JACOBIANS OF SINGULAR SPECTRAL CURVES AND COMPLETELY INTEGRABLE SYSTEMS

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Abstract   Consider an isospectral manifold formed by matrices $M \in \mathfrak{gl}_r(\mathbb{C})[x]$ with a fixed leading term. The description of such a manifold is well known in the case of a diagonal leading term with different eigenvalues. On the other hand, there are many important systems where this term has multiple eigenvalues. One approach is to impose conditions in the sub-leading term. The result is that the isospectral set is a smooth manifold, bi-holomorphic to a Zariski open subset of the generalized Jacobian of a singular curve.

Keywords: integrability; Jacobian variety

AMS 2000 *Mathematics subject classification:* Primary 37J35; 70H06

1. Introduction

Consider a Hamiltonian system written in the form of a Lax pair, i.e. $(\mathrm{d}/\mathrm{d}t)A(x) = [A(x), B(x)]$, where $A(x)$ and $B(x)$ are complex matrix polynomials in a variable x, i.e.

$$A(x) = A_d x^d + A_{d-1} x^{d-1} + \cdots + A_0, \quad A_i \in \mathfrak{gl}_r(\mathbb{C}),$$
$$B(x) = B_l x^l + B_{l-1} x^{l-1} + \cdots + B_0, \quad B_i \in \mathfrak{gl}_r(\mathbb{C}).$$

The coefficients A_i, B_j are matrices which depend on the dynamical variables. The coefficients of the spectral polynomial $P(x, y) = \det(A(x) - y \operatorname{Id}_r)$ are the first integrals of the system. It is easy to see that the matrix A_d is invariant of time, consequently it is natural to fix it. Denote by M_P^J the isospectral variety

$$M_P^J = \{A(x) : \det(A(x) - y \operatorname{Id}_r) = P(x, y) = 0 \text{ and } A_d = J\}.$$

The affine curve $X_{\mathrm{a}} = \{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\}$ is embedded in $\mathcal{O}_{\overline{\mathbb{P}}}(d)$. Let $\overline{X}_{\mathrm{a}}$ be the compactification of X_{a}, we will call it the spectral curve associated to the matrix polynomial $A(x)$. If the affine curve X_{a} is smooth, then it is easy to check that the variety M_P^J is smooth (L. Gavrilov, personal communication; see also [8]).
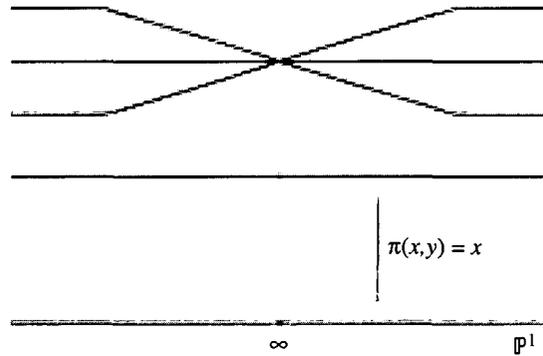
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Figure 1. Spectral curve with an ordinary 3-tuple point at infinity.

Several articles on the same topic have appeared. Adler and van Moerbeke [2] fixed the matrix $A_d = J$ and searched to calculate the variety $M_P^J/\mathbb{P}Gl_r(\mathbb{C}, J)$ ($\mathbb{P}Gl_r(\mathbb{C}, J)$ corresponds to a subgroup of the projective group $\mathbb{P}Gl_r(\mathbb{C})$ formed by matrices which commute with J; moreover, this subgroup acts on M_P^J by conjugation) when the matrix J is diagonal with simple eigenvalues. So the spectral curve becomes smooth (even at infinity). However, it is interesting to consider singularities at infinity, because a lot of integrable systems possess an affine smooth spectral curve and at infinity this curve is singular (this happens when the matrix J has eigenspace, the dimension of which is greater than one), for instance the Kowalevskaya top [5], the anharmonic oscillator, the Garnier system and integrable quartic [14, p. 172].

If the affine curve X_a is smooth, van Moerbeke and Mumford [18] describe the quotient variety $M_P^J/\mathbb{P}Gl_r(\mathbb{C}, J)$ as a Zariski open subset in the Jacobian $J(X)$. If the affine curve X_a is not smooth, they express the variety $M_P^J/\mathbb{P}Gl_r(\mathbb{C}, J)$ as a Zariski open subset in a generalized Jacobian. The difference between the present paper and [18] is that we describe M_P^J (not the quotient variety $M_P^J/\mathbb{P}Gl_r(\mathbb{C}, J)$), and, furthermore, the affine curve X_a is smooth but its compactification is singular at infinity.

The article of Adams, Harnad and Hurtubise [1] gives an expression of the same quotient variety $M_P^J/\mathbb{P}Gl_r(\mathbb{C}, J)$ as an affine part of the usual Jacobian $J(X)$. This result is also proved by Beauville [3].

In [6, 9, 13], the authors describe abstract Hamiltonian systems. On the other hand, Hitchin [9] noted that it is important to explicitly find the Hamiltonian differential equations describing these systems. Here, we consider concrete realization of integrable systems. The difference between our systems and the integrable systems of Hitchin type described in the literature [6, 13] is the line bundles, which we consider, are neither stable nor semi-stable; moreover, Hitchin's systems linearize on abelian varieties but our systems linearize on generalized Jacobians.

To conclude, we mention a recent article of Gavrilov [8], who uses smooth spectral curves and shows that the variety M_P^J is an open subset of a commutative algebraic group (the generalized Jacobian of $\overline{X}_a$).

The purpose of the present article is to describe the variety M_P^J in the case where the matrix J is diagonalizable and has multiple eigenvalues (then the matrix J is not regular).*

Suppose that the affine curve X_a is smooth and at $Q_i \in \overline{X}_a$ (where $x(Q_i) = \infty$), $\overline{X}_a$ is locally a normal crossing of many branches. Let $\mathcal{F} = \{(\mathbb{C}^r, J), (E, K)\}$ be the data at infinity of $\overline{X}_a$ (J is an endomorphism $J \in \mathrm{End}(\mathbb{C}^r)$, E a vector space and $K \in \mathrm{End}(E)$, see Definition 3.2. We assume that J is diagonalizable and K is diagonalizable with distinct eigenvalues. Let X be the smooth compactification of $\overline{X}_a$. Let $P_1, P_2, \ldots, P_r$ be the 'infinite' points of the curve X and $X' \sim (X, \sum_i P_i)$ the singular curve obtained from X identifying the points $P_1, P_2, \ldots, P_r$ with a single point noted ∞. Define

$$M_P^{\mathcal{F}} = \{A(x) \in M_P^J : \mathrm{pr}_E . A_{d-1}|_E = K\},$$

where pr_E is the projection on E and $|_E$ is the restriction to E. Denote further by Θ' the canonical Riemann theta divisor formed by special line bundles on X' of degree p_a (the arithmetic genus of X'), that is to say $\dim H^1(X', L') \neq 0$.

The main result of this article is the following theorem.

