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ON MAXIMAL TORSION RADICALS

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Let R be an associative ring with identity, and let ${}_R\mathscr{M}$ denote the category of unital left R-modules. It is known that if R is a commutative, Noetherian ring, then the maximal torsion radicals of ${}_R\mathscr{M}$ correspond to the minimal prime ideals of R. In fact, Năstăsescu and Popescu [**15**, Proposition 2.7] have given a more general result valid for arbitrary commutative rings. This paper investigates maximal torsion radicals over rings not necessarily commutative. It is shown that if R is right Noetherian, then the maximal torsion radicals of ${}_R\mathscr{M}$ correspond to the minimal prime ideals of R, directly generalizing the result in the commutative case. (In fact, the result holds under somewhat weaker chain conditions.) If R satisfies the maximum condition for two-sided ideals, every proper torsion radical is contained in a maximal torsion radical, and the maximal torsion radicals correspond to prime ideals minimal in a more general sense defined in this paper. If R is left Noetherian, then a torsion radical of ${}_R\mathscr{M}$ is maximal if and only if it is defined by the injective envelope of R/A for a critical prime left ideal A minimal in the same sense.

In general, the following conditions are equivalent for a torsion radical σ of ${}_R\mathscr{M}$:

(a) σ is a maximal torsion radical;

(b) in the quotient category determined by σ, every nonzero injective object is a cogenerator;

(c) σ is saturated and every nonzero torsionfree, injective left module over the ring of quotients $Q_\sigma(R)$ is faithful;

(d) σ is saturated and every nonzero torsionfree, injective left $R/\sigma(R)$-module is faithful.

When the quotient category is equivalent to the category of left $Q_\sigma(R)$-modules (that is, when the quotient functor is naturally isomorphic to $Q_\sigma(R) \otimes_R -$), then condition (b) can be used to show that σ is a maximal torsion radical if and only if the Jacobson radical $J(Q_\sigma(R))$ is right T-nilpotent and $Q_\sigma(R)/J(Q_\sigma(R))$ is simple Artinian. Moreover, if in this situation R is left Noetherian, then $Q_\sigma(R)$ is left Artinian.

In the first section we show that maximal radicals correspond to prime ideals. We investigate general properties of maximal torsion radicals in the second section. In the final section we introduce a relation $\leqq$ for left ideals, which reduces to ordinary containment when applied to prime ideals of a commutative ring, and use this relation to characterize maximal torsion

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radicals of certain noncommutative rings. We also give some results for hereditary, Noetherian rings. The reader is referred to the books by Lambek [**8**] and Mitchell [**13**] for any undefined terms. The term torsion radical is used by Maranda [**11**]. Equivalent notions are those of torsion theory [**9**] and idempotent kernel functor [**6**].

1. Maximal radicals and prime ideals. A subfunctor ρ of the identity on ${}_R\mathcal{M}$ (for all $M \in {}_R\mathcal{M}$, $\rho(M)$ is a submodule of M, and if $f \in \mathrm{Hom}_R(M, N)$, then $f(\rho(M)) \subseteq \rho(N)$) is called a radical of ${}_R\mathcal{M}$ if $\rho(M/\rho(M)) = 0$ for all $M \in {}_R\mathcal{M}$, and a torsion radical if in addition $\rho(M_0) = M_0 \cap \rho(M)$ for all submodules $M_0 \subseteq M$. A radical is proper if it is not the identity functor on ${}_R\mathcal{M}$, or equivalently, if $\rho(R) \neq R$. If ρ and σ are radicals with $\rho(M) \subseteq \sigma(M)$ for all $M \in {}_R\mathcal{M}$, we write $\rho \leqq \sigma$, and if ρ is a (torsion) radical then we call ρ a maximal (torsion) radical if ρ is proper and for any other (torsion) radical σ with $\rho \leqq \sigma$, either $\rho = \sigma$ or σ is the identity on ${}_R\mathcal{M}$. If ρ is a radical, then a module ${}_RM$ is ρ-torsion if $\rho(M) = M$, ρ-torsionfree if $\rho(M) = 0$, and a submodule M_0 of M is ρ-dense if M/M_0 is ρ-torsion and ρ-closed if M/M_0 is ρ-torsionfree. A (left) ideal A of R is a maximal ρ-closed (left) ideal if it is maximal in the set of proper ρ-closed (left) ideals.

For any module ${}_RN$, $\mathrm{rad}_N : {}_R\mathcal{M} \to {}_R\mathcal{M}$ defined by

$$\mathrm{rad}_N(M) = \bigcap_{f \in \mathrm{Hom}_R(M,N)} \ker(f)$$

for all $M \in {}_R\mathcal{M}$, is a radical. Note that $\mathrm{rad}_N(R)$ is just $\mathrm{Ann}(N)$. If ${}_RN$ is injective, then rad_N is a torsion radical, and moreover, every torsion radical is of this form for an appropriate injective module. If ρ is a radical and ${}_RN$ is ρ-torsionfree, then for any module ${}_RM$ and $f \in \mathrm{Hom}_R(M, N)$ we must have $f(\rho(M)) \subseteq \rho(N) = 0$, and thus $\rho \leqq \mathrm{rad}_N$. On the other hand, if $\rho \leqq \mathrm{rad}_N$, then $\mathrm{rad}_N(N) = 0$ implies $\rho(N) = 0$. Thus $\rho \leqq \mathrm{rad}_N$ if and only if $\rho(N) = 0$. This will prove to be useful in the characterization of maximal radicals. A characterization of prime ideals similar to that of Theorem 1.3 was given by the author in an earlier paper [**3**], without reference to maximal radicals. We use $E(M)$ to denote the injective envelope in ${}_R\mathcal{M}$ of the module ${}_RM$.

1.1. LEMMA. *Let A be an ideal of R. Then the following conditions are equivalent:*

(a) *A is a prime ideal.*
(b) *If $0 \neq N \in {}_R\mathcal{M}$ and $\mathrm{rad}_N \geqq \mathrm{rad}_{E(R/A)}$, then $\mathrm{Ann}(N) \subseteq A$.*
(c) *If $0 \neq N \in {}_R\mathcal{M}$ and $\mathrm{rad}_N \geqq \mathrm{rad}_{R/A}$, then $\mathrm{Ann}(N) = A$.*

Proof. (a) $\Rightarrow$ (b). Assume that A is a prime ideal and that $\mathrm{rad}_N \geqq \mathrm{rad}_{E(R/A)}$. Then $\mathrm{rad}_{E(R/A)}(N) = 0$, and so there exists $0 \neq f \in \mathrm{Hom}_R(N, E(R/A))$, which implies that $f(N) \cap R/A \neq 0$, since $E(R/A)$ is an essential extension of R/A. Now $\mathrm{Ann}(N) \cdot (f(N) \cap R/A) = 0$ implies $\mathrm{Ann}(N) \subseteq A$, since A is prime.

