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THE BEHAVIOUR OF THE FOURTH TYPE OF LAURICELLA'S HYPERGEOMETRIC SERIES IN n VARIABLES NEAR THE BOUNDARIES OF ITS CONVERGENCE REGION

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Abstract

For Lauricella's hypergeometric function $F_D^{(n)}$ of n variables, we prove two formulas exhibiting its behaviour near the boundaries of the n-dimensional region of convergence of the multiple series defining it. Each of these results can be applied to deduce the corresponding properties of several simpler hypergeometric functions of one, two, and more variables.

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1. Introduction

In terms of the Pochhammer symbol $(\lambda)_m$ defined by

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } m = 0, \\ \lambda(\lambda+1)\cdots(\lambda+m-1), & \text{if } m = 1, 2, 3, \ldots, \end{cases}$$

the Gauss hypergeometric function $F(a, b; c; z)$ is given by the power series:

$$F(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!} = {}_2F_1(a, b; c; z), \tag{1}$$

which converges (absolutely), for all values of the parameters a, b, and c, in the open unit disk $\mathscr{U} = \{z : |z| < 1\}$, provided that no zeros appear in the denominator of (1), that is, provided that $c \neq 0, -1, -2, \ldots$. Indeed, in the theory of hypergeometric functions, it is fairly well known that the Gaussian series (1) converges also when

$$z = 1 \quad \text{if} \quad \operatorname{Re}(c - a - b) > 0,$$

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and we have the Gauss summation theorem:

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\qquad \operatorname{Re}(c-a-b)>0. \tag{2}$$

In case, however, that $\operatorname{Re}(c-a-b)\leq 0$, the value $F(a,b;c;1)$ does not exist, and we have the order estimates:

$$F(a,b;c;1-\rho)=\begin{cases} O(\log\rho), & (\rho\to 0+),\ \text{if } c-a-b=0,\\ O(\rho^{\operatorname{Re}(c-a-b)}), & (\rho\to 0+),\ \text{if } \operatorname{Re}(c-a-b)<0.\end{cases}$$

The multiple hypergeometric series (see [3, p. 113] and [11, p. 33])

$$F_D^{(n)}(a,b_1,\dots,b_n;c;x_1,\dots,x_n)=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a)_{m_1+\dots+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

defining Lauricella's fourth function $F_D^{(n)}$ of n variables, converges absolutely inside the n-dimensional cuboid:

$$|x_j|<1\qquad (j=1,\dots,n),$$

and has the following values [cf. Equation (2)]:

$$F_D^{(n)}(a,b_1,\dots,b_n;c;1,x_2,\dots,x_n)=\frac{\Gamma(c)\Gamma(c-a-b_1)}{\Gamma(c-a)\Gamma(c-b_1)}F_D^{(n-1)}(a,b_2,\dots,b_n;c-b_1;x_2,\dots,x_n),$$
$$\max\{|x_2|,\dots,|x_n|\}<1;\quad \operatorname{Re}(c-a-b_1)>0;$$

$$F_D^{(n)}(a,b_1,\dots,b_n;c;1,1,x_3,\dots,x_n)=\frac{\Gamma(c)\Gamma(c-a-b_1-b_2)}{\Gamma(c-a)\Gamma(c-b_1-b_2)}F_D^{(n-2)}(a,b_3,\dots,b_n;c-b_1-b_2;x_3,\dots,x_n),$$
$$\max\{|x_3|,\dots,|x_n|\}<1;\quad \operatorname{Re}(c-a-b_1-b_2)>0;$$

$$F_D^{(n)}(a,b_1,\dots,b_n;c;1,\dots,1,x_n)=\frac{\Gamma(c)\Gamma(c-a-b_1-\dots-b_{n-1})}{\Gamma(c-a)\Gamma(c-b_1-\dots-b_{n-1})}F(a,b_n;c-b_1-\dots-b_{n-1};x_n),$$
$$|x_n|<1;\quad \operatorname{Re}(c-a-b_1-\dots-b_{n-1})>0;$$

and [3, p. 150]

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; 1, \ldots, 1) = \frac{\Gamma(c)\Gamma(c-a-b_1-\cdots-b_n)}{\Gamma(c-a)\Gamma(c-b_1-\cdots-b_n)}, \quad \operatorname{Re}(c-a-b_1-\cdots-b_n) > 0.$$

