ON A KIND OF HOMOTOPY MANIFOLD

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1. Introduction. In a recent paper **(6)**, S. T. Hu investigated the initial projection from the mth enveloping space of a topological space X into X and proved that, under some local conditions on X, the initial projection is a fibering. In a subsequent paper **(7)**, Hu showed that the terminal projection from the mth enveloping space is a fibering without assuming the local conditions on X and in **(8)** he used the terminal projection from the second enveloping space in his topological immersion theorem.

The objective of the present paper is to give a simultaneous study of both the initial and terminal projections from a subspace $Z(X)$ of the second enveloping space of a space X. $Z(X)$ is shown to be an isotopy functor and a kind of homotopy manifold is obtained by imposing conditions in terms of these two projections on X. Specifically, our principal result shows that such spaces, which include manifolds, are homotopically homogeneous in the sense of M. L. Curtis **(1; 2)**.

2. The subspace $Z(X)$**.** Let X be an arbitrary topological space. By means of the diagonal embedding we can identify X with the diagonal of the product $X \times X$. The second enveloping space $E(X)$ of X is the subspace of the space of paths in $X \times X$ with the compact–open topology, which consists of all paths $\sigma\colon I \to X \times X$ such that $\sigma(t) \in X$ if and only if $t = 0$. The initial projection $p\colon E(X) \to X$ is defined by taking $p(\sigma) = \sigma(0)$ for every σ in $E(X)$. The second residual space $R(X)$ is the space $X \times X - X$ and the terminal projection $q\colon E(X) \to R(X)$ is the map defined by taking $q(\sigma) = \sigma(1)$ for every σ in $E(X)$. Define $\pi\colon R(X) \to X$ by projection on the first coordinate and let $Z(X)$ denote the subspace of $E(X)$ which consists of all paths $\sigma \in E(X)$ such that $p(\sigma) = \pi q(\sigma)$.

Throughout this paper denote by

$$p\colon Z(X) \to X \qquad \text{and} \qquad q\colon Z(X) \to R(X)$$

the restrictions of p and q, respectively, to the subspace $Z(X)$.

Now, let $f\colon X \to Y$ be an embedding of a space X into a space Y. Then f defines an embedding

$$f \times f\colon X \times X \to Y \times Y$$

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given by $(f \times f)(x_1, x_2) = (fx_1, fx_2)$. Let $\sigma \in Z(X)$ and consider the composition $\tau = (f \times f) \circ \sigma$. An easy verification shows that $\tau \in Z(Y)$ and therefore f defines an embedding

$$Z(f): Z(X) \to Z(Y).$$

Furthermore, by obvious modifications of the discussion in **(5**, §6**)**, we have the following proposition.

Proposition 2.1. *$Z(X)$ is an isotopy functor of the category of topological spaces and embeddings.*

3. Projectionally homogeneous spaces. A topological space X is said to be *projectionally homogeneous* if the following two conditions on the initial and terminal projections are satisfied.

(PH1) *$p: Z(X) \to X$ has a slicing structure $\{\omega, \phi_U\}$ in the sense of* Hu **(4)**.

(PH2) *For each $U \in \omega$, $q\phi_U(x, \sigma) = q\phi_U(x, \tau)$ whenever $q(\sigma) = q(\tau)$ and the function*

$$q_U: U \times q[p^{-1}(U)] \to R(X)$$

defined by

$$q_U(x, q(\sigma)) = q\phi_U(x, \sigma) \qquad (x \in U, q(\sigma) \in q[p^{-1}(U)])$$

is continuous.

If X is an ANR (metric), then by **(11)**, (PH1) is equivalent to the condition that $p: Z(X) \to X$ have the path lifting property.

We consider some examples of projectionally homogeneous spaces.

