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MEAN GROWTH OF HARMONIC FUNCTIONS OF BEURLING TYPE

BY

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ABSTRACT. A harmonic function on the unit disc is of Beurling type ω if its Fourier (or Taylor) coefficients grow no faster than $\exp \omega(|n|)$ as $|n| \to \infty$, where ω is a given increasing, concave function with $\omega(x)/x \downarrow 0$ as $x \to \infty$. These harmonic functions are characterized by the growth rate of their L_1-norms on circles of radius r as $r \to 1$. The classical Schwartz result follows as a corollary by taking $\omega(x) = \log(1+x)$. The Gevrey case is also included in the general result if one uses $\omega(x) = x^\alpha$, $0 < \alpha < 1$.

The purpose of this note is to give a characterization of (complex valued) harmonic (which includes analytic) functions in the unit disc with Beurling distributional bounary values in terms of the growth rate of their L_1-norm on concentric circles of radius r, as r approaches 1.

In order to study different families of harmonic functions in the unit disc, one must first *define* the families of interest. There are many ways to do this, however three are predominantly used:

(i) specify the "boundary values";
(ii) classify the Fourier coefficients;
(iii) restrict the growth rate of certain integral means.

Method (iii) is of course used as the starting point for H^p-theory. On the other hand one can describe the family of *all* harmonic functions in the unit disc using method (ii) simply by requiring such a function $u(r, t) = u_r(t)$ to be representable by a series

(1) $$u(r,t) = \sum_n a_n r^{|n|} e^{int}, \qquad 0 \le r < 1, \quad t \text{ real},$$

where the sequence of *Fourier coefficients* $\{a_n\}$ are restricted by

(2) $$\limsup_{|n| \to \infty} |a_n|^{1/|n|} \le 1.$$

The series in (1) will converge uniformly on compact subsets of the unit disc.

Describing a class of harmonic functions using method (i) usually means identifying the "boundary values" as an element of the dual space of some

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topological vector space of functions defined on the unit circle, this latter space being called the test space for the space of boundary values. One then shows that the functions $u_r(\cdot) = u(r, \cdot)$ converge as $r \to 1$ to the "boundary values" in one (or more) of the topologies that can be put on the dual space.

Once a family of harmonic functions has been *defined* using one of the three methods above, one is then often interested in *characterizing* the family by using one of the remaining two methods. A well known characterization using (i) and (iii) is the following (see [4] or [5]).

A harmonic function $u(r, t)$ is the Poisson–Stieltjes integral of a finite, complex Borel measure μ on the unit circle if and only if the functions u_r are bounded in L_1-norm. Here the space of boundary values, i.e. measures, is the dual of the test space of continuous functions on the unit circle with sup norm, and the functions, or more properly the measures $u_r(t)\,dt$, converge to μ in the weak-$*$ topology on the space of measures.

A simple characterization using methods (i) and (ii) can be found in [7, p. 44] where it is shown that a harmonic function $u(r, t)$ given by (1) has *bounded* Fourier coefficients $\{a_n\}$, if and only if the family of functions $u_r(\cdot) = u(r, \cdot)$ converge weak-$*$ as $r \to 1$, to a *pseudo-measure*, which is an element of the dual of the test space of *absolutely convergent Fourier series*, with the usual l_1-norm.

Although not so well known, in 1968 Johnson [6] also used methods (i) and (ii) to characterize the space of *all* harmonic functions, defined by (1) and (2) (method (ii)), by describing a certain locally convex topology on the test space of real analytic functions on the unit circle. The rather large dual space constitutes the space of boundary values of all harmonic functions in a generalized distributional sense.

Proceeding in this spirit, a harmonic function $u(r, t)$ is said to be of *Schwartz type*, if the functions u_r converge weak-$*$ to a Schwartz distribution as $r \to 1$. Having thus defined this class using method (i), the class can be characterized using methods (ii) *or* (iii).

Firstly, a harmonic function u is of Schwartz type if and only if its Fourier coefficients $\{a_n\}$ are of *slow-growth:*

$$(3) \qquad a_n = O(1+|n|)^\lambda, \qquad |n| \to \infty,$$

for some positive constant λ [8, chapter 11].

Secondly, using the fact that a Schwartz distribution is the finite order distributional derivative of a continuous function, it is easy to show that a function u is of Schwartz type if and only if the L_1-norms of the functions u_r are of *polynomial growth* (or *slow growth*). That is to say,

$$(4) \qquad \|u_r\|_{L_1} = O(1-r)^{-\lambda}, \qquad r \to 1,$$

for some positive constant λ.