In general, we have

$$\begin{aligned} &F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) \\ &= \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{(a)_{m_1+\cdots+m_k}(b_1)_{m_1}\ldots(b_k)_{m_k}}{(c)_{m_1+\cdots+m_k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_k^{m_k}}{m_k!} \\ &\quad \times F_D^{(n-k)}(a+m_1+\cdots+m_k, b_{k+1}, \ldots, b_n; c+m_1+\cdots+m_k; x_{k+1}, \ldots, x_n), \\ &\qquad (k = 0, 1, \ldots, n), \end{aligned} \tag{3}$$

where $F_D^{(0)} = 1$, $F_D^{(1)} = F$ and $F_D^{(2)} = F_1$ in terms of Appell's double hypergeometric function F_1 defined by [1, p. 14]

$$F_1(a, b_1, b_2; c; x, y) = \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2}}{(c)_{m_1+m_2}} \frac{x^{m_1}}{m_1!}\frac{y^{m_2}}{m_2!}, \quad \max\{|x|, |y|\} < 1;$$

and

$$\begin{aligned} &F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) \\ &= \sum_{m_1,\ldots,m_{n-1}=0}^{\infty} F(a+m_1+\cdots+m_{n-1}, b_1+\ldots+b_n+m_1+\cdots+m_{n-1}; \\ &\qquad c+m_1+\ldots+m_{n-1}; x_n) \\ &\quad \times \frac{(a)_{m_1+\cdots+m_{n-1}}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}}{(c)_{m_1+\cdots+m_{n-1}}} \frac{(x_1-x_n)^{m_1}}{m_1!}\cdots\frac{(x_{n-1}-x_n)^{m_{n-1}}}{m_{n-1}!}. \end{aligned} \tag{4}$$

This last identity (4) would follow from (3) if we employ the well-known transformation:

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right)$$

and its straightforward consequence [3, p. 149]:

$$\begin{aligned} &F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) \\ &= (1-x_1)^{-a} F_D^{(n)}\left(a, c-b_1-\cdots-b_n, b_2, \ldots, b_n; c; \frac{x_1}{x_1-1}, \frac{x_1-x_2}{x_1-1}, \ldots, \frac{x_1-x_n}{x_1-1}\right). \end{aligned}$$

The behaviour of the Gauss function (1), and that of many of its generalizations in two and three variables, near the boundaries of the regions of convergence of the

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series defining them, are known (see, for example, [6, 7, 8, 9]). Many of the results exhibiting such behaviour have also been used in solving various boundary value problems involving the celebrated Euler-Darboux equation and in the study of certain operators of fractional calculus (*cf.* [5] and [12]). In particular, Saigo [9] gave three formulas for the Lauricella function $F_D^{(3)}(x, y, z)$ which exhibit its behaviour

(i) near the corner point $x = y = z = 1$,
(ii) near the edge $y = z = 1$, and
(iii) near the side $z = 1$

of the unit cube representing the region of convergence of the triple series defining the function. The object of the present paper is to derive similar results in the n-dimensional case. With a view to stating our main results as precisely as practicable, we shall need a number of definitions, notation, and conventions from the theory of multivariable hypergeometric functions (see [2, 11] for details). For the sake of convenience and ready reference, we recall some of these definitions as follows.

Srivastava and Daoust ([10]; see also [11, p. 37]) defined a general multivariable hypergeometric function by means of the multiple series

$$
\begin{aligned}
&F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
(5)\qquad &= F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\left(\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\varphi'];\ \ldots;\ [(b^{(n)}):\varphi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ \ldots;\ [(d^{(n)}):\delta^{(n)}]; \end{matrix}\ x_1,\ldots,x_n\right) \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} \Omega(m_1,\ldots,m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},
\end{aligned}
$$

where

$$
\Omega(m_1,\ldots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\varphi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\varphi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}
$$

with real and nonnegative coefficients

$$
\begin{aligned}
&\theta_j^{(k)},\quad j=1,\ldots,A; &&\varphi_j^{(k)},\quad j=1,\ldots,B^{(k)}; \\
&\psi_j^{(k)},\quad j=1,\ldots,C; &&\delta_j^{(k)},\quad j=1,\ldots,D^{(k)}
\end{aligned}
$$

for $k = 1, \ldots, n$. Here, for convenience, (a) abbreviates the array of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b'), (c), *et cetera*.

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We shall also need the following alternative notations for some obvious special cases of (5).

$$M^{(n)}\begin{bmatrix} a;\ b\colon\ c_1;\ \cdots;\ c_n; & \\ & x_1,\ldots,x_n \\ d;\ e\colon —;\ \cdots;\ —; & \end{bmatrix}$$

$$(6)\quad = F^{2:1;\cdots;1}_{2:0;\cdots;0}\begin{pmatrix} [a\colon 1,\ldots,1,0],\ [b\colon 0,\ldots,0,1,1]\colon [c_1\colon 1];\ \cdots;\ [c_n\colon 1]; & \\ & x_1,\ldots,x_n \\ [d\colon 1,\ldots,1,0],\ [e\colon 0,\ldots,0,1,1]\colon \quad —;\ \cdots;\quad —; & \end{pmatrix}$$

$$(n \geq 2),$$

which, for $n = 2, 3, 4$, contains some Kampé de Fériet functions $F^{p:r;u}_{q:s;v}$, some Srivastava functions $F^{(3)}$, and the function $F^{(4)}$ (see [9]). In particular, we have

$$M^{(2)}\begin{bmatrix} a;\ b\colon\ c_1;\ c_2; & \\ & x_1,x_2 \\ d;\ e\colon —;\ —; & \end{bmatrix} = F^{1:2;1}_{1:1;0}\begin{bmatrix} b\colon a, c_1;\ c_2; & \\ & x_1,x_2 \\ e\colon\quad d;\ —; & \end{bmatrix},$$