A space X is *locally homogeneous* if, for every $x_0 \in X$, there exists a neighbourhood U of x_0 in X together with a continuous map $M: X \times U \times U \to X$ which satisfies the following conditions:

(LH1) *For every pair of points a, b in U, the map $M[a, b]: X \to X$ defined by $M[a, b](x) = M(x, a, b)$ is a homeomorphism on X satisfying $M[a, b](a) = b$.*

(LH2) *If $a \in U$, $M[a, a]$ is the identity map on X.*

Proposition 3.1. *If X is locally homogeneous, then it is projectionally homogeneous.*

Proof. Let $x_0 \in X$ and choose a neighbourhood U of x_0 in X with a continuous map $M: X \times U \times U \to X$ satisfying (LH1, 2). In order to prove that $p: Z(X) \to X$ has a slicing structure, we define a continuous map

$$\phi_U: U \times p^{-1}(U) \to Z(X)$$

as follows. For each $x \in U$ and $\sigma \in p^{-1}(U)$, $\phi_U(x, \sigma)$ is the path $\tau: I \to X \times X$ given by $\tau_i(t) = M[\sigma(0), x](\sigma_i(t))$, $i = 1, 2$, where $\sigma_i(t)$ and $\tau_i(t)$ denote the

ith coordinates of the points $\sigma(t)$ and $\tau(t)$ in $X \times X$. By (LH1) and the equality $\sigma_1(1) = \sigma(0)$, it follows that $\tau \in Z(X)$. That U is a slicing neighbourhood and ϕ_U is a slicing function can be readily checked using (LH1, 2). This verifies (PH1).

The condition (LH1) and the continuity of M implies (PH2). This completes the proof of 3.1.

Since topological groups and locally euclidean spaces are examples of locally homogeneous spaces, we have the following corollary.

COROLLARY 3.2. *Topological groups and locally euclidean spaces are projectionally homogeneous.*

PROPOSITION 3.3. *If X is a non-degenerate compact metric* AR, *then it is not projectionally homogeneous.*

Proof. Suppose X satisfies (PH1). By **(4)**,

$$p: Z(X) \to X$$

has the absolute covering homotopy property. Since X is contractible, there is a homotopy

$$h_t: X \to X \qquad (0 \leqslant t \leqslant 1)$$

such that h_0 is the identity map and $h_1(X) = x_0 \in X$. Define a homotopy

$$f_t: X \to X \qquad (0 \leqslant t \leqslant 1)$$

by taking $f_t(x) = h_{(1-t)}(x)$ for every x in X. Since $\{h_t\}$ is a contraction of the non-degenerate space X into x_0, there is a path $\xi: I \to X$ such that $\xi(0) = x_0$ and $\xi(t) \neq x_0$ for every $t > 0$. Define a path $\sigma: I \to X \times X$ by taking

$$\sigma(t) = (x_0, \xi(t)) \qquad (t \in 1).$$

A straightforward verification shows that $\sigma \in p^{-1}(x_0) \subset Z(X)$. Let

$$g: X \to Z(X)$$

denote the constant map $g(X) = \sigma$. Then f_t is a homotopy of the map $f = p \circ g$. Therefore there exists a homotopy

$$g_t: X \to Z(X)$$

which covers f_t, i.e. $pg_t = f_t$ for each $t \in 1$. Then the continuous map $g: X \to X$ satisfying $qg_1(x) = (x, g(x)) \in R(X)$ $(x \in X)$ contradicts the fact that X has the fixed-point property. This completes the proof of 3.3.

Combining 3.1 and 3.3, we obtain the following slightly stronger assertion of E. Fadell **(3)** in which he considers connected locally homogeneous spaces.

COROLLARY 3.4. *There exist no non-degenerate locally homogeneous compact metric* AR'*s*.

4. Homotopically homogeneous spaces. In **(1)**, Curtis defines a space X to be *homotopically homogeneous* if $(R(x), \pi, X)$ is a Hurewicz fibre space, that is, if $\pi: R(X) \to X$ has the path lifting property.

We give a sufficient condition for a projectionally homogeneous space to be homotopically homogeneous.