Theorem 1.1. *The variety $M_P^{\mathcal{F}}$ is smooth and bi-holomorphic to a Zariski open subset of generalized Jacobian of X', $J(X') - \Theta'$.*

2. The generalized Jacobian

Now, we introduce the notation of this article [17]. The construction and properties of generalized Jacobians are due to Rosenlicht [15,16] (even if the generalized Jacobian has already been used by Jacobi [10]) and Lang [11,12]; they rely on the theory of abelian varieties, developed by Weil [20]. We consider X to be a smooth compact irreducible algebraic curve. Let P_i for $1 \leqslant i \leqslant s$ be some distinct points on X. We put $\mathfrak{m} = \sum_{i=1}^s n_i P_i$ and $n_i \in \mathbb{N}^*$, $\forall i = 1, 2, \ldots, s$. We call $S(D)$ the support of some divisor D on X, i.e. $S(\mathfrak{m}) = \{P_1, P_2, \ldots, P_s\}$, denote $X_{\mathrm{reg}} = X - S(\mathfrak{m})$ and $r = \sum_{i=1}^s n_i$.

Definition 2.1. To the pair $(X, \mathfrak{m})$ we associate a singular curve $X' = X_{\mathrm{reg}} \bigcup \infty$. The curve X' is, as a topological space, just X with its points $P_1, \ldots, P_s$ identified with a single point ∞. The structure sheaf $\mathcal{O}'$ of X' is defined in the following way. Let $\mathcal{O}_{X'}$ be the direct image of the structure sheaf $\mathcal{O}_X$ under canonical projection $X \to X'$. Then

$$\mathcal{O}_P' = \begin{cases} \mathcal{O}_P, & \text{if } P \in X_{\mathrm{reg}}, \\ \mathbb{C} + i_\infty, & \text{if } P = \infty, \end{cases}$$

where i_∞ is the ideal of $\mathcal{O}_\infty$ formed by the functions f having a zero at P_i of order at least n_i.

We define the sheaf $\mathcal{L}'(D)$, where D is a divisor on X such that $S(D) \bigcap S(\mathfrak{m}) = \emptyset$, by

$$\mathcal{L}'(D)_P = \begin{cases} \mathcal{L}(D)_P, & \text{if } P \in X_{\mathrm{reg}}, \\ \mathcal{O}_\infty', & \text{if } P = \infty. \end{cases}$$

* A matrix is regular if all eigenspaces are of dimension one.

The sheaf $\mathcal{L}'(D)$ is a locally free $\mathcal{O}'$-module of rank one and there is a one-to-one correspondence between isomorphism classes of invertible sheaves $\mathcal{L}'$ on X' and isomorphism classes of line bundles L' over X'.

Let

$$L'(D) = H^0(X', \mathcal{L}'(D)), \qquad\qquad I'(D) = H^1(X', \mathcal{L}'(D)),$$
$$l'(D) = \dim_{\mathbb{C}} L'(D), \qquad\qquad i'(D) = \dim_{\mathbb{C}} I'(D).$$

As the sheaf $\mathcal{O}_X/\mathcal{O}'_X$ is coherent, we put $\delta_P = \dim_{\mathbb{C}}(\mathcal{O}_P/\mathcal{O}'_P)$ with $P \in X'$, the arithmetic genus p_{a} (dimension of $H^1(X', \mathcal{O}')$) of the singular curve X' is obtained from the geometric genus p_{g} of X by the relation

$$p_{\mathrm{a}} = p_{\mathrm{g}} + \delta_\infty.$$

In fact,

$$\delta_\infty = \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_\infty}{\mathbb{C} + i_\infty}\right) = \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_\infty}{i_\infty}\right) - 1 = \deg(\mathfrak{m}) - 1,$$

then

$$p_{\mathrm{a}} = p_{\mathrm{g}} + \deg(\mathfrak{m}) - 1$$

or

$$p_{\mathrm{a}} = p_{\mathrm{g}} + r - 1.$$

A divisor D on X with $S(D) \bigcap S(\mathfrak{m}) = \emptyset$ verifies

$$l'(D) - i'(D) = \deg(D) + 1 - p_{\mathrm{a}}.$$

Now we define the equivalence relation $\overset{\mathfrak{m}}{\sim}$.

Definition 2.2. Let D_1 and D_2 be two divisors on X with $S(D_1) \bigcap S(\mathfrak{m}) = \emptyset$ and $S(D_2) \bigcap S(\mathfrak{m}) = \emptyset$. Then $D_1 \overset{\mathfrak{m}}{\sim} D_2$, provided that there exists a global meromorphic function f on X, such that $(f) = D_1 - D_2$ and $v_{P_i}(f - 1) \geqslant n_i$, $i = 1, 2, \ldots, s$ ($v_P(\cdot)$ is the order function at P).

Definition 2.3. We call the subgroup $\mathrm{Pic}^0(X')$ of $\mathrm{Pic}(X') := \mathrm{Div}(X')/\overset{\mathfrak{m}}{\sim}$ formed by the divisors D on X with $S(D) \bigcap S(\mathfrak{m}) = \emptyset$ and $\deg(D) = 0$ the generalized Jacobian of X', denoted $J(X')$.

It is known that $J(X')$ is an extension of $J(X)$, the Jacobian of X, by the algebraic group $(\mathbb{C}^*)^{s-1} \times \mathbb{C}^{r-s}$:

$$0 \to (\mathbb{C}^*)^{s-1} \times \mathbb{C}^{r-s} \to J(X') \overset{\phi}{\to} J(X) \to 0.$$

Remark 2.4. There is a one-to-one correspondence between isomorphism classes of invertible sheaves $\mathcal{L}$ on X' and isomorphism classes of line bundles L over X'.

3. The spectral curve

Let x be an affine coordinate on $\mathbb{P}^1$. We now consider the vector space M of all complex matrix polynomials $A(x)$ in a variable x, of fixed degree d ($d \geqslant 1$) and dimension r ($r \geqslant 3$)

$$A(x) = A_d x^d + A_{d-1} x^{d-1} + \cdots + A_0, \quad A_i \in \mathfrak{gl}_r(\mathbb{C}).$$

Let

$$M_P = \{A(x) \in M : \det(A(x) - y\,\mathrm{Id}_r) = P(x,y) = 0\}$$

and M^J be the affine subspace formed by matrices with a fixed leading coefficient $A_d = J$. We put

$$M_P^J = M_P \cap M^J = \{A(x) \in M^J : \det(A(x) - y\,\mathrm{Id}_r) = P(x,y) = 0\},$$

where $P(x,y)$ is an irreducible fixed polynomial,

$$P(x,y) = \det(A(x) - y\,\mathrm{Id}_r) = (-1)^r y^r + s_1(x) y^{r-1} + \cdots + s_r(x),$$

with $\deg(s_i(x)) \leqslant i.d$. Let $\mathbb{P}^2(d)$ be the d-weighted projective space defined by $(\mathbb{C}^3 - \{(0,0,0)\})/ \sim$, where $(x,y,z) \sim (\lambda x, \lambda^d y, \lambda z)$ with $\lambda \in \mathbb{C}^*$.