$$M^{(3)}\begin{bmatrix} a;\ b\colon\ c_1;\ c_2;\ c_3; & \\ & x_1,x_2,x_3 \\ d;\ e\colon —;\ —;\ —; & \end{bmatrix} = F^{(3)}\begin{bmatrix} —::\ a;\ b;\ —;\ c_1;\ c_2;\ c_3; & \\ & x_1,x_2,x_3 \\ —::\ d;\ e;\ —;\ —;\ —;\ —; & \end{bmatrix},$$

and

$$M^{(4)}\begin{bmatrix} a;\ b\colon\ c_1;\ c_2;\ c_3;\ c_4; & \\ & x_1,x_2,x_3,x_4 \\ d;\ e\colon —;\ —;\ —;\ —; & \end{bmatrix}$$

$$= F^{(4)}\begin{bmatrix} —:::\ a;\ —;\ —;\ —::\ —;\ —;\ —;\ —;\ b\colon\ c_1;\ c_2;\ c_3;\ c_4; & \\ & x_1,x_2,x_3,x_4 \\ —:::\ d;\ —;\ —;\ —::\ —;\ —;\ —;\ —;\ e\colon —;\ —;\ —;\ —; & \end{bmatrix};$$

$$N^{(n)}\begin{bmatrix} a\colon b_1, b_2;\ c_2;\ \cdots;\ c_n; & \\ & x_1,\ldots,x_n \\ d\colon\quad e;\ —;\ \cdots;\ —; & \end{bmatrix}$$

$$= F^{1:2;1;\cdots;1}_{1:1;0;\cdots;0}\begin{pmatrix} [a\colon 1,\ldots,1]\colon [b_1\colon 1], [b_2\colon 1];\ [c_2;\ 1];\ \cdots;\ [c_n\colon 1]; & \\ & x_1,\ldots,x_n \\ [d\colon 1,\ldots,1]\colon\quad [e\colon 1];\quad —;\ \cdots;\quad —; & \end{pmatrix},$$

which, in the case $n = 1$, becomes the Clausen function ${}_3F_2$.

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We now state our main results involving the Lauricella function $F_D^{(n)}$:

(7)

$$
\begin{aligned}
&F_D^{(n)}(a, b_1, \dots, b_n; a+b_n; x_1, \dots, x_{n-1}, 1-\rho) \\
&\quad = -\frac{\Gamma(a+b_n)}{\Gamma(a)\Gamma(b_n)} \prod_{i=1}^{n-1} \left\{(1-x_i)^{-b_i}\right\} \left[2\gamma + \psi(a) + \psi(b_n) + \log \rho\right] \\
&\qquad - \frac{\Gamma(a+b_n)}{\Gamma(a)\Gamma(b_n)} \\
&\qquad \times \left\{ \frac{b_1}{a} x_1 M^{(2)} \left[\begin{matrix} a;\ b_1+1\colon\ 1;\ 1; \\ a+1;\ 2\colon —;\ —; \end{matrix} \ x_1, x_1 \right] \right. \\
&\qquad\quad + M^{(2)} \left[\begin{matrix} a;\ b_1\colon\ 1;\ 1; \\ a+1;\ 1\colon —;\ —; \end{matrix} \ x_1, x_1 \right] \sum_{r=2}^{n-1} \frac{b_r}{a} x_r \prod_{i=2}^{r-1} \left\{(1-x_i)^{-b_i}\right\} F(b_r+1, 1; 2; x_r) \\
&\qquad\quad + \sum_{r=2}^{n-1} \sum_{s=2}^{r-1} \frac{b_r b_s}{a+1} x_r x_s \\
&\qquad\qquad \times M^{(s+1)} \left[\begin{matrix} a+1;\ b_s+1\colon\ b_1;\ \cdots;\ b_{s-1};\ 1;\ 1; \\ a+2;\ 2\colon —;\ \cdots;\ —;\ —;\ —; \end{matrix} \ x_1, \dots, x_{s-1}, x_s, x_s \right] \\
&\qquad\qquad \times \prod_{i=s+1}^{r-1} \left\{(1-x_i)^{-b_i}\right\} F(b_r+1, 1; 2; x_r) \\
&\qquad\quad + \sum_{r=2}^{n-1} \frac{b_r(b_r+1)}{2(a+1)} x_r^2 \\
&\qquad\qquad \left. \times M^{(r+1)} \left[\begin{matrix} a+1;\ b_r+2\colon\ b_1;\ \cdots;\ b_{r-1};\ 1;\ 1; \\ a+2;\ 3\colon —;\ \cdots;\ —;\ —;\ —; \end{matrix} \ x_1, \dots, x_{r-1}, x_r, x_r \right] \right\} \\
&\qquad + o(1), \qquad (\rho \to 0+);
\end{aligned}
$$