(b) $\Rightarrow$ (c). If $\text{rad}_N \geqq \text{rad}_{R/A}$, then certainly $\text{rad}_N \geqq \text{rad}_{E(R/A)}$, and so by assumption $\text{Ann}(N) \subseteq A$. But $\text{rad}_{R/A}(N) = 0$ implies that for each $x \in N$ we must have $Ax = 0$, since otherwise there exists $f \in \text{Hom}_R(N, R/A)$ with $Af(x) = f(Ax) \neq 0$, a contradiction. Thus $\text{Ann}(N) = A$.

(c) $\Rightarrow$ (a). Let B and C be ideals of R with $BC \subseteq A$. If $A \subsetneq C$, then $C/A \neq 0$ and $\text{rad}_{C/A} \geqq \text{rad}_{R/A}$. By assumption $A = \text{Ann}(C/A) \supseteq B$, and this is sufficient to show that A is a prime ideal, completing the proof.

1.2. PROPOSITION. *Let A be an ideal of R. The following conditions are equivalent:*

(a) *A is a prime ideal.*

(b) *A is a maximal σ-closed ideal for a torsion radical σ.*

(c) *A is a maximal ρ-closed ideal for a radical ρ.*

Proof. (a) $\Rightarrow$ (b). If A is a prime ideal, let $\sigma = \text{rad}_{E(R/A)}$. Then A is σ-closed, and if B is any proper σ-closed ideal, then $\text{rad}_{R/B} \geqq \text{rad}_{E(R/A)}$, and Lemma 1.1 implies that $B = \text{Ann}(R/B) \subseteq A$.

(b) $\Rightarrow$ (c). Assume that A is a maximal σ-closed ideal and let $\rho = \text{rad}_{R/A}$. Then $\rho \geqq \sigma$ and every ρ-closed ideal is also σ-closed, so A must be a maximal ρ-closed ideal.

(c) $\Rightarrow$ (a). If A is a maximal ρ-closed ideal and $\text{rad}_N \geqq \text{rad}_{R/A}$ for some $0 \neq N \in {}_R\mathscr{M}$, then $\text{Ann}(N) \supseteq A$ and $\text{Ann}(N)$ is ρ-closed since $\text{rad}_N \geqq \rho$. By assumption we must have $\text{Ann}(N) = A$, and then Lemma 1.1 implies that A is a prime ideal.

1.3. THEOREM. *Let ρ be a radical of ${}_R\mathscr{M}$, with $\rho(R) = A$. Then ρ is a maximal radical $\Leftrightarrow$ $\rho = \text{rad}_{R/A}$ and A is a prime ideal.*

Proof. ($\Rightarrow$) Suppose that ρ is a maximal radical. Then $A \neq R$ and $\rho(R/A) = 0$ implies $\rho \leqq \text{rad}_{R/A}$, so $\rho = \text{rad}_{R/A}$ since ρ is maximal. Furthermore, A is a maximal ρ-closed ideal since any larger ρ-closed ideal would determine a larger radical. The previous proposition then shows that A is a prime ideal.

($\Leftarrow$) If A is a prime ideal and $\rho = \text{rad}_{R/A}$, then the proof of Proposition 1.2 shows that A is a maximal ρ-closed ideal. If α is a radical with $\alpha \geqq \rho$, then $\alpha(R)$ is ρ-closed and contains A. Hence either $\alpha(R) = R$ or $\alpha(R) = A$ and $\alpha \leqq \text{rad}_{R/A} = \rho$.

1.4. COROLLARY. *Every proper radical of ${}_R\mathscr{M}$ is contained in a maximal radical $\Leftrightarrow$ for each $0 \neq M \in {}_R\mathscr{M}$ there exists a submodule $M_0 \subseteq M$ such that* $\text{Ann}(M_0)$ *is a prime ideal.*

Proof. ($\Rightarrow$) If $0 \neq M \in {}_R\mathscr{M}$ and every proper radical is contained in a maximal radical, then the theorem shows that there exists a prime ideal P with $\text{rad}_{R/P} \geqq \text{rad}_M$. Let

$$M_0 = \{m \in M : Pm = 0\}.$$

It can easily be shown that $\text{Ann}(M_0) = P$.

($\Leftarrow$) Let ρ be a proper radical with $\rho(R) = A$. By assumption there exists a left ideal $A \subsetneqq B$ with $\mathrm{Ann}(B/A) = P$ a prime ideal. Thus $\rho \leqq \mathrm{rad}_{R/A} \leqq \mathrm{rad}_{R/P}$, and $\mathrm{rad}_{R/P}$ is a maximal radical.

An ideal K of R is called a torsion ideal if there exists a torsion radical σ with $\sigma(R) = K$. It can be shown that K is a torsion ideal if and only if $K = \sigma(R)$ for $\sigma = \mathrm{rad}_{E(R/K)}$, and thus K is a torsion ideal if and only if $\mathrm{Ann}(E(R/K)) = K$. Other equivalent conditions are given by Lambek [**9**].

1.5. PROPOSITION. *Let K be a proper torsion ideal of R. Then K is a prime ideal $\Leftrightarrow$ $\rho(R) = K$ for all proper radicals ρ such that $\rho \geqq \mathrm{rad}_{E(R/K)}$.*

Proof. ($\Rightarrow$) If ρ is a proper radical with $\rho \geqq \mathrm{rad}_{E(R/K)}$, then

$$\rho(R) \supseteq \mathrm{rad}_{E(R/K)}(R) = \mathrm{Ann}(E(R/K)) = K.$$

Since K is a prime ideal, Proposition 1.2 shows that K is a maximal $\mathrm{rad}_{E(R/K)}$-closed ideal, and therefore $\rho(R) = K$, since $\rho(R)$ is $\mathrm{rad}_{E(R/K)}$-closed.

($\Leftarrow$) This follows immediately from Lemma 1.1.

