so $ABB^{CT}A^{CT} = AB^TB^CA^{CT} = B^TA^TA^CB^C = (AB)^T(AB)^C$. (The fact that A is normal is not used in the latter.)

It is also possible for the product of two normal matrices A and B to be quasi-normal. If $Q = HU = UH^T$ is quasi-normal and if $A = U$ and $B = H$ this is so or if $KV = VK^T$ is quasi-normal and if $A = UK = KU$ is normal with K hermitian and V and U unitary, for $B = V$, $AB = (UK)V = K(UV) = (UV)K^T$ is quasi-normal. But if in the first example, U^2H is not normal, then HU is not quasi-normal (see section 2, c)) so that BA is not necessarily quasi-normal though AB is. When A alone is normal an analog of Theorem 2 can be obtained which states the following: If A is normal, then AB and AB^T are quasi-normal if and only if $ABB^{CT} = B^TB^CA$, $BB^{CT}A = AB^TB^C$, and $B^CAA^{CT} = A^TA^CB^C$. (The proof is not included here because of its similarity to that above.) When B is quasi-normal, two of these conditions merge into one in Theorem 7.

It is possible for the product of two quasi-normal matrices to be quasi-normal, but no such simple analogous necessary and sufficient conditions as exhibited above are available. This may be seen as follows. Two non-real complex commutative matrices $S = S^T$ and $T = T^T$ can form a quasi-normal (and non-real symmetric) matrix ST (such that TS is also quasi-normal) which need not be normal. Then two symmetric matrices:

$$X = \begin{bmatrix} i & i+i \\ 1+i & -i \end{bmatrix}, \qquad Y = \begin{bmatrix} 1+2i & 3-4i \\ 3-4i & -(1+2i) \end{bmatrix}$$

are such that $XY = Z$ is real, normal and quasi-normal (and not symmetric). Finally, if U and V are two complex unitary matrices of the same order, they can be chosen so UV is non-real complex, normal and quasi-normal. If $A = S \dotplus X \dotplus U$ and $B = T \dotplus Y \dotplus V$, $AB = ST \dotplus XY \dotplus UV$ where A and B are quasi-normal as in AB (but not symmetric). A simple inspection of these matrices shows that relations on the order of $(B^TB^C)A = A(BB^{CT}) = (BB^{CT})A$ and $(A^TA^C)B^C = (AA^{CT})B^C = B^C(AA^{CT})$ do not necessarily hold; these are sufficient, however, to guarantee that AB is quasi-normal (as direct verification from the definition will show).

References

[1] R. Bellman, *Introduction to Matrix Analysis* (McGraw-Hill, New York, 1960).
[2] K. Morita, 'Über normale antilineare Transformationen', *J. Acad. Proc. Tokyo*, 20 (1944), 715–720.
[3] I. Schur, 'Ein Satz über quadratische Formen mit komplexen Koeffizienten', *Amer. J. Math.* 67 (1945), 472–480.
[4] J. Stander and N. Wiegmann, 'Canonical Forms for Certain Matrices under Unitary Congruence', *Can. J. Math.* 12 (1960), 427–445.
[5] U. Wegner and J. Wellstein, 'Bermerkung zur Transformationen von komplexen symmetrischen Matrizen', *Monatshefte für Math. and Physik.* 40 (1933), 319–332.
[6] N. Wiegmann, 'Normal Products of Matrices', *Duke Math. Journal* 15 (1948), 633–638.

California State College
Dominguez Hills

https://doi.org/10.1017/S1446788700006716 Published online by Cambridge University Press