(8)

$$
\begin{aligned}
&F_D^{(n)}(a, b_1, \dots, b_n; a+b_{k+1}+\cdots+b_n; x_1, \dots, x_k, 1-\rho x_{k+1}, \dots, 1-\rho x_n) \\
&\quad = \frac{\Gamma(a+b_{k+1}\dots+b_n)}{\Gamma(a)\Gamma(b_{k+1}+\cdots+b_n+1)} \sum_{s=1}^{n-k-1} b_{n-s} \left(1-\frac{x_{n-s}}{x_n}\right) \prod_{i=1}^{k} \left\{(1-x_i)^{-b_i}\right\} \\
&\qquad \times N^{(s)} \left[\begin{matrix} 1\colon\ b_{n-s}+1, 1;\ b_{n-s+1};\ \cdots;\ b_{n-1}; \\ b_{k+1}+\cdots+b_{n+1}\colon\ 2;\ —;\ \cdots;\ —; \end{matrix} \ 1-\frac{x_{n-s}}{x_n}, \dots, 1-\frac{x_{n-1}}{x_n} \right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\Gamma(a+b_{k+1}+\cdots+b_n)}{\Gamma(a)\Gamma(b_{k+1}+\cdots+b_n)} \prod_{i=1}^{k} \left\{(1-x_i)^{-b_i}\right\} \\
&\qquad \times \left[2\gamma + \psi(a) + \psi(b_{k+1}+\cdots+b_n) + \log \rho x_n\right] \\
&+ o(1), \qquad (\rho \to 0+), \quad (k = 0, 1, \ldots, n-2),
\end{aligned}
$$

where, as usual, $\psi(z)$ denotes the Psi function defined by

$$
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(z)dz,
$$

and

(9) $$\gamma = -\psi(1) \cong 0.5772156649\cdots$$

denotes the Euler-Mascheroni constant.

In our derivations of these properties of the Lauricella function $F_D^{(n)}$, we shall also require the following formulas which are given, for example, in [4]:

(10) $$
\begin{aligned}
F(a,b;a+b;z) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(m!)^2} \\
&\quad \times \left[2\psi(m+1) - \psi(a+m) - \psi(b+m) - \log(1-z)\right](1-z)^m;
\end{aligned}
$$

(11) $$
\begin{aligned}
F(a,b;a+b-n;z) &= \frac{\Gamma(n)\Gamma(a+b-n)}{\Gamma(a)\Gamma(b)} (1-z)^{-n} \sum_{m=0}^{n-1} \frac{(a-n)_m (b-n)_m}{m!(1-n)_m} (1-z)^m \\
&\quad + \frac{(-1)^n \Gamma(a+b-n)}{\Gamma(a-n)\Gamma(b-n)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m!(m+n)!} (1-z)^m \\
&\quad \times \left[\psi(m+1) + \psi(m+n+1) - \psi(a+m) - \psi(b+m) - \log(1-z)\right].
\end{aligned}
$$

2. Derivation of Formula (7)

Applying the formulas (10) and (3), we have

$$
\begin{aligned}
&F_D^{(n)}(a, b_1, \ldots, b_n; a+b_n; x_1, \ldots, x_{n-1}, 1-\rho) \\
&= \frac{\Gamma(a+b_n)}{\Gamma(a)\Gamma(b_n)} \sum_{m_1,\ldots,m_{n-1},k=0}^{\infty} \frac{(a)_{m_1+\ldots+m_{n-1}+k} (b_1)_{m_1} \cdots (b_{n-1})_{m_{n-1}} (b_n)_k}{(a)_{m_1+\ldots+m_{n-1}}} \\
&\quad \times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!} \frac{\rho^k}{(k!)^2} \\
&\quad \times \left[2\psi(k+1) - \psi(a+m_1+\ldots+m_{n-1}+k) - \psi(b_n+k) - \log \rho\right] \\
&= \frac{\Gamma(a+b_n)}{\Gamma(a)\Gamma(b_n)} \{2S_1 - S_2 - S_3 - S_4\}, \quad \text{say.}
\end{aligned}
$$

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Let us now evaluate each series S_i $(i = 1, 2, 3, 4)$. Firstly, the sum S_4 is partitioned into two terms for $k = 0$ and $k \geq 1$, and we have

$$S_4 = \log\rho \prod_{i=1}^{n-1}\left\{(1-x_i)^{-b_i}\right\} + \rho a b_n \log\rho \sum_{m_1,\ldots,m_{n-1},k=0}^{\infty} \frac{(a+1)_{m_1+\ldots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n+1)_k}{(a)_{m_1+\ldots+m_{n-1}}} \times \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{\rho^k}{\{(k+1)!\}^2}.$$