Definition 3.1. Let $X_a = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}$ be the curve associated to the irreducible polynomial $P(x,y)$. The spectral curve is the closure $\overline{X}_a$ of X_a in $\mathbb{P}^2(d)$.

Now fix a spectral polynomial $P(x,y) = \det(A(x) - y\,\mathrm{Id})$, and assume that A_d is diagonalizable.

Let

$$E_d = \mathbb{C}^r, \qquad u_d = A_d.$$

Let $u_d \in \mathrm{End}(E_d)$ be an endomorphism of E_d and $Sp(u_d) = \{\lambda_1, \ldots, \lambda_k\}$ of respectively order $s_1, \ldots, s_k$, we introduce the set

$$I_d = \{i : \dim \mathrm{Ker}(u_d - \lambda_i) > 1\}.$$

If $I_d \neq \emptyset$, then we can define

$$E_{d-1} := \bigoplus_{i \in I_d} \mathrm{Ker}(u_d - \lambda_i)^{s_i},$$

and $u_{d-1} \in \mathrm{End}(E_{d-1})$ by

$$u_{d-1} := \bigoplus_{i \in I_d} u_{d-1}^i : E_{d-1} \to E_{d-1},$$

where $u_{d-1}^i, i \in I_d$ is the endomorphism

$$
\begin{array}{rcccc}
u_{d-1}^i : \mathrm{Ker}(u_d - \lambda_i)^{s_i} & \to & E_d & \to & \mathrm{Ker}(u_d - \lambda_i)^{s_i} \\
v & \mapsto & A_{d-1}(v) & \mapsto & pr(A_{d-1}(v)).
\end{array}
$$

Definition 3.2. The data $\mathcal{F}$ of the spectral curve at infinity is the set

$$\mathcal{F} := \{(E_d, u_d), (E_{d-1}, u_{d-1})\}.$$

Definition 3.3. The projective sub-group of $\mathbb{P}Gl_r(\mathbb{C})$ such that the data $\mathcal{F}$ is invariant under the conjugation action of this sub-group is denoted $\mathbb{P}Gl_r(\mathbb{C}; \mathcal{F})$

$$\mathbb{P}Gl_r(\mathbb{C}; \mathcal{F}) = \{R \in \mathbb{P}Gl_r(\mathbb{C}) : \forall k \in \{d, d-1\} \ R_{E_k} u_k|_{E_k} = u_k\}$$

and

$$\begin{array}{cccccc} R_{E_k} : & E_k & \to & E_d & \to & E_k \\ & v & \mapsto & RvR^{-1} & \mapsto & pr(RvR^{-1}). \end{array}$$

We assume that the curve $\overline{X}_a - \sum_{i \in I_d} Q_i$ is smooth (where $x(Q_i) = \infty$ and $y(Q_i) = \lambda_i$) and, at Q_i, $\overline{X}_a$ is a normal crossing of s_i branches. Let $\mathcal{F} := \{(E_d, u_d), (E_{d-1}, u_{d-1})\}$ be the data of the spectral curve at infinity where u_d is diagonalizable and u_{d-1} is diagonalizable with distinct eigenvalues. The geometric genus of $\overline{X}_a$ is

$$p_g = \tfrac{1}{2}(r-1)(rd-2) - \sum_{i \in I_d} \delta_{Q_i} = \tfrac{1}{2}(r-1)(rd-2) - \sum_{i \in I_d} \tfrac{1}{2} s_i(s_i - 1).$$

Let X be the smooth compactification of $\overline{X}_a$. When we desingularize $\overline{X}_a$, each point Q_i gives s_i distinct points on X.

The projection

$$\pi : \mathbb{P}^2(d) - \{[0,1,0]\} \to \mathbb{P}^1$$
$$[x, y, z] \mapsto [x, z]$$

identifies the surface $\mathbb{P}^2(d) - \{[0,1,0]\}$ with the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$, and, as $[0,1,0] \notin X$, then X is naturally embedded in $\mathcal{O}_{\mathbb{P}^1}(d)$ too. The induced projection

$$\pi : X \to \mathbb{P}^1$$

is a ramified covering of degree r, and over the affine plane $\mathbb{C}$ it is simply the first projection

$$\pi : X \to \mathbb{C}$$
$$(x, y) \mapsto x.$$

Definition 3.4. If $\pi^{-1}([1,0]) = \sum_{i=1}^r P_i$, then we define the modulus $\mathfrak{m}$ to be the effective divisor $\mathfrak{m} = \sum_{i=1}^r P_i$. We denote by $X' = X_{\mathfrak{m}}$ the singular curve obtained from X when we identify its points $P_1, P_2, \ldots, P_r$ with a single point ∞. The arithmetic genus of X' is given by $p_a = p_g + r - 1$.

Remark 3.5. Suppose the multiplicity of λ_{i_0} is greater than 1 and $(u_{d-1})^{i_0}$ is homologous to

$$\left(\begin{array}{ccc|c} \alpha_1 & & & \\ & \ddots & & * \\ & & \alpha_{s_{i_0}} & \\ \hline & * & & * \end{array} \right),$$

and the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{s_{i_0}}$ are fixed and distinct. The coefficient α_j is the coefficient of z in the Puiseux expansion of y, i.e. $y = \lambda_{i_0} + \alpha_j z + \cdots$.

Definition 3.6. Let $P(x, y) \in \mathbb{C}[x, y]$ and $\mathcal{F} = \{(\mathbb{C}^r, J), (E, K)\}$ be the data at infinity of the spectral curve. We introduce the variety $M_P^{\mathcal{F}}$ defined by

$$M_P^{\mathcal{F}} = \{A(x) \in M_P^J : A(x) = Jx^d + A_{d-1}x^{d-1} + \cdots + A_0, \ A_{d-1}|_E = K\}.$$

For $A(x) \in M_P^{\mathcal{F}}$ and $R \in \mathbb{P}Gl_r(\mathbb{C}, \mathcal{F})$, then $RA(x)R^{-1} \in M_P^{\mathcal{F}}$, it is easy to see that we can choose a suitable base such that J and K are simultaneously diagonalized.

4. The eigenvector bundle

For each non-singular point $p \in X_a$ which is not a branching point of X, there is a one-dimensional eigenspace $l(x) \subset \mathbb{C}^r$ of $A(x(p))$ with eigenvalue $y(p)$. This gives a holomorphic line bundle on X_a defined everywhere except for singular points and branching points. Let X be the non-singular compact model of X_a, the eigenvector bundle extends to a holomorphic line bundle l on the whole smooth curve X. Indeed, the mapping $p \to l(p)$ determines a meromorphic mapping

$$X \to \mathbb{C}\mathbb{P}^{r-1}$$
$$x \mapsto [f_1(x), f_2(x), \ldots, f_r(x)].$$

Since any such mapping is actually holomorphic, the eigenvector bundle extends to X. We denote its dual by L. If D is the minimal effective divisor such that $\forall i$, $(f_i) \geqslant -D$, then $L = L_D$. If $S(D) \bigcap S(\mathfrak{m}) \neq \emptyset$, L_D does not define a line bundle on X', that is why we must normalize the bundle L_D to define the eigenvector bundle on X'.