2. Maximal torsion radicals. We next characterize maximal torsion radicals by giving conditions on the ring of quotients and on the quotient category determined by the torsion radical. For a torsion radical σ, a module ${}_RN$ is called σ-injective if for each module ${}_RM$ and each homomorphism $f \in \mathrm{Hom}_R(M_0, N)$ such that M_0 is a σ-dense submodule of M, there exists an extension of f to M. The quotient category determined by σ is the full subcategory of σ-torsionfree, σ-injective R-modules, and will be denoted by ${}_R\mathcal{M}_\sigma$. The quotient functor $Q_\sigma : {}_R\mathcal{M} \to {}_R\mathcal{M}_\sigma$ is a left adjoint to the inclusion, and is defined on modules as follows: for $M \in {}_R\mathcal{M}$ let $M/\sigma(M) = M_t$, and then $Q_\sigma(M)$ is the submodule of $E(M_t)$ for which $Q_\sigma(M)/M_t = \sigma(E(M_t)/M_t)$. $Q_\sigma(R)$, which we abbreviate to R_σ, has a ring structure and is called the ring of left quotients of R determined by σ.

A torsion radical σ is said to be saturated if $\sigma \geqq \alpha$ for every torsion radical α such that $\alpha(R) = \sigma(R)$. (See [**17**].) For any torsion radical σ with $\sigma(R) = K$, we have $\sigma(E(R/K)) = 0$ and hence $\sigma \leqq \mathrm{rad}_{E(R/K)}$. It follows immediately that σ is saturated if and only if $\sigma = \mathrm{rad}_{E(R/K)}$. The complete ring of left quotients of R, which we denote by $Q_{\max}(R)$, is the ring of left quotients determined by $\sigma = \mathrm{rad}_{E(R)}$. For this torsion radical σ we abbreviate the terms σ-torsion, σ-torsionfree, σ-dense, and σ-closed to torsion, torsionfree, dense, and closed, respectively. For a torsion ideal K of R, $K = \mathrm{Ann}(E(R/K))$, and so $E(R/K)$ is an R/K-module. Thus the R/K-injective envelope of R/K and the R-injective envelope of R/K coincide. This shows that if σ is a saturated torsion radical with $\sigma(R) = K$, then $\sigma = \mathrm{rad}_{E(R/K)}$ and $R_\sigma = Q_{\max}(R/K)$.

2.1. PROPOSITION. *Let σ be a torsion radical. Then σ is a maximal torsion radical $\Leftrightarrow$ σ is saturated and $\alpha(R) = \sigma(R)$ for all proper torsion radicals α such that $\alpha \geqq \sigma$.*

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Proof. ($\Rightarrow$) If $\sigma(R) = K$, then $\sigma \leqq \mathrm{rad}_{E(R/K)}$, and so if σ is a maximal torsion radical it follows that $\sigma = \mathrm{rad}_{E(R/K)}$, and σ is saturated. If α is a proper torsion radical with $\alpha \geqq \sigma$, then we must have $\alpha = \sigma$ and in particular $\alpha(R) = \sigma(R)$.

($\Leftarrow$) If α is a proper torsion radical with $\alpha \geqq \sigma$, then by assumption $\alpha(R) = \sigma(R)$, and so $\alpha \leqq \sigma$ since σ is saturated.

If σ is a maximal torsion radical with $\sigma(R) = K$, then K must be maximal in the set of σ-closed torsion ideals. In fact, an ideal K is the torsion ideal determined by a maximal torsion radical if and only if K is maximal in the set of σ-closed torsion ideals for some torsion radical σ.

2.2. THEOREM. *Let σ be a saturated torsion radical with $\sigma(R) = K$. If K is a prime ideal, then σ is a maximal torsion radical. The converse holds if K is a semiprime ideal.*

Proof. If K is a prime ideal, then since σ is saturated, the proof of Proposition 1.2 shows that K is a maximal σ-closed ideal, and so by the preceding remarks σ is a maximal torsion radical.

Conversely, if σ is a maximal torsion radical, then to show that K is a maximal σ-closed ideal and therefore prime, it is sufficient to prove the following: if K is a semiprime torsion ideal, then every σ-closed ideal A such that $A \supseteq K$ is a torsion ideal of R. Accordingly, assume that K is a semiprime torsion ideal and that $A \supseteq K$ is a σ-closed ideal. Since K is σ-torsion and $E(R/A)$ is σ-torsionfree, $\mathrm{Hom}_R(K, E(R/A)) = 0$, and so we shall prove that

$$\mathrm{Hom}_R(A/K, E(R/A)) = 0,$$

since this implies that $\mathrm{Hom}_R(A, E(R/A)) = 0$ and therefore A is a torsion ideal. Now if $0 \neq f \in \mathrm{Hom}_R(A/K, E(R/A))$, then since $E(R/A)$ is $\mathrm{rad}_{E(R/K)}$-torsionfree there exists $g \in \mathrm{Hom}_R(E(R/A), E(R/K))$ with $gf \neq 0$. Consequently there exists a left ideal B with $K \subseteq B \subseteq A$ such that $f(B/K) \subseteq R/A$ and $0 \neq gf(B/K) \subseteq R/K$. Let C be the left ideal containing K with $C/K = gf(B/K)$. Then $B \subseteq A$ implies $B \cdot f(B/K) \subseteq B \cdot (R/A) = 0$ and hence $B \cdot C \subseteq K$. Since K is a semiprime ideal, this implies that $C \cdot B \subseteq K$, and thus

$$C \cdot gf(B/K) \subseteq gf(C \cdot (B/K)) = 0,$$

which shows that $C^2 \subseteq K$. Since K is semiprime, this implies that $C \subseteq K$, a contradiction. Thus $\mathrm{Hom}_R(A, E(R/A)) = 0$ and we conclude that K is a prime ideal.

Recall that in a category with direct products an object W is a cogenerator if and only if each object of the category can be embedded in a direct product of copies of W. (See [**13**].) Note that for modules ${}_RM$ and ${}_RN$, $\mathrm{rad}_N(M) = 0$ if and only if M can be embedded in a direct product of copies of N.

2.3. LEMMA. *Let σ be a saturated torsion radical with $\sigma(R) = K$. The following conditions are equivalent for a module $M \in {}_R\mathcal{M}$:*

(a) *M is a σ-torsionfree, injective R-module.*
(b) *M is an injective object in ${}_R\mathcal{M}_\sigma$.*
(c) *M is a torsionfree, injective R_σ-module.*
(d) *M is a torsionfree, injective R/K-module.*

Proof. Since σ is saturated, $\sigma = \text{rad}_{E(R/K)}$, and since K is a torsion ideal, $E(R/K)$ is the R/K-injective envelope of R/K. Furthermore, since $R_\sigma = Q_{\max}(R/K)$, $E(R/K)$ is the R_σ-injective envelope of R_σ, as shown by Lambek in [**8**, Proposition 2, § 4.3]. Each of the conditions of the lemma can thus be seen to be equivalent to the condition that M is an R-direct summand of a direct product of copies of $E(R/K)$.