By noting that [see also Equation (9)]

$$\psi(z+k) = \psi(z) + \sum_{l=0}^{k-1}\frac{(z)_l}{(z)_{l+1}},$$

we have

$$\begin{aligned} S_1 &= \sum_{m_1,\ldots,m_{n-1},k=0}^{\infty} \frac{(a)_{m_1+\ldots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n)_k}{(a)_{m_1+\ldots+m_{n-1}}} \times \left(-\gamma + \sum_{l=0}^{k-1}\frac{(1)_l}{(2)_l}\right)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{\rho^k}{(k!)^2} \\ &= -\gamma \prod_{i=1}^{n-1}\left\{(1-x_i)^{-b_i}\right\} \\ &\quad -\gamma\rho a b_n \sum_{m_1,\ldots,m_{n-1},k=0}^{\infty} \frac{(a+1)_{m_1+\ldots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n+1)_k}{(a)_{m_1+\ldots+m_{n-1}}} \times \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_{n-1}^{m_{n-1}}}{m_{n-1}!}\frac{\rho^k}{\{(k+1)!\}^2} \\ &\quad +\rho a b_n \sum_{m_1,\ldots,m_{n-1},l,p=0}^{\infty} \frac{(a+1)_{m_1+\ldots+m_{n-1}+l+p}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n+1)_{l+p}(1)_l}{(a)_{m_1+\ldots+m_{n-1}}\left\{(2)_{l+p}\right\}^2(2)_l\, m_1!\cdots m_{n-1}!} \times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^{l+p}, \end{aligned}$$

where we have set $k = l + p + 1$ in the last term.

Similarly, we have

$$S_3 = \sum_{m_1,\ldots,m_{n-1},k=0}^{\infty} \frac{(a)_{m_1+\ldots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n)_k}{(a)_{m_1+\ldots+m_{n-1}}\, m_1!\cdots m_{n-1}!(k!)^2} \times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^k\left[\psi(b_n) + \frac{1}{b_n}\sum_{l=0}^{k-1}\frac{(b_n)_l}{(b_n+1)_l}\right]$$

$$= \psi(b_n)\prod_{i=1}^{n-1}\{(1-x_i)^{-b_i}\}$$

$$+\rho a b_n \psi(b_n) \sum_{m_1,\dots,m_{n-1},k=0}^{\infty} \frac{(a+1)_{m_1+\dots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n+1)_k}{(a)_{m_1+\dots+m_{n-1}}\, m_1!\cdots m_{n-1}!\,\{(2)_k\}^2} \times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^k$$

$$+\rho a \sum_{m_1,\dots,m_{n-1},l,p=0}^{\infty} \frac{(a+1)_{m_1+\dots+m_{n-1}+l+p}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n+1)_{l+p}(b_n)_l}{(a)_{m_1+\dots+m_{n-1}}\, m_1!\cdots m_{n-1}!\,\left\{(2)_{l+p}\right\}^2 (b_n+1)_l} \times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^{l+p}.$$

Finally, S_2 is written in the form:

$$S_2 = \sum_{m_1,\dots,m_{n-1},k=0}^{\infty} \frac{(a)_{m_1+\dots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n)_k}{(a)_{m_1+\dots+m_{n-1}}\, m_1!\cdots m_{n-1}!(k!)^2} x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^k \times\left(\psi(a)+\frac{1}{a}\sum_{l=0}^{m_1+\cdots+m_{n-1}+k-1}\frac{(a)_l}{(a+1)_l}\right)$$

$$= \psi(a)\prod_{i=1}^{n-1}\{(1-x_i)^{-b_i}\}$$

$$+\rho a b_n \psi(a) \sum_{m_1,\dots,m_{n-1},k=0}^{\infty} \frac{(a+1)_{m_1+\dots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n+1)_k}{(a)_{m_1+\dots+m_{n-1}}\, m_1!\cdots m_{n-1}!\,\{(2)_k\}^2} \times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^k$$

$$+S_2^*$$

where, for convenience,

(12)

$$S_2^* = \frac{1}{a}\sum_{m_1+\cdots+m_{n-1}+k\neq 0}\ \sum_{l=0}^{m_1+\cdots+m_{n-1}+k-1} \frac{(a)_{m_1+\dots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n)_k(a)_l}{(a)_{m_1+\dots+m_{n-1}}\, m_1!\cdots m_{n-1}!(k!)^2(a+1)_l} \times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^k$$

$$= \frac{1}{a}\sum_{m_1=1}^{\infty}\sum_{l=0}^{m_1-1}\frac{(b_1)_{m_1}(a)_l}{(a+1)_l\, m_1!}x_1^{m_1}$$

$$+\frac{1}{a}\sum_{r=2}^{n-1}\ \sum_{m_1,\dots,m_{r-1}=0}^{\infty}\ \sum_{m_r=1}^{\infty}\ \sum_{l=0}^{m_1+\cdots+m_r-1}\frac{(b_1)_{m_1}\cdots(b_r)_{m_r}(a)_l}{(a+1)_l\, m_1!\cdots m_r!}x_1^{m_1}\cdots x_r^{m_r}$$

$$+\frac{1}{a}\sum_{k=1}^{\infty}\ \sum_{m_1,\dots,m_{n-1}=0}^{\infty}\ \sum_{l=0}^{m_1+\cdots+m_{n-1}+k-1}\frac{(a)_{m_1+\cdots+m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n)_k(a)_l}{(a)_{m_1+\cdots+m_{n-1}}\, m_1!\cdots m_{n-1}!(k!)^2(a+1)_l} \times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^k.$$