Proposition 4.1. *Let* $f(x, y) = (f_1, f_2, \ldots, f_r)$ *be an eigenvector of* $A(x) \in M_P^{\mathcal{F}}$ *normalized by the condition*

$$f_1 + f_2 + \cdots + f_r \equiv 1.$$

Let D *be the minimal divisor, such that* $(f_i) \geqslant -D$, $i = 1, 2, \ldots, r$. *Then we get* $S(D) \bigcap S(\mathfrak{m}) = \emptyset$ *and we define (the dual to the) eigenvector bundle on the singularized spectral curve* X'.

Proof. If $j \in I_d$, we shall write the polynomial $P(x, y, z)$ near the points $Q_j = (x = \infty, y = \lambda_j)$. This point give s_j distinct points $P_{n_1}, \ldots, P_{n_{s_j}}$ when we desingularize $\overline{X}_a$. Without loss of generality, we shall assume that $\lambda_j = 0$. The polynomial $P(x, y, z)$ is homogeneous in $\mathbb{P}^2(d)$ of degree rd:

$$P(x, y, z) = \det(z^d A(x/z) - y \operatorname{Id}_r).$$

Write $P(1, y, z)$ as the sum of homogeneous polynomials in ascending degrees

$$P(1, y, z) = P_{s_j}(y, z) + P_{s_j+1}(y, z) + \cdots + P_{rd}(y, z).$$

Precisely

$$P_{s_j}(y, z) = \prod_{i \neq j} \lambda_i^{s_i} \det(zu_{d-1}^j - y\,\mathrm{Id}_{s_j}),$$

$$P_{s_j}(y, z) = \prod_{i \neq j} \lambda_i^{s_i} \prod_{i=1}^{s_j}(z\alpha_i - y).$$

Let ϕ_{n_i} be the eigenvector of $A(x)$ near the points P_{n_i} for $i = 1, 2, \ldots, s_j$. We have

$$y = \alpha_{n_i} z + \cdots.$$

We put $\phi_{n_i} = e_{n_i} + v_{n_i} z + \cdots$, and get

$$(A_d + A_{d-1}z + \cdots)(e_{n_i} + v_{n_i}z + \cdots) = (\alpha_{n_i}z + \cdots)(e_{n_i} + v_{n_i}z + \cdots),$$

or, componentwise,

$$A_d e_{n_i} = 0,$$
$$A_d v_{n_i} + A_{d-1}e_{n_i} = \alpha_{n_i}e_{n_i}.$$

We get that e_{n_i} corresponds to the eigenvector of $(\delta_k^{n_i})_{1 \leqslant k \leqslant r}$.

Notice the vectors $f(P_i) = (f_1(P_i), \ldots, f_r(P_i))$ are fixed and depend on the spectral curve $\overline{X}_a$ at infinity. Let $c \in \mathbb{C}^r$ and consider the hyperplane $H_c = \{v \in \mathbb{C}^r : \langle c, v \rangle = c_1 v_1 + \cdots + c_r v_r = 0\}$. To normalize the eigenvector f, we choose $c \in \mathbb{C}^r$ such that $f(P_i) \notin H_c$ (in our case, we can choose $c = (1, \ldots, 1)$), and the normalization of f is given by $\langle c, f \rangle \equiv 1$. f has a pole at P_i if and only if the line determined by $f(P_i)$ is contained in the plane $c_1 f_1 + c_2 f_2 + \cdots + c_r f_r = 0$, so with the above normalization P_i is not a pole for f.   □

If D is the pole divisor of the normalized eigenvector f, then denote by $L = L_D$ the line bundle and by $\mathcal{L} = \mathcal{L}_D$ the corresponding sheaf of sections.

5. The dimension of $H^0(X, \mathcal{L}(D))$

To calculate $\dim H^0(X, \mathcal{L}(D))$ we follow [18]. Let $(x)_\infty = \sum_{i=1}^r P_i$ and introduce $X_0 = X - \{P_1, P_2, \ldots, P_r\}$ and let R_0 be the algebra of regular functions in X_0. Clearly $R_0 = \mathbb{C}[x, y]$. Put $R = \mathbb{C}[x]$. Let V denote the subspace of $H^0(X_0, \mathcal{L}(D))$ generated by linear coordinates in $\mathbb{C}^r$. Now we prove that the natural mapping $r : V \otimes R \to H^0(X_0, \mathcal{L}(D))$ is surjective. $r(V \otimes R)$ is an R_0-module if $f = (f_1, \ldots, f_r)$ is the standard basis in V, then $yf = A(x)f$ so that $yf_i \in r(V \otimes R)$. Suppose that $r(V \otimes R)$ is a proper R_0-submodule in $H^0(X_0, \mathcal{L}(D))$. Then there is a maximal ideal I_{m} in R_0 such that

$$r(V \otimes R) \subset I_{\mathrm{m}} H^0(X_0, \mathcal{L}(D)).$$

A maximal ideal I_{m} in R_0 is naturally associated with a point $p \in X_0$:

$$I_{\mathrm{m}} = \{g \in R_0, g(p) = 0\}.$$

All functions in $r(V \otimes R)$ would vanish at p, while $1 = \sum_{i=1}^{r} f_i$ contradicts this fact. Hence, any function of $f \in L(D)$ is an element in $r(V \otimes R)$:

$$f = \sum_{i=1}^{r} b_i f_i + x \left(\sum_{i=1}^{r} b_i^1 f_i \right) + \cdots + x^{\alpha} \left(\sum_{i=1}^{r} b_i^{\alpha} f_i \right).$$

Moreover,

$$f x^{-\alpha}(P_j) = \sum_{i=1}^{r} b_i f_i x^{-\alpha}(P_j) + x^{1-\alpha}(P_j) \left(\sum_{i=1}^{r} b_i^1 f_i(P_j) \right) + \cdots + \left(\sum_{i=1}^{r} b_i^{\alpha} f_i(P_j) \right).$$

We get

$$\sum_{i=1}^{r} b_i^{\alpha} f_i(P_j) = 0, \quad 1 \leqslant j \leqslant r.$$

The r eigenvectors $(f_i(P_j))_{1 \leqslant i \leqslant r}$ are independent because the matrix J is diagonalizable, and $\det(f_i(P_j))_{1 \leqslant i,j \leqslant r} \neq 0$, implying $b_i^{\alpha} = 0$ for $1 \leqslant i \leqslant r$. In the same way, $b_i^j = 0$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant \alpha$, proving that

$$f = \sum_{i=1}^{r} b_i f_i.$$

Therefore $\dim H^0(X, \mathcal{L}(D)) = r$.