2.4. THEOREM. *Let σ be a torsion radical with $\sigma(R) = K$. The following conditions are equivalent:*
(a) *σ is a maximal torsion radical.*
(b) *In ${}_R\mathcal{M}_\sigma$ every nonzero injective object is a cogenerator.*
(c) *σ is saturated and every nonzero torsionfree, injective left R_σ-module is faithful.*
(d) *σ is saturated and every nonzero torsionfree, injective left R/K-module is faithful.*

Proof. (a) $\Rightarrow$ (b). Let M be an injective object in ${}_R\mathcal{M}_\sigma$. By Lemma 2.3, ${}_RM$ is σ-torsionfree and injective, so $\sigma(M) = 0$ implies $\sigma \leqq \text{rad}_M$, and then since rad_M defines a torsion radical of ${}_R\mathcal{M}$ and σ is a maximal torsion radical, we must have $\sigma = \text{rad}_M$. For any object N in ${}_R\mathcal{M}_\sigma$, $\text{rad}_M(N) = \sigma(N) = 0$, and so N can be embedded in a direct product of copies of M. Thus M is a cogenerator in ${}_R\mathcal{M}_\sigma$.

(b) $\Rightarrow$ (c). $E(R/K)$ is σ-torsionfree and injective, so it is an injective object in ${}_R\mathcal{M}_\sigma$ and by assumption a cogenerator in ${}_R\mathcal{M}_\sigma$. Since every object in ${}_R\mathcal{M}_\sigma$ can be embedded in a direct product of copies of $E(R/K)$, it follows that every σ-torsionfree R-module can also be embedded in a direct product of copies of $E(R/K)$. Thus $\text{rad}_{E(R/K)}(M) = 0$ for all modules ${}_RM$ with $\sigma(M) = 0$, and so $\text{rad}_{E(R/K)} \leqq \sigma$. But since $\text{rad}_{E(R/K)} \geqq \sigma$ always holds, we have equality, and σ is saturated. If M is a torsionfree, injective R_σ-module, then by Lemma 2.3 it is an injective object in ${}_R\mathcal{M}_\sigma$, and so by assumption M is a cogenerator in ${}_R\mathcal{M}_\sigma$. In particular, R_σ can be embedded in a direct product of copies of M, and this shows that M is faithful as an R_σ-module.

(c) $\Rightarrow$ (d). This follows immediately from Lemma 2.3, since $R/K \subseteq R_\sigma$, and thus any faithful R_σ-module is faithful as an R/K-module.

(d) $\Rightarrow$ (a). If α is a proper torsion radical with $\alpha \geqq \sigma$, then $E(R/\alpha(R))$ is injective and σ-torsionfree, so it is injective and torsionfree as an R/K-module, and by assumption it must be a faithful R/K-module. Thus

$$\alpha(R) = \text{Ann}(E(R/\alpha(R))) = K,$$

and since σ is saturated, Proposition 2.1 shows that σ is a maximal torsion radical.

Applying the theorem to $\mathrm{rad}_{E(R)}$, which is saturated, we obtain the following corollary. Note that if S is any subring of $Q_{\max}(R)$ with $R \subseteq S \subseteq Q_{\max}(R)$, then S satisfies the conditions of the corollary if and only if R satisfies the conditions, since $Q_{\max}(S) = Q_{\max}(R)$.

2.5. COROLLARY. *Every nonzero torsionfree, injective left R-module is faithful* $\Leftrightarrow \mathrm{rad}_{E(R)}$ *is a maximal torsion radical of* ${}_R\mathcal{M}$.

Lambek and Michler [**10**] call a left ideal A critical if it is maximal in the set of proper σ-closed left ideals for some torsion radical σ. In this case σ can be taken to be $\mathrm{rad}_{E(R/A)}$, since A is σ-closed and therefore $\sigma \leqq \mathrm{rad}_{E(R/A)}$. (If R is commutative, then the notions of critical ideal and prime ideal coincide, as shown by Proposition 1.2.) With $\sigma = \mathrm{rad}_{E(R/A)}$, it can be seen that A is a critical left ideal if and only if $Q_\sigma(R/A)$ is a simple object in ${}_R\mathcal{M}_\sigma$. In [**16**] Popescu calls a torsion radical σ a prime torsion radical if ${}_R\mathcal{M}_\sigma$ has a cogenerator which is the injective envelope of a simple object in ${}_R\mathcal{M}_\sigma$, and he shows that this is equivalent to the original definition given by Goldman in [**6**]. It is evident that σ is a prime torsion radical if and only if $\sigma = \mathrm{rad}_{E(R/A)}$ for a critical left ideal A. Theorem 2.4(b) establishes the connection between maximal torsion radicals and prime torsion radicals.

2.6. COROLLARY. *Let σ be a torsion radical of ${}_R\mathcal{M}$.*

(a) *If σ is a maximal torsion radical, then σ is a prime torsion radical $\Leftrightarrow$ there exists a σ-closed critical left ideal.*

(b) *If σ is a prime torsion radical, then σ is a maximal torsion radical $\Leftrightarrow$ for each nonzero σ-torsionfree injective module ${}_RM$ there exists $m \in M$ such that* $\mathrm{Ann}(m)$ *is a critical left ideal.*

Proof. If σ is a maximal torsion radical, then every nonzero injective in ${}_R\mathcal{M}_\sigma$ is a cogenerator, and so σ is a prime torsion radical if and only if ${}_R\mathcal{M}_\sigma$ has a simple object. This occurs if and only if there exists a σ-closed critical left ideal.

If σ is a prime torsion radical, then every nonzero injective in ${}_R\mathcal{M}_\sigma$ is a cogenerator if and only if every nonzero injective in ${}_R\mathcal{M}_\sigma$ contains a submodule simple in ${}_R\mathcal{M}_\sigma$.