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Making use of the summation formula:

$$\sum_{m_1,\dots,m_n,l=0}^{\infty} \Lambda(m_1,\dots,m_n,l)$$
$$= \sum_{l,q,m_2,\dots,m_n=0}^{\infty} \Lambda(l+q,m_2,\dots,m_n,l)$$
$$+ \sum_{s=2}^{n} \sum_{\substack{m_1,\dots,m_{s-1},p_s,\\ k_s,m_{s+1},\dots,m_n=0}}^{\infty} \Lambda(m_1,\dots,m_{s-1},p_s+k_s+1,m_{s+1},\dots,m_n,$$
$$m_1+\dots+m_{s-1}+p_s+1),$$

we find from (12) that

$$S_2^* = \frac{b_1}{a}x_1 \sum_{l,q=0}^{\infty} \frac{(b_1+1)_{l+q}(a)_l}{(2)_{l+q}(a+1)_l} x_1^{l+q}$$
$$+ \sum_{r=2}^{n-1} \frac{b_r}{a}x_r \sum_{m_1,\dots,m_r=0}^{\infty} \sum_{l=0}^{m_1+\dots+m_r} \frac{(b_1)_{m_1}\cdots(b_{r-1})_{m_{r-1}}(b_r+1)_{m_r}(a)_l}{(a+1)_l\, m_1!\cdots m_{r-1}!(2)_{m_r}} x_1^{m_1}\cdots x_r^{m_r}$$
$$+ b_n\rho \sum_{m_1,\dots,m_{n-1},k=0}^{\infty} \sum_{l=0}^{m_1+\dots+m_{n-1}+k} \frac{(a+1)_{m_1+\dots m_{n-1}+k}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_n+1)_k(a)_l}{(a)_{m_1+\dots+m_{n-1}}(a+1)_l\, m_1!\cdots m_{n-1}!\,\{(2)_k\}^2}$$
$$\times x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}\rho^k$$
$$= \frac{b_1}{a}x_1 \sum_{l,q=0}^{\infty} \frac{(b_1+1)_{l+q}(a)_l}{(2)_{l+q}(a+1)_l} x_1^{l+q}$$
$$+ \sum_{r=2}^{n-1} \frac{b_r}{a}x_r \left\{ \sum_{l,q,m_2,\dots,m_r=0}^{\infty} \frac{(b_1)_{l+q}(b_2)_{m_2}\cdots(b_{r-1})_{m_{r-1}}(b_r+1)_{m_r}(a)_l}{(a+1)_l(1)_{l+q}\, m_2!\cdots m_{r-1}!(2)_{m_r}} x_1^{l+q}x_2^{m_2}\cdots x_r^{m_r} \right.$$
$$+ \sum_{s=2}^{r-1} \sum_{\substack{m_1,\dots,m_{s-1},p_s,\\ k_s,m_{s+1},\dots,m_r=0}}^{\infty} \frac{(b_1)_{m_1}\cdots(b_{s-1})_{m_{s-1}}(b_s)_{p_s+k_s+1}}{(a+1)_{m_1+\dots+m_{s-1}+p_s+1}\, m_1!\cdots m_{s-1}!}$$
$$\times \frac{(b_{s+1})_{m_{s+1}}\cdots(b_{r-1})_{m_{r-1}}(b_r+1)_{m_r}(a)_{m_1+\dots+m_{s-1}+p_s+1}}{(1)_{p_s+k_s+1}\, m_{s+1}!\cdots m_{r-1}!(2)_{m_r}}$$
$$\times x_1^{m_1}\cdots x_{s-1}^{m_{s-1}}x_s^{p_s+k_s+1}x_{s+1}^{m_{s+1}}\cdots x_r^{m_r}$$
$$+ \sum_{m_1,\dots,m_{r-1},p_r,k_r=0}^{\infty} \frac{(b_1)_{m_1}\cdots(b_{r-1})_{m_{r-1}}(b_r+1)_{p_r+k_r+1}(a)_{m_1+\dots+m_{r-1}+p_r+1}}{(a+1)_{m_1+\dots+m_{r-1}+p_r+1}\, m_1!\cdots m_{r-1}!(2)_{p_r+k_r+1}}$$
$$\left. \times x_1^{m_1}\cdots x_{r-1}^{m_{r-1}}x_r^{p_r+k_r+1} \right\}$$
$$+ o(1), \qquad (\rho \to 0+).$$