6. Calculation of deg(D)

We consider $E = D + n \sum_{i=1}^{r} P_i$ with n large enough. Clearly $\dim L(E) \geqslant r(n+1)$, since every function $x^i f_j$ for $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant r$ belongs to that space. But every function $f \in L(E)$ belongs to $r(V \otimes R)$. So that

$$f = \sum_{i=1}^{r} b_i f_i + x \left(\sum_{i=1}^{r} b_i^1 f_i \right) + \cdots + x^{\alpha} \left(\sum_{i=1}^{r} b_i^{\alpha} f_i \right).$$

Using the same process, it is easy to show that $\alpha \leqslant n$. Therefore, the dimension of $L(E)$ is $r(n + 1)$. Using the Riemann–Roch Theorem on X we get

$$rn + \deg(D) - p_{\mathrm{g}} + 1 = \dim L(E) = r(n + 1).$$

So that

$$\deg(D) = p_{\mathrm{g}} + r - 1.$$

7. Statement of the main result

Firstly, we state the next proposition.

Proposition 7.1. *We define Θ as the canonical Riemann theta divisor of $J^{p_a}(X)$ formed by special line bundles L_D of degree p_a, i.e. $\dim H^1(X, \mathcal{L}(D)) \neq 0$. If $L_D \in \mathrm{Pic}^{p_a}(X) - \Theta$, then the sheaf $\pi_* \mathcal{L}(D)$ (π is the projection $X \to \mathbb{P}^1$ induced by $[x, y, z] \mapsto [x, z]$) is a trivial $\mathcal{O}_{\mathbb{P}^1}$-module of rank r:*

$$\pi_* \mathcal{L}(D) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}.$$

Then we have a bi-holomorphic map

$$\{a \text{ matrix } A(x) \in M_P \text{ up to conjugation by a matrix in } \mathbb{P}GL_r(\mathbb{C})\}$$

$$\downarrow$$

$$\{an \text{ isomorphism class of a line bundle } L \in \mathrm{Pic}^{p_a}(X) - \Theta\}.$$

Proof. The demonstration of [3] works with little modifications. Let L be a line bundle on X of degree $\deg(L) = p_a$ and $\dim H^0(X, \mathcal{L}) = r$. By the Riemann–Roch Theorem,[*] $\chi(\mathcal{L}) = \deg(L) - p_g + 1 = r$. Moreover,

$$\dim H^0(X, \mathcal{L}) = \dim H^0(\mathbb{P}^1, \pi_* \mathcal{L}),$$

$$\chi(\mathcal{L}) = \chi(\pi_* \mathcal{L}) = \deg(\pi_* \mathcal{L}) + (1 - p_g(\mathbb{P}^1)) \operatorname{rank}(\pi_* \mathcal{L}) = \deg(\pi_* \mathcal{L}) + r.$$

Then we obtain that $\pi_* \mathcal{L}$ is a locally trivial $\mathcal{O}_{\mathbb{P}^1}$-module of rank r, degree 0 and it has r holomorphic sections, so

$$\pi_* \mathcal{L} = \underbrace{\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}}_{r}.$$

The invertible sheaf $\mathcal{L}$ on X can be equivalently described as a locally trivial $\mathcal{O}_{\mathbb{P}^1}$-module $\pi_* \mathcal{L}$ equipped with an additional structure of a $\pi_* \mathcal{O}$-module, or, equivalently, a homomorphism of algebras $a : \pi_* \mathcal{O} \to \mathrm{End}(\pi_* \mathcal{L})$. To describe the homomorphism a amounts to giving a linear map

$$A : \pi_* \mathcal{L} \to \pi_* \mathcal{L}(d),$$

that is to say, a polynomial $r \times r$ matrix $A(x)$ of degree d. Clearly, $A(x)$ satisfies $P(x, A(x)) = 0$ and as $P(x, y)$ is irreducible over $\mathbb{C}(x)$, then by the Cayley–Hamilton Theorem the spectral polynomial of $A(x)$ is $P(x, y)$. Note that the matrix $A(x)$ is determined only modulo an automorphism of $\pi_* \mathcal{L}$, and hence up to conjugation by a matrix $R \in GL_r(\mathbb{C})$. $\qquad\square$

Denote

$$\mathbb{P}GL_r(\mathbb{C}; \mathcal{F}) = \{R \in \mathbb{P}GL_r(\mathbb{C}) : R \text{ is a diagonal matrix}\}.$$

---

[*] $\chi(\mathcal{L})$ is the Euler–Poincaré characteristic defined for a line bundle L on X by

$$\chi(\mathcal{L}) = \dim H^0(X, \mathcal{L}) - \dim H^1(X, \mathcal{L}).$$

The group $\mathbb{P}GL_r(\mathbb{C}; \mathcal{F})$ acts freely and properly on the smooth manifold $M_P^{\mathcal{F}}$ by conjugation. We consider the following commutative diagram

$$
\begin{array}{ccccc}
M_P^{\mathcal{F}} & \xrightarrow{\;\varphi\;} & M_P^{\mathcal{F}}/\mathbb{P}GL_r(\mathbb{C}; \mathcal{F}) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle l'} & & \Big\downarrow{\scriptstyle l} & & \\
0 \longrightarrow (\mathbb{C}^*)^{r-1} \longrightarrow & J(X') - \Theta' & \xrightarrow{\;\phi\;} & J(X) - \Theta & \longrightarrow 0
\end{array}
$$

where $\Theta' = \phi^{-1}(\Theta)$. We are ready to state the following theorem.

Theorem 7.2. *Denote by $\Theta' = \phi^{-1}(\Theta)$ the theta divisor formed by line bundles $L' \in \mathrm{Pic}^{p_a}(X')$ such that $\dim H^1(X', L') \neq 0$. The variety $M_P^{\mathcal{F}}$ is smooth and bi-holomorphic to $J(X') - \Theta'$.*

Proof. Let $\mathrm{Pic}^{p_a}(X')$ be the shifted Picard group $\mathrm{Pic}^0(X')$ of degree p_a line bundles on X'. It is isomorphic to the Jacobian variety $J(X')$, and $J(X') - \Theta'$ is the subset of line bundles $L' \in \mathrm{Pic}^{p_a}(X')$ with one non-zero holomorphic section. We get a mapping

$$\{\text{a matrix } A(x) \in M_P^{\mathcal{F}}\}$$

$$\Big\downarrow{\scriptstyle l'}$$

$$\{\text{an isomorphism class of a line bundle } L \in \mathrm{Pic}^{p_a}(X') - \Theta'\}.$$

In fact it is an isomorphism, to show that we follow [8] and prove the fibres $\varphi^{-1}(b)$ and $\phi^{-1} \circ l(b)$ have the same dimension and l' is an injective morphism of algebraic groups. The dimension of the fibres is $r - 1$. Now check that l' is injective. Let L' be a line bundle on X' of degree $\deg(L') = p_a$ and $\dim H^0(X', \mathcal{L}') = 1$. By the Riemann–Roch Theorem $\chi(\mathcal{L}') = \deg(L') - p_a + 1 = 1$. Moreover,

$$\dim H^0(X', \mathcal{L}') = \dim H^0(\mathbb{P}^1, \pi_* \mathcal{L}'),$$
$$\chi(\mathcal{L}') = \chi(\pi_* \mathcal{L}') = \deg(\pi_* \mathcal{L}') + (1 - p_g(\mathbb{P}^1)) \operatorname{rank}(\pi_* \mathcal{L}') = \deg(\pi_* \mathcal{L}') + r.$$