The singular submodule $Z({}_RM)$ of a module $M \in {}_R\mathcal{M}$ is the set of elements of M whose annihilator is an essential left ideal of R. The Goldie torsion radical G of ${}_R\mathcal{M}$ is defined for ${}_RM$ by letting $G(M)$ be the submodule of M for which $G(M)/Z(M) = Z(M/Z(M))$. It is well-known that G defines a torsion radical of ${}_R\mathcal{M}$, with $G \geqq \mathrm{rad}_{E(R)}$ since every dense left ideal of R is essential, and that $G = \mathrm{rad}_{E(R)}$ if and only if $Z(R) = 0$. Furthermore, $G(M) = M$ if and only if $Z(M)$ is essential in M. (See [**4**].) We note that, as remarked by Alin and Armendariz [**1**], for any commutative local ring R with nilpotent maximal ideal, $Z(R)$ is essential in R and the only proper torsion radical of ${}_R\mathcal{M}$ is the zero functor, so $\mathrm{rad}_{E(R)}$ is a maximal torsion radical.

2.7. PROPOSITION. *If σ is a maximal torsion radical with $\sigma(R) = K$, then either $Z({}_{R/K}R/K) = 0$ or $Z({}_{R/K}R/K)$ is essential in R/K.*

Proof. If σ is a maximal torsion radical, then by Theorem 2.4 every nonzero torsionfree, injective left R/K-module is faithful. Thus $\mathrm{rad}_{E(R/K)}$ defines a maximal torsion radical for ${}_{R/K}\mathscr{M}$. If G is the Goldie torsion radical for ${}_{R/K}\mathscr{M}$, then $G \geqq \mathrm{rad}_{E(R/K)}$, and so either $G = \mathrm{rad}_{E(R/K)}$ and $Z({}_{R/K}R/K) = 0$ or else G is the identity on ${}_{R/K}\mathscr{M}$ and $Z({}_{R/K}R/K)$ is essential in R/K.

2.8. THEOREM. *Let σ be a saturated torsion radical with $\sigma(R) = K$. If $Z({}_{R/K}R/K) = 0$, then σ is a maximal torsion radical $\Leftrightarrow$ R_σ is a prime ring.*

Proof. ($\Leftarrow$) If σ is a maximal torsion radical, then by Theorem 2.4 every nonzero torsionfree, injective left R_σ-module is faithful. As in the proof of Theorem 2.2, to show that R_σ is a prime ring it is sufficient to show that every closed ideal of R_σ is a torsion ideal. Since σ is saturated, $R_\sigma = Q_{\max}(R/K)$, and if $Z({}_{R/K}R/K) = 0$, then it is well-known that R_σ must be von Neumann regular. This yields the desired conclusion, since every ideal of a von Neumann regular ring is a torsion ideal [**9**, Proposition 2.3].

($\Leftarrow$) If R_σ is a prime ring, then it follows from Theorem 2.2, Corollary 2.5, and Theorem 2.4 that σ is a maximal torsion radical, without assuming that $Z({}_{R/K}R/K) = 0$.

Alin and Armendariz [**1**] have investigated the condition that $\mathrm{rad}_{E(R)} \geqq \sigma$ for every proper torsion radical σ, and they have shown [**1**, Theorem 2.2] that for commutative rings whose singular ideal is not essential this condition characterizes integral domains. It follows from our results that for the same class of rings the weaker condition that $\mathrm{rad}_{E(R)}$ is a maximal torsion radical also characterizes integral domains. For any ring R, $\mathrm{rad}_{E(R)} \geqq \sigma$ for every proper torsion radical σ of ${}_R\mathscr{M}$ if and only if 0 is the only proper torsion ideal of R, and since the torsion ideals of R are precisely the annihilators of injective modules, this occurs if and only if every nonzero injective left R-module is faithful. Any prime von Neumann regular ring R which is not simple has nontrivial torsion ideals and yet $\mathrm{rad}_{E(R)}$ is a maximal torsion radical. This shows that the two conditions are not always equivalent.

2.9. PROPOSITION. *Every nonzero injective left R-module is faithful $\Leftrightarrow$ every nonzero torsionfree, injective left R-module is faithful and every torsion ideal of ${}_RR$ is closed.*

Proof. ($\Rightarrow$) This is clear since by assumption the only torsion ideals are 0 and R.

($\Leftarrow$) If ${}_RM$ is injective, then $\mathrm{Ann}(M)$ is a torsion ideal, and so by assumption it is closed. If $M \neq 0$ then $E(R/\mathrm{Ann}(M))$ is a nonzero torsionfree injective, and so

$$\mathrm{Ann}(M) = \mathrm{Ann}(E(R/\mathrm{Ann}(M))) = 0.$$

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2.10. Proposition. *Let K be a torsion ideal of R. Consider the following conditions:*

(a) *Every nonzero injective left R/K-module is faithful.*

(b) *K is a maximal torsion ideal of R.*

(c) *Every nonzero torsionfree, injective left R/K-module is faithful.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *If R is left hereditary, then* (a) *and* (b) *are equivalent. If every torsion ideal of R/K is closed, then the conditions are all equivalent.*

Proof. (a) $\Rightarrow$ (b). If we assume (a), then the only torsion ideal of R/K is the zero ideal. We shall show that if A is a torsion ideal of R with $A \supseteq K$, then A/K is a torsion ideal of R/K, and thus K must be a maximal torsion ideal of R. Now A is a torsion ideal of R if and only if $\mathrm{Hom}_R(A, E(R/A)) = 0$ [**9**, Corollary, p. 33], and this is equivalent to the condition that $\mathrm{Hom}_R(A_0, R/A) = 0$ for all left ideals $A_0 \subseteq A$ [**9**, Proposition 0.2]. This implies that

$$\mathrm{Hom}_{R/K}(A_0/K, R/A) = \mathrm{Hom}_R(A_0/K, R/A) = 0$$

for all left ideals A_0 with $K \subseteq A_0 \subseteq A$, and since $(R/K)/(A/K) \approx R/A$, this shows that A/K is a torsion ideal of R/K.

(b) $\Rightarrow$ (c). If σ is a proper torsion radical such that $\sigma \geqq \mathrm{rad}_{E(R/K)}$, then $\sigma(R) \supseteq K$, and since K is a maximal torsion ideal, $\sigma(R) = K$. Proposition 2.1 shows that $\mathrm{rad}_{E(R/K)}$ is a maximal torsion radical, and so the desired conclusion follows from Theorem 2.4.