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By using the notation (6), we may write

$$S_2^* = \frac{b_1}{a} x_1 M^{(2)} \begin{bmatrix} a;\ b_1+1:\ 1;\ \ 1; & \\ & x_1, x_1 \\ a+1; \quad 2: —; —; & \end{bmatrix}$$

$$+M^{(2)} \begin{bmatrix} a;\ b_1:\ 1;\ \ 1; & \\ & x_1, x_1 \\ a+1;\ \ 1: —; —; & \end{bmatrix} \sum_{r=2}^{n-1} \frac{b_r}{a} x_r \prod_{i=2}^{r-1} \left\{(1-x_i)^{-b_i}\right\} F(b_r+1, 1; 2; x_r)$$

$$+\sum_{r=2}^{n-1} \sum_{s=2}^{r-1} \frac{b_r b_s}{a+1} x_r x_s$$

$$\times M^{(s+1)} \begin{bmatrix} a+1;\ b_s+1:\ b_1;\ \cdots;\ b_{s-1};\ \ 1;\ \ 1; & \\ & x_1, \ldots, x_{s-1}, x_s, x_s \\ a+2; \quad 2: —;\ \cdots;\ \ —; —; —; & \end{bmatrix}$$

$$\times \prod_{i=s+1}^{r-1} \left\{(1-x_i)^{-b_i}\right\} F(b_r+1, 1; 2; x_r)$$

$$+\sum_{r=2}^{n-1} \frac{b_r(b_r+1)}{2(a+1)} x_r^2$$

$$\times M^{(r+1)} \begin{bmatrix} a+1;\ b_r+2:\ b_1;\ \cdots;\ b_{r-1};\ \ 1;\ \ 1; & \\ & x_1, \ldots, x_{r-1}, x_r, x_r \\ a+2; \quad 3: —;\ \cdots;\ \ —; —; —; & \end{bmatrix}$$

$$+o(1), \qquad (\rho \to 0+).$$

Thus, gathering the results obtained above, we have the formula (7).

3. Derivation of Formula (8)

Making use of (3), (4), and (11), we find that

$$F_D^{(n)}(a, b_1, \ldots, b_n; a+b_{k+1}+\cdots+b_n; x_1, \ldots, x_k, 1-\rho x_{k+1}, \ldots, 1-\rho x_n)$$

$$= \sum_{m_1, \ldots, m_{n-1}=0}^{\infty} \frac{(a)_{m_1+\cdots+m_{n-1}} (b_1)_{m_1} \cdots (b_{n-1})_{m_{n-1}}}{(a+b_{k+1}+\cdots+b_n)_{m_1+\cdots+m_{n-1}} m_1! \cdots m_{n-1}!}$$

$$\times F(a+m_1\cdots+m_{n-1}, b_{k+1}+\cdots+b_n+m_{k+1}+\cdots+m_{n-1};$$

$$a+b_{k+1}+\cdots+b_n+m_1+\cdots+m_{n-1}; 1-\rho x_n)$$

$$\times x_1^{m_1}\cdots x_k^{m_k}(x_n-x_{k+1})^{m_{k+1}}\cdots(x_n-x_{n-1})^{m_{n-1}} \rho^{m_{k+1}+\cdots+m_{n-1}}$$

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$$= \frac{\Gamma(a+b_{k+1}+\cdots+b_n)}{\Gamma(a)\Gamma(b_{k+1}+\cdots+b_n)} \sum_{m_1,\ldots,m_k=0}^{\infty} \sum_{m_{k+1}+\cdots+m_{n-1}\neq 0} \sum_{m_n=0}^{m_{k+1}+\cdots+m_{n-1}-1}$$

$$\frac{(a)_{m_1+\cdots+m_k+m_n}(b_{k+1}+\cdots+b_n)_{m_n}(m_{k+1}+\cdots+m_{n-1}-m_n-1)!(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}}{(a)_{m_1+\cdots+m_k}(b_{k+1}+\cdots+b_n)_{m_{k+1}+\cdots+m_{n-1}}\, m_1!\cdots m_n!}$$

$$\times x_1^{m_1}\cdots x_k^{m_k}\left(1-\frac{x_{k+1}}{x_n}\right)^{m_{k+1}}\cdots\left(1-\frac{x_{n-1}}{x_n}\right)^{m_{n-1}} x_n^{m_n}(-\rho)^{m_n}$$

$$-\frac{\Gamma(a+b_{k+1}+\cdots+b_n)}{\Gamma(a)\Gamma(b_{k+1}+\cdots+b_n)}$$

$$\times \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_{k+1}+\cdots+b_n)_{m_{k+1}+\cdots+m_n}}{(a)_{m_1+\cdots+m_k}(b_{k+1}+\cdots+b_n)_{m_{k+1}+\cdots+m_{n-1}}(m_{k+1}+\cdots+m_n)!m_1!\cdots m_n!}$$

$$\times x_1^{m_1}\cdots x_k^{m_k}(x_{k+1}-x_n)^{m_{k+1}}\cdots(x_{n-1}-x_n)^{m_{n-1}}x_n^{m_n}\rho^{m_{k+1}+\cdots+m_n}$$

$$\times\Big[\log\rho x_n+\psi(a+m_1+\cdots+m_n)+\psi(b_{k+1}+\cdots+b_n+m_{k+1}+\cdots+m_n)$$