Then we obtain that $\pi_* \mathcal{L}'$ is a locally trivial $\mathcal{O}_{\mathbb{P}^1}$-module of rank r, degree $1 - r$ and has one holomorphic section, so

$$\pi_* \mathcal{L}' = \underbrace{\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-1)}_{r}.$$

The invertible sheaf $\mathcal{L}'$ on X' can be equivalently described as a locally trivial $\mathcal{O}_{\mathbb{P}^1}$-module $\pi_* \mathcal{L}'$ equipped with an additional structure of a $\pi_* \mathcal{O}'$-module, therefore a homomorphism of algebras $a : \pi_* \mathcal{O}' \to \operatorname{End}(\pi_* \mathcal{L}')$. To describe the homomorphism a we shall give a linear map

$$A : \pi_* \mathcal{L}' \to \pi_* \mathcal{L}'(d),$$

that is to say, a polynomial $r \times r$ matrix $A(x)$ of degree d. Clearly, $A(x)$ satisfies $P(x, A(x)) = 0$ and as $P(x, y)$ is irreducible over $\mathbb{C}(x)$, then by the Cayley–Hamilton

Theorem the spectral polynomial of $A(x)$ is $P(x, y)$. Note that the matrix $A(x)$ is determined only modulo an automorphism of $\pi_* \mathcal{L}'$. As the vector $1 = \sum f_i \in H^0(\mathbb{P}^1, \pi_* \mathcal{L}')$ in the base $f_1, f_2, \ldots, f_r$, is the vector $(1, 1, \ldots, 1)$, then $A(x)$ is determined up to conjugation by a constant matrix R with eigenvector $(1, 1, \ldots, 1)$. Thus if $A'(x)$ is another matrix in $M_P^{\mathcal{F}}$ which defines the same eigenvector sheaf $\mathcal{L}'$, then $A'(x) = RA(x)R^{-1}$. As R commutes with the leading term J, $R_{E_{d-1}}$ commutes with u_{d-1} and $(1, 1, \ldots, 1)$ is an eigenvector of R, then $R = \mathrm{Id} \in \mathbb{P}GL_r(\mathbb{C})$.

Finally it remains to show that $M_P^{\mathcal{F}}$ is smooth. We choose $a \in \mathbb{P}^1$, $a \neq [0 : 1]$ generic (i.e. such that $A(a)$ has a simple spectrum), the vector fields

$$\frac{\mathrm{d}}{\mathrm{d}t} A(x) = \left[\frac{A^k(a)}{x - a}, A(x) \right], \quad k \in \mathbb{N},$$

are tangent to the fibre $\varphi^{-1}(b)$. It is shown in [**14**] that these vector fields induce translation invariant vector fields on $J(X')$ (although the results are formulated only on $J(X)$). The direction of these vector fields is moreover computed (formula (8.5) on p. 177 of [**14**]). These formulae imply that these vector fields span, for a generic, the tangent space to the generalized Jacobian $J(X')$. $\square$

8. Examples

Example 1: a harmonic oscillator with two degrees of freedom

Consider the simplest case, where $A(x)$ is written in the base $\{e_1, e_2, e_3\}$:

$$A(x) = A_1 x + A_0 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & a_0 \end{array} \right) x + \left(\begin{array}{cc|c} a_1 & 0 & y_1 \\ 0 & a_2 & y_2 \\ \hline x_1 & x_2 & a_3 \end{array} \right).$$

The spectral polynomial $P(x, y) = \det(A(x) - y \,\mathrm{Id}_r)$ is

$$P(x, y) = h_1 + h_2 y + h_3 y^2 - y^3 + x(h_4 + h_5 y + a_0 y^2).$$

When $a_1 = a_2$, the polynomial $P(x, y)$ is not irreducible, so consider the case $a_1 \neq a_2$. Let

$$X_a = \{(x, y) : P(x, y) = 0\} \subset \mathbb{P}^2.$$

Let $\overline{X}_a$ denote the closure of the affine curve X_a in the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. We choose $a_0 \in \mathbb{C}^*$ and $(h_1, h_2, h_3, h_4, h_5) \in \mathbb{C}^5$ such that $\overline{X}_a - Q$ is smooth, and then at Q $(x(Q) = \infty$ and $y(Q) = 0)$, the curve $\overline{X}_a$ is locally a normal crossing of two branches. Denote by X the smooth compactification of $\overline{X}_a$. The double point $Q \in \overline{X}_a$ gives two smooth points $P_1, P_2 \in X$ on $\overline{X}_a$. Therefore, X is topologically a sphere. Let $P_3 = (x = \infty, y = a_0)$, the modulus is $\mathfrak{m} = P_1 + P_2 + P_3$ and call $X' = X_{\mathfrak{m}}$,

$$\begin{aligned} P(x, y, z) = \ & a_0 a_1 a_2 x z^2 - a_0 (a_1 + a_2) xyz + a_0 xy^2 - y^3 \\ & + (a_1 + a_2 + a_3) y^2 z + (x_1 y_1 + x_2 y_2 - a_1 a_2 - a_1 a_3 - a_2 a_3) yz^2 \\ & + (a_1 a_2 a_3 - a_2 x_1 y_1 - a_1 x_2 y_2) z^3. \end{aligned}$$

The data of $\overline{X}_a$ at infinity is

$$\mathcal{F} = \left\{ (\{e_1, e_2, e_3\}, A_1), \left(\{e_1, e_2\}, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right) \right\}.$$

The variety $M_P^{\mathcal{F}}$ is defined by

$$M_P^{\mathcal{F}} = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \in \mathbb{C}^7 : \begin{aligned} a_1 a_2 a_3 - a_2 x_1 y_1 - a_1 x_2 y_2 &= h_1 \\ y_1 x_1 + y_2 x_2 - a_1 a_2 - a_1 a_3 - a_2 a_3 &= h_2 \\ a_1 + a_2 + a_3 &= h_3 \\ a_0 a_1 a_2 &= h_4 \\ -a_0(a_1 + a_2) &= h_5 \end{aligned} \right\}.$$

The Hamiltonian system on $M_P^{\mathcal{F}}$ is $\dot{A}_0 = [A_0, A_1]$, or, in others words,

$$\dot{y}_1 = a_0 y_1,$$
$$\dot{y}_2 = a_0 y_2,$$
$$\dot{x}_1 = -a_0 x_1,$$
$$\dot{x}_2 = -a_0 x_2,$$
$$\dot{a}_1 = 0,$$
$$\dot{a}_2 = 0,$$
$$\dot{a}_3 = 0.$$

The Hamiltonian function is

$$H(A(x)) = \tfrac{1}{2} Res_{x=0} \operatorname{tr}(A^2(x)) \frac{\mathrm{d}x}{x},$$

so

$$H(A(x)) = y_1 x_1 + y_2 x_2.$$

The Poisson bracket $\{\cdot, \cdot\}_1$ is given by

$$\{y_1, x_1\}_1 = -a_0, \quad \{y_2, x_2\}_1 = -a_0, \quad \{y_i, x_j\}_1 = \{x_i, y_j\}_1 = 0, \quad \text{for } i \neq j.$$

The functions a, a_1, a_2 and a_3 are Casimir functions for this structure.