(b) $\Rightarrow$ (a). Assume that R is left hereditary. Condition (a) will follow from (b) if we can show that if A is an ideal of R with $A \supseteq K$, and A/K is a torsion ideal of R/K, then A is a torsion ideal of R. As above, if A/K is a torsion ideal of R/K, then $\mathrm{Hom}_R(A_0/K, R/A) = 0$ for all left ideals A_0 with $K \subseteq A_0 \subseteq A$. Since K is a torsion ideal, $\mathrm{Hom}_R(K_0, R/K) = 0$ for all left ideals $K_0 \subseteq K$. By assumption every left ideal is projective, and so this implies $\mathrm{Hom}_R(K_0, R/A) = 0$. Thus

$$\mathrm{Hom}_R(K, E(R/A)) = 0 \quad \text{and} \quad \mathrm{Hom}_R(A/K, E(R/A)) = 0,$$

and therefore $\mathrm{Hom}_R(A, E(R/A)) = 0$ and A is a torsion ideal of R.

(c) $\Rightarrow$ (a). Assume that every torsion ideal of R/K is closed. The result follows from Proposition 2.9.

2.11. Proposition. *Let σ be a saturated torsion radical with $\sigma(R) = K$. If $Z({}_{R/K}R/K) = 0$, then R_σ is a simple ring $\Leftrightarrow$ every nonzero injective left R_σ-module is faithful.*

Proof. If σ is saturated, then $R_\sigma = Q_{\max}(R/K)$, and so R_σ is von Neumann regular if $Z(R/K) = 0$. As remarked previously every ideal of a von Neumann ring is a torsion ideal, and so R_σ is simple if and only if the only torsion ideals of R_σ are 0 and R_σ.

Năstăsescu and Popescu [**14**] call a ring R left semi-Artinian if every nonzero left R-module contains a simple submodule. They show [**14**, Proposition 3.2] that R is left semi-Artinian if and only if $R/J(R)$ is left semi-Artinian and $J(R)$ is right T-nilpotent, where $J(R)$ is the Jacobson radical of R. (An ideal A is right T-nilpotent if for each sequence $\{a_i\}_{i=1}^{\infty}$ of elements of A there exists an integer n such that $a_n a_{n-1} \ldots a_1 = 0$.) The only proper torsion radical of ${}_R\mathcal{M}$ is the zero functor if and only if every nonzero injective left R-module is a cogenerator, since $\mathrm{rad}_M = 0$ if and only if ${}_RM$ is a cogenerator for ${}_R\mathcal{M}$. Gardner has shown [**5**, Proposition 2] that this occurs if and only if R is left semi-Artinian and has a unique (up to isomorphism) simple left module. Moreover, in this case $R/J(R)$ is simple Artinian [**5**, Theorem 6]. It is well-known that the inclusion functor from ${}_R\mathcal{M}_\sigma$ to the category of left R_σ-modules is an equivalence if and only if Q_σ is naturally isomorphic to $R_\sigma \otimes_R -$. When this occurs, Theorem 2.4 takes the following form. (Note that the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) hold for all torsion radicals σ.)

2.12 THEOREM. *Let σ be a torsion radical of ${}_R\mathcal{M}$ such that $Q_\sigma \approx R_\sigma \otimes {}_R-$. The following conditions are equivalent:*

(a) *σ is a maximal torsion radical.*

(b) *$R_\sigma/J(R_\sigma)$ is simple Artinian and $J(R_\sigma)$ is right T-nilpotent.*

(c) *σ is saturated and every nonzero injective left R_σ-module is faithful.*

Proof. (a) $\Rightarrow$ (b). By assumption ${}_R\mathcal{M}_\sigma$ is equivalent to the category of left R_σ-modules, and so Theorem 2.4(b) shows that every nonzero injective left R_σ-module is a cogenerator. The results of [**5**] and [**14**] then imply that $R_\sigma/J(R_\sigma)$ is simple Artinian and $J(R_\sigma)$ is right T-nilpotent.

(b) $\Rightarrow$ (c). If R_σ satisfies the conditions of (b), then every nonzero injective left R_σ-module is a cogenerator and therefore faithful. The proof of Theorem 2.4 shows that σ is saturated, since $E(R/\sigma(R))$ is a cogenerator.

(c) $\Rightarrow$ (a). This follows immediately from Theorem 2.4.

3. Maximal torsion radicals and minimal primes in Noetherian rings.

3.1. THEOREM. *If R is left Noetherian and σ is a maximal torsion radical of ${}_R\mathcal{M}$ such that $Q_\sigma \approx R_\sigma \otimes_R -$, then R_σ is left Artinian.*

Proof. If R is left Noetherian and $Q_\sigma \approx R_\sigma \otimes_R -$, then R_σ is a left localization in the sense of Silver, and consequently R_σ is left Noetherian [**18**, Proposition 1.6]. By Theorem 2.12, R_σ is left semi-Artinian, and a left Noetherian, left semi-Artinian ring is left Artinian.

3.2. PROPOSITION. *Let R satisfy the maximum condition for ideals.*

(a) *Every proper radical of ${}_R\mathcal{M}$ is contained in a maximal radical.*

(b) *Every proper torsion radical of ${}_R\mathcal{M}$ is contained in a maximal torsion radical.*

Proof. (a) If ρ is any proper radical, then there exists by assumption an ideal A maximal in the set of proper ρ-closed ideals. By Proposition 1.2, A is a prime ideal, and by Theorem 1.3, $\mathrm{rad}_{R/A}$ is a maximal radical. We then have $\rho \leqq \mathrm{rad}_{R/A}$, since A is ρ-closed.

(b) If σ is any proper torsion radical, then there exists an ideal K maximal in the set of proper σ-closed torsion ideals. Then $\sigma \leqq \mathrm{rad}_{E(R/K)}$, since K is σ-closed. If α is a proper torsion radical with $\mathrm{rad}_{E(R/K)} \leqq \alpha$, then $\alpha(R)$ is α-closed, and hence σ-closed. Since $\alpha(R) \supseteq K$, we must have $\alpha(R) = K$, and then Proposition 2.1 implies that $\mathrm{rad}_{E(R/K)}$ is a maximal torsion radical.

The torsion radical $\mathrm{rad}_{E(R)}$ is contained in a maximal torsion radical of ${}_R\mathscr{M}$ if and only if there exists a maximal closed torsion ideal, and if R is semiprime, then this occurs if and only if there exists a maximal closed ideal, since the proof of Theorem 2.2 shows that every closed ideal of R is a torsion ideal. If R is commutative and semiprime, then $Q_{\max}(R)$ is self-injective, and the closed ideals of $Q_{\max}(R)$ are precisely the annihilator ideals. Since in this case the lattice of annihilator ideals of R is isomorphic to the lattice of annihilator ideals of $Q_{\max}(R) = Q$, any commutative semiprime ring with no maximal annihilator ideal yields an example of a ring Q for which there is no maximal torsion radical containing $\mathrm{rad}_{E(Q)}$.