PROPOSITION 4.1. *If a paracompact Hausdorff space X is projectionally homogeneous and the terminal projection*

$$q: Z(X) \to R(X)$$

maps $Z(X)$ onto $R(X)$, then X is homotopically homogeneous.

Proof. Since X is paracompact Hausdorff, it suffices to show that the map $\pi: R(X) \to X$ has a slicing structure. For this purpose consider a slicing structure $\{\omega, \phi_U\}$ of the projection $p: Z(X) \to X$. Define a function

$$\psi_U: U \times \pi^{-1}(U) \to R(X) \qquad (U \in \omega)$$

as follows. Take $x \in U$ and $r \in \pi^{-1}(U)$. Since q is surjective, there is a path $\sigma \in Z(X)$ for which $q(\sigma) = r$. By the properties of the slicing function ϕ_U of p, $p\phi_U(x, \sigma) = x$. Thus $q\phi_U(x, \sigma) = (x, y) \in R(X)$ where y is some point in $\pi^{-1}(x)$. Set $\psi_0(x, r) = (x, y)$. By the first part of (PH2), it follows that $\psi_U(x, r)$ is independent of the choice of σ satisfying $q(\sigma) = r$ and thus is a well-defined function. This completes the construction of ψ_U. By (PH2), one can see that ψ_U is continuous. Obviously

$$\pi\psi_U(x, r) = x$$

for every $x \in U$ and $r \in \pi^{-1}(U)$. Since $\phi_U(p(\sigma), \sigma) = \sigma$ for every $\sigma \in p^{-1}(U)$, it follows that

$$\psi_U(\pi(r), r) = r$$

for every $r \in \pi^{-1}(U)$. Therefore $\{\omega, \psi_U\}$ is a slicing structure for π. This completes the proof.

The following corollary is a consequence of 4.1 and a slight modification of the proof of **(7**, 4.3**)**.

COROLLARY 4.2. *If a paracompact Hausdorff space X is projectionally homogeneous, then it is homotopically homogeneous provided the following two conditions are satisfied:*

(i) *X is pathwise accessible* **(7**, §4**)**.

(ii) *$R(X)$ is pathwise connected.*

THEOREM 4.3. *If a paracompact Hausdorff space X is pathwise connected and projectionally homogeneous, then it is homotopically homogeneous.*

Proof. According to 4.1 it suffices to show that the projection $q: Z(X) \to R(X)$ is surjective. For this purpose let (x, y) be any point in $R(X)$. Since X is pathwise connected, there exists a path $\tau: I \to X$ such that $\tau(0) = y$ and $\tau(1) = x$. Let

$$t_0 = \min\{t \in I | \tau(t) = x\}.$$

Since $x \neq y$, $t_0 > 0$. Define a path $\xi: I \to X$ by taking $\xi(t) = \tau(t_0(1 - t))$. Then ξ is a path from x to y with $\xi(t) \neq x$ for $t > 0$. Next, define a path

$$\sigma: I \to X \times X$$

by taking $\sigma(t) = (x, \xi(t))$ for every $t \in I$. One can easilv verify that $\sigma \in p^{-1}(x)$ and that $q(\sigma) = (x, y)$. This completes the proof of 4.3.

It is clear that if X is locally conic **(6**, §8**)**, then it is locally contractible and hence locally pathwise connected. Thus by **(9**, p. 89**)** we have the following corollary.

COROLLARY 4.4. *If a connected paracompact Hausdorff space X is locally conic and projectionally homogeneous, then it is homotopically homogeneous.*

By **(1**, §4**)** and 4.4, we have the following two theorems.

THEOREM 4.5. *If X is a connected, compact metric, finite-dimensional, locally conic, projectionally homogeneous space, then X is a Kosiński r-space* **(10)**.

THEOREM 4.6. *A connected finite-dimensional projectionally homogeneous simplicial polytope is a homotopically homogeneous simplicial polytope, an r-simplicial polytope, a homotopy manifold, and hence a homology generalized manifold.*

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