Remark 8.1. We can take

$$H_k(A(x)) = \tfrac{1}{2} Res_{x=0} \operatorname{tr}(A^2(x)) \frac{\mathrm{d}x}{x^k}, \quad k = 2, 3,$$

as the Hamiltonian function for the above system. Following the scheme described by Reyman [14], we get other compatible Poisson structures. For instance,

$$H_2 = a_0 a_3,$$

with the Poisson bracket

$\{\cdot,\cdot\}_2$	a_0	a_1	a_2	a_3	y_1	y_2	x_1	x_2
a_0	0	0	0	0	0	0	0	0
a_1	0	0	0	0	y_1	0	$-x_1$	0
a_2	0	0	0	0	0	y_2	0	$-x_2$
a_3	0	0	0	0	$-y_1$	$-y_2$	x_1	x_2
y_1	0	$-y_1$	0	y_1	0	0	a_1-a_3	0
y_2	0	0	$-y_2$	y_2	0	0	0	a_2-a_3
x_1	0	x_1	0	$-x_1$	a_3-a_1	0	0	0
x_2	0	0	x_2	$-x_2$	0	a_3-a_2	0	0

or again we may consider

$$H_3 = \tfrac{1}{2}a_0^2,$$

with the following Poisson bracket

$\{\cdot,\cdot\}_3$	a_0	a_1	a_2	a_3	y_1	y_2	x_1	x_2
a_0	0	0	0	0	$-y_1$	$-y_2$	x_1	x_2
a_1	0	0	0	0	0	0	0	0
a_2	0	0	0	0	0	0	0	0
a_3	0	0	0	0	0	0	0	0
y_1	y_1	0	0	0	0	0	0	0
y_2	y_2	0	0	0	0	0	0	0
x_1	$-x_1$	0	0	0	0	0	0	0
x_2	$-x_2$	0	0	0	0	0	0	0

The components of the eigenvector of $A(x)$ are given by

$$f_1 = y_1(a_2 - y), \qquad f_2 = (a_1 - y)y_2, \, f_3 = (a_1 - y)(a_2 - y).$$

We normalize these functions by dividing by $g = f_1 + f_2 + f_3 = y^2 - (a_1 + a_2 + y_1 + y_2)y + a_1a_2 + a_1y_2 + y_1a_2$. This polynomial g has two roots denoted y^0 and y^1. At each value of y, there is one value of x, we call x^0 the value of x for y^0 and x^1 for y^1. Let $Q_0 = (x^0, y^0)$ and $Q_1 = (x^1, y^1)$. We get

$$(f_k/g) \geqslant -Q_0 - Q_1, \quad k = 1, 2, 3.$$

We put $D = Q_0 + Q_1$, $\deg(D) = 2$. Using Theorem 7.2, we get that the variety $M_P^{\mathcal{F}}$ is an open given by $J^2(X') - \Theta'.^*$

---

* Let $J^{p_a}(X') = \mathrm{Pic}^{p_a}(X')$ be the variety (isomorphic to the generalized Jacobian $J(X')$) formed by line bundles of degree p_a (the arithmetic genus of the singularized curve X') over X'.

Example 2: integrable perturbation of the harmonic oscillator with five degrees of freedom

Now consider the Hamiltonian function

$$H = (a_1 + a_2)p_1^2 + (a_1 + b_2)p_2^2 + (b_1 + a_2)p_3^2 + (b_1 + b_2)p_4^2 + (a_2 + b_2)p_5^2$$
$$- (a_1 + a_2)q_1^2 - (a_1 + b_2)q_2^2 - (b_1 + a_2)q_3^2 - (b_1 + b_2)q_4^2 - (a_2 + b_2)q_5^2$$

defined on the co-tangent bundle $T^*\mathbb{C}^5$ of $\mathbb{C}^5$, equipped with a canonical symplectic form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3 + dp_4 \wedge dq_4 + dp_5 \wedge dq_5$. Under a simple transformation

$$p_i = \frac{1}{\sqrt{2}}(x_i + y_i), \quad q_i = \frac{1}{\sqrt{2}}(-x_i + y_i), \quad i = 1, \ldots, 4,$$
$$p_5 = \tfrac{1}{2}(x_5 + y_5), \qquad q_5 = \tfrac{1}{2}(-x_5 + y_5),$$

the Hamiltonian function becomes

$$H = (a_1 + a_2)x_1y_1 + (a_1 + b_2)x_2y_2 + (b_1 + a_2)x_3y_3 + (b_1 + b_2)x_4y_4 + (a_2 + b_2)x_5y_5,$$

and the symplectic structure becomes:

$\{\cdot,\cdot\}_4$	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5
x_1	0	0	0	0	0	1	0	0	0	0
x_2	0	0	0	0	0	0	1	0	0	0
x_3	0	0	0	0	0	0	0	1	0	0
x_4	0	0	0	0	0	0	0	0	1	0
x_5	0	0	0	0	0	0	0	0	0	2
y_1	−1	0	0	0	0	0	0	0	0	0
y_2	0	−1	0	0	0	0	0	0	0	0
y_3	0	0	−1	0	0	0	0	0	0	0
y_4	0	0	0	−1	0	0	0	0	0	0
y_5	0	0	0	0	−2	0	0	0	0	0

(8.1)

We perturb H, adding terms of order three

$$H' = (a_1 + a_2)x_1y_1 + (a_1 + b_2)x_2y_2 + (b_1 + a_2)x_3y_3 + (b_1 + b_2)x_4y_4$$
$$+ (a_2 + b_2)x_5y_5 + (x_1x_2 + x_3x_4)y_5 + (y_1y_2 + y_3y_4)x_5.$$

Proposition 8.2. *The system* $(H', \{\cdot,\cdot\}_4, \mathbb{R}^{10})$ *is algebraically completely integrable.*

Note in the base $\{e_1, e_2, e_3, e_4\}$

$$A(x) = A_1 x + A_0 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) x + \left(\begin{array}{cc|cc} a_1 & 0 & y_1 & x_2 \\ 0 & b_1 & y_3 & x_4 \\ \hline x_1 & x_3 & a_2 & x_5 \\ y_2 & y_4 & y_5 & b_2 \end{array} \right).$$