The result that maximal torsion radicals of a commutative, Noetherian ring correspond to minimal prime ideals can be generalized in several ways to the noncommutative case. These involve prime left ideals and a more general notion of minimality. Lambek and Michler [**10**] call a left ideal P of R prime if for all $a, b \in R$, $aRb \subseteq P$ implies $a \in P$ or $b \in P$. For a left ideal A of R and $r \in R$, we let

$$Ar^{-1} = \{x \in R : xr \in A\}.$$

3.3. Proposition. *Let A and B be left ideals of R. The following conditions are equivalent:*

(a) *For all left ideals A_1 such that $A \subsetneqq A_1$, there exist $r \in A_1 \backslash A$ and $s \in R\backslash B$ such that $Ar^{-1} \subseteq Bs^{-1}$.*

(b) *For all left ideals A_1 such that $A \subsetneqq A_1$, there exists a left ideal A_2 such that $A_1 \supseteq A_2 \supseteq A$ and* $\mathrm{Hom}_R(A_2/A, R/B) \neq 0$.

(c) $\mathrm{rad}_{E(R/A)} \geqq \mathrm{rad}_{E(R/B)}$.

Proof. (a) $\Rightarrow$ (b). If $A \subsetneqq A_1$, then by assumption there exist $r \in A_1\backslash A$ and $s \in R\backslash B$ with $Ar^{-1} \subseteq Bs^{-1}$. Let A_2 be the left ideal generated by A and r. Then the homomorphism $f : A_2/A \to R/B$ defined by $f(xr + A) = xs + B$ is well-defined and nonzero.

(b) $\Rightarrow$ (c). Note that $\mathrm{rad}_{E(R/A)} \geqq \mathrm{rad}_{E(R/B)}$ if and only if

$$\mathrm{rad}_{E(R/B)}(R/A) = 0.$$

Let $A_1 \supseteq A$ with $A_1/A = \mathrm{rad}_{E(R/B)}(R/A)$. If $A_1/A \neq 0$, then there exists by assumption a left ideal A_2 with $A_1 \supseteq A_2 \supseteq A$ and a homomorphism $0 \neq f \in \mathrm{Hom}_R(A_2/A, R/B)$. Then f extends by the injectivity of $E(R/B)$ to $g : R/A \to E(R/B)$ with $g(A_2/A) \neq 0$, a contradiction.

(c) ⇒ (a). If $A \subsetneqq A_1$, then $\mathrm{rad}_{E(R/B)}(R/A) = 0$ implies there exists $f \in \mathrm{Hom}_R(R/A, E(R/B))$ with $f(A_1/A) \neq 0$. Since $E(R/B)$ is an essential extension of R/B,

$$f(A_1/A) \cap R/B \neq 0,$$

so $f(r + A) = s + B \neq 0$ for some $r \in A_1 \backslash A$ and $s \in R \backslash B$, and then $Ar^{-1} \subseteq Bs^{-1}$.

3.4. *Definition.* For left ideals A and B of R, we say $A \leqq B$ if for each left ideal A_1 with $A \subsetneqq A_1$ there exist $r \in A_1 \backslash A$ and $s \in R \backslash B$ such that $Ar^{-1} \subseteq Bs^{-1}$.

If A and B are left ideals containing an ideal K, then Proposition 3.3 (b) shows that $A \leqq B$ in ${}_R\mathscr{M}$ if and only if $A/K \leqq B/K$ in ${}_{R/K}\mathscr{M}$. If A is an ideal with $A_1/A = \mathrm{rad}_{E(R/B)}(R/A)$, then A_1/A is invariant under R-endomorphisms of R/A, and so A_1 must be an ideal. Thus the proof of Proposition 3.3 shows that if A is an ideal, then $A \leqq B$ if and only if the condition of Definition 3.4 is satisfied for each ideal A_1, with $A \subsetneqq A_1$. If P is a prime ideal in a commutative ring, then $Pr^{-1} = P$ for all $r \notin P$, and so for prime ideals P_1 and P_2 in a commutative ring, $P_1 \leqq P_2$ if and only if $P_1 \subseteq P_2$. The following lemma shows that in general if P_1 and P_2 are prime ideals, then $P_1 \leqq P_2$ implies $P_1 \subseteq P_2$, and so any minimal prime ideal is minimal with respect to the relation $\leqq$.

3.5. LEMMA. *Let A be an ideal of R, and let B be a left ideal of R.*

(a) *If $A \leqq B$ and B is a prime left ideal, then $A \subseteq B$.*

(b) *If $A \subseteq B$ and A is a prime ideal such that R/A satisfies the ascending chain condition for annihilator right ideals, then $A \leqq B$.*

Proof. (a) If $A \leqq B$, then there exist $r \in R \backslash A$ and $s \in R \backslash B$ with $Ar^{-1} \subseteq Bs^{-1}$. For $a \in A$, $aR \subseteq A \subseteq Ar^{-1} \subseteq Bs^{-1}$ implies that $aRs \subseteq B$. Since B is a prime left ideal and $s \notin B$, we must have $a \in B$. Thus $A \subseteq B$.

(b) Since $A \leqq B$ in ${}_R\mathscr{M}$ if and only if $0 \leqq B/A$ in ${}_{R/A}\mathscr{M}$, it suffices to show that in a prime ring R with ascending chain condition on annihilator right ideals, 0 is the only proper torsion ideal, since then $\mathrm{rad}_{E(R)} \geqq \mathrm{rad}_{E(R/C)}$, or equivalently, $0 \leqq C$, for all left ideals C of R. This result is due to E. P. Armendariz (stated in a letter to the author), and can be shown using the methods of Lambek. Proposition 2.2 of [**9**] shows that if R is a semiprime ring with ascending chain condition on annihilator right ideals, then every torsion ideal of ${}_RR$ is of Type 1, i.e., every torsion ideal K satisfies the condition that for all $r \notin K$, there exists $s \in R$ such that $sK = 0$ and $sr \notin K$. This shows that for all $r \notin K$ there exists a right R-homomorphism $f \in \mathrm{Hom}((R/K)_R, R_R)$ such that $f(r) = sr \neq 0$, and so K is closed as a right ideal. But R is a prime ring, and by Proposition 1.2, 0 is a maximal closed ideal of R_R and so $K = 0$ or $K = R$.