It is the purpose of this note to use only methods (ii) and (iii) to characterize the broader class of harmonic functions of *Beurling type.*

Let $\omega(x)$, $0 \le x < \infty$, be any real valued function with the following properties:

(α) ω is increasing, concave down, $\omega(0) = 0$.
(β) $\omega(x)/x \downarrow 0$ as $x \to \infty$.
(γ) $\omega(x) \ge a + b\log(1+x)$,

for some constants a and positive b.

We define a harmonic function $u(r, t)$ given by (1) to be of *Beurling type* ω if its Fourier coefficients a_n are of ω*-slow growth*:

$$a_n = O(e^{\lambda\omega(|n|)}), \qquad |n| \to \infty, \tag{5}$$

for some positive constant λ.

We see that a harmonic function of Schwartz type is of Beurling type ω if one takes $\omega(x) = \log(1+x)$, a function which satisfies (α) and (β). Condition (γ) guarantees that the class of Beurling harmonic functions is a proper generalization of the class of Schwartz harmonic functions. The Beurling condition (β) is customarily replaced by the more restrictive requirement that

$$\int_1^\infty \frac{\omega(x)}{x^2}\, dx < \infty,$$

however the main result below remains valid with the slightly weaker hypothesis given by (β) above.

By using (5) as a *definition* for functions of Beurling type, we bypass the need to define Beurling distributions as continuous linear functionals on certain Beurling test spaces. A complete description of these test spaces and distributions as well as a characterization in terms of condition (5), can be found in the authors' paper [3] (see also [1] and [2]).

THEOREM. *Suppose* $u(r, t) = u_r(t)$ *is harmonic in the unit disc, and* ω *satisfies* (α), (β) *and* (γ). *Then* u *is of Beurling type* ω *if and only if there exist positive constants* λ *and* Λ *such that*

$$\|u_r\|_{L_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(r, t)|\, dt = O(e^{\Lambda\omega(X)}), \tag{6}$$

as $r \to 1$, *where* $X = X(r)$ *is the unique solution to*

$$\frac{\omega(X)}{X} = \frac{1}{2\lambda}\log\frac{1}{r}. \tag{7}$$

Proof. Suppose first the coefficients a_n of $u(r, t)$ are of ω-slow growth. Then the series in (1) converges uniformly in each subdisc of radius $r < 1$, for the series is dominated by a constant time $\sum r^{|n|}e^{\lambda\omega(|n|)}$. But this series converges for each $0<r<1$ by the root test and condition (β).

It follows that

$$\|u_r\|_{L_1}\le c\sum_{n=0}^{\infty} r^n e^{\lambda\omega(n)}=cS(r), \qquad 0\le r<1,$$

and the problem is to decipher the asymptotic behavior of $S(r)$ as $r\to 1$.

First of all, since $\omega(x)$ is concave, (7) has solutions for each $r<1$ (and sufficiently close to 1), and since by condition (β), $\omega(x)$ cannot be linear for all x, the solution is unique (again for r sufficiently close to 1, since ω may start out linear).

Set $r=e^{-t}$, $0<t<\infty$. Then for t small enough, $\lambda\omega(x)-tx\ge 0$ on some interval $0\le x\le x_0$. If $\xi=\xi(t)$ is the first value of x where $\lambda\omega(x)-tx$ attains its maximum, and if $X=X(t)$ is the unique solution to $2\lambda\omega(X)/X=t=\log 1/r$, then $\xi(t)<x_0<X(t)$. Hence

$$\begin{aligned} S(r)=\sum_{n=0}^{\infty} e^{\lambda\omega(n)-tn} &= \sum_{n=0}^{[\xi]-1} e^{\lambda\omega(n)-tn}+\sum_{n=[\xi]+2}^{\infty} e^{\lambda\omega(n)-tn} \\ &\quad + e^{\lambda\omega([\xi])-t[\xi]}+e^{\lambda\omega([\xi]+1)-t([\xi]+1)} \\ &\le \int_0^{\infty} e^{\lambda\omega(x)-tx}\,dx+2e^{\lambda\omega(\xi)-t\xi}. \end{aligned}$$

On the other hand,

$$\int_0^X e^{\lambda\omega(x)-tx}\,dx\le e^{\lambda\omega(X)}\int_0^X e^{-tx}\,dx<\frac{1}{t}\,e^{\lambda\omega(X)}.$$