The Hamiltonian system is $\dot{A}_0 = [A_1, A_0^2]$

$$\dot{x}_1 = (a_1 + a_2)x_1 + y_2 x_5, \qquad\qquad \dot{y}_1 = -(a_1 + a_2)y_1 - x_2 y_5,$$
$$\dot{x}_2 = (a_1 + b_2)x_2 + y_1 x_5, \qquad\qquad \dot{y}_2 = -(a_1 + b_2)y_2 - x_1 y_5,$$
$$\dot{x}_3 = (a_2 + b_1)x_3 + y_4 x_5, \qquad\qquad \dot{y}_3 = -(a_2 + b_1)y_3 - x_4 y_5,$$
$$\dot{x}_4 = (b_1 + b_2)x_4 + y_3 x_5, \qquad\qquad \dot{y}_4 = -(b_1 + b_2)y_4 - x_3 y_5,$$
$$\dot{x}_5 = 2(a_2 + b_2)x_5 + 2(x_1 x_2 + x_3 x_4), \qquad \dot{y}_5 = -2(a_2 + b_2)y_5 - 2(y_1 y_2 + y_3 y_4),$$
$$\dot{a}_2 = 0, \qquad\qquad\qquad\qquad\qquad\qquad \dot{b}_2 = 0.$$

The corresponding Hamiltonian function is

$$H_1(A(x)) = \tfrac{1}{3} Res_{x=0} \, \mathrm{tr}(A^3(x)) \frac{\mathrm{d}x}{x}$$
$$= (a_1 + a_2)x_1 y_1 + (a_1 + b_2)x_2 y_2 + (b_1 + a_2)x_3 y_3 + (b_1 + b_2)x_4 y_4$$
$$+ (a_2 + b_2)x_5 y_5 + (x_1 x_2 + x_3 x_4)y_5 + (y_1 y_2 + y_3 y_4)x_5,$$

with the Poisson bracket of (8.1).

Remark 8.3. As in the previous example, this system possesses another Poisson structure compatible with the first one. We find the same system with $H_2 = x_1 y_1 - x_2 y_2 + x_3 y_3 - x_4 y_4 + a_2^2 - b_2^2$ and the Poisson structure $\{\cdot, \cdot\}_5$ (see Table 1).

Fix the polynomial

$$P(x, y) = \det(A(x) - y \, \mathrm{Id}_r)$$
$$= h_1 + h_2 x + h_3 y - a_1 b_1 x^2 + h_4 xy + h_5 y^2 + (a_1 + b_1)x^2 y$$
$$+ (b_2 - a_2)xy^2 - (a_1 + b_1 + a_2 + b_2)y^3 - x^2 y^2 + y^4,$$

where $a_1 \neq b_1$ and $a_2, b_2, h_i \in \mathbb{C}$. Note that

$$X_\mathrm{a} = \{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\} \subset \mathbb{P}^2.$$

Denote by $\overline{X}_\mathrm{a}$ the closure of the affine curve X_a in the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. We choose $a_0 \in \mathbb{C}^*$ and $(h_1, h_2, h_3, h_4, h_5) \in \mathbb{C}^5$ such that $\overline{X}_\mathrm{a} - Q$ is smooth, and, at Q $(x(Q) = \infty$ and $y(Q) = 0)$, the curve $\overline{X}_\mathrm{a}$ is locally a normal crossing of two branches (i.e. $a_1 \neq a_2$). Denote by X the smooth compactification of $\overline{X}_\mathrm{a}$. The double point $Q \in \overline{X}_\mathrm{a}$ gives two smooth points, $P_1, P_2 \in X$, on $\overline{X}_\mathrm{a}$. Therefore, X is topologically a 2-torus. Let $P_3 = (x = \infty, y = 1)$ and $P_4 = (x = \infty, y = -1)$, the modulus is $\mathfrak{m} = P_1 + P_2 + P_3 + P_4$ and call $X' = X_\mathfrak{m}$. The data of $\overline{X}_\mathrm{a}$ at infinity are

$$\mathcal{F} = \left\{ (\{e_1, e_2, e_3, e_4\}, A_1), \left(\{e_1, e_2\}, \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \right) \right\}.$$

Table 1. *Poisson structure* $\{\cdot,\cdot\}_5$

$\{\cdot,\cdot\}_5$	a_1	b_1	a_2	b_2	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5
a_1	0	0	0	0	$-x_1$	x_2	0	0	0	y_1	$-y_2$	0	0	0
b_1		0	0	0	0	0	$-x_3$	x_4	0	0	0	y_3	$-y_4$	0
a_2			0	0	x_1	0	x_3	0	x_5	$-y_1$	0	$-y_3$	0	$-y_5$
b_2				0	0	$-x_2$	0	$-x_4$	$-x_5$	0	y_2	0	y_4	y_5
x_1					0	x_5	0	0	0	$a_2 - a_1$	0	0	0	$-y_2$
x_2						0	0	0	0	0	$a_1 - b_2$	0	0	y_1
x_3							0	x_5	0	0	0	$a_2 - b_1$	0	$-y_4$
x_4								0	0	0	0	0	$b_1 - b_2$	y_3
x_5									0	$-x_2$	x_1	$-x_4$	x_3	$a_2 - b_2$
y_1										0	$-y_5$	0	0	0
y_2											0	0	0	0
y_3												0	$-y_5$	0
y_4													0	0
y_5														0

The isospectral variety $M_P^{\mathcal{F}}$ is the set

$$
\left\{ (x_i, y_i) \in \mathbb{C}^{10} :
\begin{array}{c}
a_1 a_2 b_1 b_2 - b_1 b_2 x_1 y_1 - a_2 b_1 x_2 y_2 - a_1 b_2 x_3 y_3 \\
\quad - a_1 a_2 x_4 y_4 - a_1 b_1 x_5 y_5 + a_1 (x_3 x_4 y_5 + y_3 y_4 x_5) \\
\quad + b_1 (x_1 x_2 y_5 + y_1 y_2 x_5) + (x_1 y_4 - x_3 y_2)(x_4 y_1 - x_2 y_3) = h_1 \\[1.2em]
a_1 b_1 (b_2 - a_2) + a_1 (x_3 y_3 - x_4 y_4) + b_1 (x_1 y_1 - x_2 y_2) = h_2 \\[1.2em]
- a_1 b_2 (b_1 + a_2) - b_1 a_2 (a_1 + b_2) + (b_1 + b_2) x_1 y_1 \\
\quad + (b_1 + a_2) x_2 y_2 + (a_1 + b_2) x_3 y_3 + (a_1 + a_2) x_4 y_4 \\
\quad + (b_1 + a_1) x_5 y_5 - (y_1 y_2 + y_3 y_4) x_5 - (x_1 x_2 + x_3 x_4) y_5 = h_3 \\[1.2em]
(a_1 + b_1)(a_2 - b_2) - x_1 y_1 + x_2 y_2 - x_3 y_3 + x_4 y_4 = h_4 \\[1.2em]
a_1 a_2 + b_1 b_2 + (a_1 + a_2)(b_1 + b_2) - x_1 y_1 \\
\quad - x_2 y_2 - x_3 y_3 - x_4 y_4 - x_5 y_5 = h_5
\end{array}
\right\}.
$$

Using Theorem 7.2, we state the result

$$
M_P^{\mathcal{F}} = J^5(X') - \Theta'.
$$

Remark 8.4. Using the same approach, we will be able to construct other completely integrable systems.

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