3.6. THEOREM. (a) *If R satisfies the maximum condition for ideals, then there is a one-to-one correspondence between maximal torsion radicals of ${}_R\mathscr{M}$ and prime ideals of R minimal with respect to the relation $\leqq$.*

(b) *If R is right Noetherian, then there is a one-to-one correspondence between maximal torsion radicals of ${}_R\mathscr{M}$ and minimal prime ideals of R.*

Proof. (a) If σ is a maximal torsion radical of ${}_R\mathscr{M}$, then by Proposition 3.2, σ is contained in a maximal radical, and thus $\sigma \leqq \text{rad}_{E(R/P)}$ for a prime ideal P. Since σ is maximal, we must have $\sigma = \text{rad}_{E(R/P)}$, and it is clear from Proposition 3.3 (c) that P is minimal with respect to $\leqq$. If P_1 and P_2 are prime ideals with $\text{rad}_{E(R/P_1)} = \text{rad}_{E(R/P_2)}$, then $P_1 \leqq P_2$ and $P_2 \leqq P_1$, which shows by Lemma 3.5 that $P_1 = P_2$. Thus the correspondence between maximal torsion radicals and prime ideals minimal with respect to $\leqq$ is one-to-one, since for all σ, $\sigma \leqq \text{rad}_{E(R/P)}$, with P minimal.

(b) If R is right Noetherian, then for every prime ideal P, R/P satisfies the ascending chain condition for annihilator right ideals, and so Lemma 3.5 shows that the prime ideals minimal with respect to $\leqq$ are precisely the prime ideals minimal with respect to $\subseteq$. The result then follows from part (a).

Let R be the ring of all linear transformations of a vector space of countably infinite dimension. The only ideals of R are 0, R, and the ideal F of all transformations of finite rank [**7**, Chapter IV, §17, Theorem 1], and so R certainly satisfies the maximum condition for ideals. Since R is von Neumann regular, F is a torsion ideal [**9**, Proposition 2.3]. It can be seen easily that F is a prime ideal, and so both $\text{rad}_{E(R/F)}$ and $\text{rad}_{E(R)}$ are maximal torsion radicals of ${}_R\mathscr{M}$. This example shows that the maximum condition for ideals is not sufficient to guarantee a one-to-one correspondence between maximal torsion radicals and minimal prime ideals. The proof of Theorem 3.6 shows that to guarantee this one-to-one correspondence it is sufficient to assume that every proper radical of ${}_R\mathscr{M}$ is contained in a maximal radical and that R/P satisfies the ascending chain condition for annihilator right ideals, for every minimal prime ideal P of R.

3.7. COROLLARY. *Let σ be a torsion radical of ${}_R\mathscr{M}$, with $\sigma(R) = K$. If R is commutative, Noetherian, then the following conditions are equivalent:*

(a) *σ is a maximal torsion radical.*

(b) *$\sigma = \text{rad}_{E(R/P)}$ for a minimal prime ideal P of R.*

(c) *σ is saturated and every nonzero injective R/K-module is faithful.*

(d) *σ is saturated and K is a maximal torsion ideal.*

Proof. Conditions (a) and (b) are equivalent by Theorem 3.6. Conditions (a), (c), and (d) are equivalent by Theorem 2.4 and Proposition 2.10, since [**9**, Proposition 2.2 (1)] together with the proof of Lemma 3.5 shows that every torsion ideal of R/K is closed.

If R is left Noetherian, then Corollary 2.6 shows that every maximal torsion radical σ is a prime torsion radical. Thus $\sigma = \text{rad}_M$ for an indecomposable, injective module ${}_RM$ [**11**, Corollary 1.2], and then $\sigma = \mathrm{rad}_{E(R/A)}$, for a critical prime left ideal A [**10**, Corollary 2.14]. For the sake of completeness we give here a short proof that if R is left Noetherian, then for any nonzero injective module ${}_RM$, there exists a critical prime left ideal A with $\mathrm{rad}_{E(R/A)} \geqq \mathrm{rad}_M$. If $\sigma = \mathrm{rad}_M$, then let P be a maximal σ-closed ideal, and let A be maximal in the set of σ-closed left ideals containing P. Clearly A is critical, and for any left ideal A_1 with $A \subsetneqq A_1$, $P \subseteq \mathrm{Ann}(A_1/A)$ and $\mathrm{Ann}(A_1/A)$ is σ-closed, so $P = \mathrm{Ann}(A_1/A)$. If $a, b \in R$ and $aRb \subseteq A$, with $b \notin A$, then

$$a \in \mathrm{Ann}(Rb + A/A) = P \subseteq A,$$

so A is also prime. The next theorem follows immediately from this result.

3.8. THEOREM. *Let σ be a torsion radical of ${}_R\mathcal{M}$. If R is left Noetherian, then σ is a maximal torsion radical $\Leftrightarrow$ $\sigma = \mathrm{rad}_{E(R/A)}$ for a critical prime left ideal A minimal with respect to $\leqq$.*

We complete this section by applying some of our previous results to the case when the ring R is left hereditary and left Noetherian.

3.9. LEMMA. *If R is left hereditary and left Noetherian, and K is a torsion ideal of R, then R/K is left hereditary.*

Proof. A module ${}_RM$ is called fully divisible if it can be expressed as a quotient of a direct sum of copies of $E(R)$. Every injective module is fully divisible, and it was noted in [**2**] that every fully divisible left R-module is injective if and only if R is left hereditary and left Noetherian. If K is a torsion ideal of R, then the R/K-injective envelope and the R-injective envelope of R/K coincide, so every fully divisible R/K-module is fully divisible as an R-module. If R is left hereditary and left Noetherian, then every fully divisible R/K-module is injective as an R-module and hence as an R/K-module, so this shows that R/K is left hereditary.

3.10. THEOREM. *Let σ be a torsion radical of ${}_R\mathcal{M}$. If R is left hereditary and left Noetherian, then σ is a maximal torsion radical $\Leftrightarrow$ R_σ is a simple Artinian ring.*

Proof. Note that since R is left hereditary and left Noetherian, $Q_\sigma \approx R_\sigma \otimes_R -$. If σ is a maximal torsion radical, then Theorem 3.1 implies that R_σ is left Artinian. By Lemma 3.9, $R/\sigma(R)$ is left hereditary and therefore has zero singular ideal, and so Theorem 2.8 implies that R_σ is a prime ring. It follows that R_σ is simple Artinian, since a left Artinian ring is prime if and only if simple.

The converse follows from Theorem 2.12.

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