$$-\psi(m_n+1)-\psi(m_{k+1}+\cdots+m_n+1)\Big]$$

$$= \frac{\Gamma(a+b_{k+1}+\cdots+b_n)}{\Gamma(a)\Gamma(b_{k+1}+\cdots+b_n)}$$

$$\times\left\{S_1+S_2-S_3-\prod_{i=1}^{k}\{(1-x_i)^{-bi}\}\,[2\gamma+\psi(a)+\psi(b_{k+1}+\cdots+b_n)+\log\rho x_n]\right\},$$

say,

where, for convenience,

$$S_1=\sum_{m_1,\ldots,m_k=0}^{\infty}\sum_{m_{k+1}+\cdots+m_{n-1}\neq 0}\frac{(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(m_k+\cdots+m_{n-1}-1)!}{(b_{k+1}+\cdots+b_n)_{m_{k+1}+\cdots+m_{n-1}}\, m_1!\cdots+m_{n-1}!}$$

$$\times x_1^{m_1}\cdots x_k^{m_k}\left(1-\frac{x_{k+1}}{x_n}\right)^{m_{k+1}}\cdots\left(1-\frac{x_{n-1}}{x_n}\right)^{m_{n-1}},$$

$$S_2=\sum_{m_1,\ldots,m_k=0}^{\infty}\sum_{m_{k+1}+\cdots+m_n\geq 2}\sum_{m_n=1}^{m_{k+1}+\cdots+m_{n-1}-1}$$

$$\frac{(a)_{m_1+\cdots+m_k+m_n}(b_{k+1}+\cdots+b_n)_{m_n}(m_{k+1}+\cdots+m_{n-1}-m_n-1)!}{(a)_{m_1+\cdots+m_k}(b_{k+1}+\cdots+b_n)_{m_{k+1}+\cdots+m_{n-1}}\, m_1!\cdots m_n!}$$

$$\times x_1^{m_1}\cdots x_k^{m_k}\left(1-\frac{x_{k+1}}{x_n}\right)^{m_{k+1}}\left(\frac{x_{n-1}}{x_n}\right)^{m_{n-1}}(-\rho x_n)^{m_n},$$

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and

$$S_3 = \sum_{m_1,\ldots,m_k=0}^{\infty} \sum_{m_{k+1}+\cdots+m_n \neq 0} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}(b_{k+1}+\cdots+b_n)_{m_{k+1}+\cdots+m_n}}{(a)_{m_1+\cdots+m_k}(b_{k+1}+\cdots+b_n)_{m_{k+1}+\cdots+m_{n-1}}(1)_{m_{k+1}+\cdots+m_n}m_1!\cdots m_n!}$$
$$\times x_1^{m_1}\cdots x_k^{m_k}(x_{k+1}-x_n)^{m_{k+1}}\cdots(x_{n-1}-x_n)^{m_{n-1}}x_n^{m_n}\rho^{m_{k+1}+\cdots+m_n}$$
$$\times\big[\log\rho x_n+\psi(a+m_1+\cdots+m_n)$$
$$+\psi(b_{k+1}+\cdots+b_n+m_{k+1}+\cdots+m_n)-\psi(m_n+1)$$
$$-\psi(m_{k+1}+\cdots+m_n+1)\big].$$

Since the series S_1 is represented in the sum with respect to s from 1 to $n-k-1$, the indices being $m_{k+1}=\ldots=m_{n-s-1}=0$, $m_{n-s}\geq 1$, $m_{n-s+1}\geq 0,\ldots, m_{n-1}\geq 0$, we have

$$S_1 = \sum_{s=1}^{n-k-1}\sum_{m_1,\ldots,m_k=0}^{\infty}\sum_{m_{n-s}=1}^{\infty}\sum_{m_{n-s+1}+\cdots+m_{n-1}=0}^{\infty} \frac{(b_1)_{m_1}\cdots(b_k)_{m_k}(b_{n-s})_{m_{n-s}}\cdots(b_{n-1})_{m_{n-1}}(m_{n-s}+\cdots+m_{n-1}-1)!}{(b_{k+1}+\cdots+b_n)_{m_{n-s}+\cdots+m_{n-1}}m_1!\cdots m_k!m_{n-s}!\cdots m_{n-1}!}$$
$$\times x_1^{m_1}\cdots x_k^{m_k}\left(1-\frac{x_{n-s}}{x_n}\right)^{m_{n-s}}\left(1-\frac{x_{n-1}}{x_n}\right)^{m_{n-1}}$$
$$=\prod_{i=1}^{k}\{(1-x_i)^{-b_i}\}\sum_{s=1}^{n-k-1}\frac{b_{n-s}}{b_{k+1}+\cdots+b_n}\left(1-\frac{x_{n-s}}{x_n}\right)$$
$$\times N^{(s)}\left[\begin{matrix}1\colon b_{n-s}+1,\ 1;\ b_{n-s+1};\ \cdots;\ b_{n-1};\\ b_{k+1}+\cdots+b_n+1\colon\quad 2;\quad —;\ \cdots;\ —;\end{matrix}\ 1-\frac{x_{n-s}}{x_n},\ldots,1-\frac{x_{n-1}}{x_n}\right].$$

The series S_2 and S_3 include the terms multiplied by ρ, and the formula (8) follows readily.

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