Next if $x\ge X$, $2\lambda\omega(x)/x\le t$ and therefore $\lambda\omega(x)-tx\le -\frac{1}{2}tx$. Hence

$$\int_X^{\infty} e^{\lambda\omega(x)-tx}\,dx\le\frac{2}{t}\,e^{-tX}e^{tx/2}<\frac{2}{t}\,e^{tX/2}=\frac{2}{t}\,e^{\lambda\omega(X)}.$$

Finally since $\xi(t)<X(t)$, $2e^{\lambda\omega(\xi)-t\xi}\le 2e^{\lambda\omega(\xi)}\le 2e^{\lambda\omega(X)}<(1/t)e^{\lambda\omega(X)}$. We have shown $\|u_r\|_{L_1}\le(4/t)e^{\lambda\omega(X)}$, $t=\log 1/r$. Now by (7), $1/t=2\lambda X/\omega(X)$. But ω is increasing and by condition (γ): $\log(1+x)\le b\omega(x)+a$, $b>0$, $\omega(x)$ is increasing to ∞ as $t\to 0$. So $1/t\le 1+1/t\le e^{\alpha}e^{b\omega(1/t)}=e^{\alpha}e^{b\omega(2\lambda X/\omega(X))}\le e^{a}e^{b\omega(X)}$ as soon as $\omega(X)\ge 2\lambda$. Hence (6) follows with $\Lambda=\lambda+b$.

For the sufficiency of (6) suppose $u(r,t)=\sum_n a_n r^{|n|}e^{int}$ is harmonic in the unit disc. Then if X satisfies (7) we have

$$|a_n|=\frac{r^{-|n|}}{2\pi}\left|\int_0^{2\pi} u(r,t)e^{-int}\,dt\right|\le r^{-|n|}\,\|u_r\|_{L_1}\le cr^{-|n|}e^{\Lambda\omega(X)}.$$

But this is valid for all integers n and all $0<r<1$. For a given n, choose $r=\exp(-2\lambda\omega(|n|)/|n|)$. Hence $r^{-|n|}=\exp 2\lambda\omega(|n|)$. Also $\omega(X)/X=(1/2\lambda)\log 1/r=\omega(|n|)/|n|$. So $X=|n|$ and $e^{\Lambda\omega(X)}=e^{\Lambda\omega(|n|)}$. The conclusion is that $|a_n|\le ce^{(\Lambda+2\lambda)\omega(|n|)}$, i.e. the a_n are ω-slow growth.

REMARK. We note that not only $\|u_r\|_{L_1}$ but also $\|u_r\|_{L_\infty}$ satisfy the estimate of the theorem. The reason for this is that from (1) one sees that both $\|u_r\|_{L_\infty}$ and $\|u_r\|_{L_1}$ are dominated by a constant times $S(r)=\sum r^n e^{\lambda\omega(n)}$, and it is $S(r)$ that has been estimated in the theorem.

We conclude by considering two special cases. First of all, in the classical Schwartz case, $\omega(x)=\log(1+x)$, and although (7) cannot be solved explicitly for X, several applications of L'Hôpital's rule show X is asymptotic to $(2\lambda/t)\log(1+2\lambda/t)$, $t=\log 1/r$, as $t\to 0$ $(r\to 1)$. Choosing $\Lambda=\lambda$ in (6) we see from (7) that $\|u_r\|_{L_1}=O(e^{\lambda\omega(X)})=O(e^{Xt/2})=O(\exp\lambda\log(1+2\lambda/t))=O(1+2\lambda/t)^{\lambda}=O(1/t^{\lambda})=O(1/(1-r)^{\lambda})$ as $r\to 1$, which is the classical result.

Secondly, in the important Gevrey case, $\omega(x)=x^{\alpha}$, $0<\alpha<1$, (7) can be solved explicitly for X, and we obtain the following result.

COROLLARY. *Suppose* $u(r,t)=u_r(t)=\sum_n a_n r^{|n|}e^{int}$ *is harmonic in the unit disc, and* $0<\alpha<1$. *Then*

$$a_n=O(e^{\lambda|n|^{\alpha}}),\quad \text{as}\quad |n|\to\infty,$$

for some positive constant λ, *if and only if*

$$\|u_r\|_{L_1}=O(\exp\Lambda/(1-r)^{\alpha/(1-\alpha)}),\quad \textit{as}\quad r\to 1,$$

for some positive constant Λ.

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