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CONSTRUCTION OF NORMAL NUMBERS USING THE DISTRIBUTION OF THE *kTH* LARGEST PRIME FACTOR

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Abstract

Given an integer $q \geq 2$, a *q-normal number* is an irrational number η such that any preassigned sequence of ℓ digits occurs in the q-ary expansion of η at the expected frequency, namely $1/q^\ell$. In a recent paper we constructed a large family of normal numbers, showing in particular that, if $P(n)$ stands for the largest prime factor of n, then the number $0.P(2)P(3)P(4)\ldots$, the concatenation of the numbers $P(2), P(3), P(4), \ldots$, each represented in base q, is a q-normal number, thereby answering in the affirmative a question raised by Igor Shparlinski. We also showed that $0.P(2+1)P(3+1)P(5+1)\ldots P(p+1)\ldots$, where p runs through the sequence of primes, is a q-normal number. Here, we show that, given any fixed integer $k \geq 2$, the numbers $0.P_k(2)P_k(3)P_k(4)\ldots$ and $0.P_k(2+1)P_k(3+1)P_k(5+1)\ldots P_k(p+1)\ldots$, where $P_k(n)$ stands for the kth largest prime factor of n, are q-normal numbers. These results are part of more general statements.

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1. Introduction

Given an integer $q \geq 2$, a *q-normal number*, or simply a *normal number*, is an irrational number whose q-ary expansion is such that any preassigned sequence, of length $\ell \geq 1$, of base q digits from this expansion, occurs at the expected frequency, namely $1/q^\ell$.

Let $A_q := \{0, 1, \ldots, q-1\}$. Given an integer $\ell \geq 1$, an expression of the form $i_1 i_2 \ldots i_\ell$, where each $i_j \in A_q$, is called a *word* of length ℓ. The symbol Λ will denote the *empty word*. We let A_q^ℓ stand for the set of all words of length ℓ and A_q^* stand for the set of all the words regardless of their length.

Given a positive integer n, we write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t, \tag{1.1}$$

where $\varepsilon_i(n) \in A_q$ for $0 \leq i \leq t$ and $\varepsilon_t(n) \neq 0$. We associate with this representation the word

$$\overline{n} = \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) = \varepsilon_0\varepsilon_1\ldots\varepsilon_t \in A_q^{t+1}.$$

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Moreover, in the case $n \le 0$, we set $\overline{n} = \Lambda$.

Let $P(n)$ stand for the largest prime factor of $n \ge 2$, with $P(1) = 1$. In a recent paper [2], we showed that if $F \in \mathbb{Z}[x]$ is a polynomial of positive degree with $F(x) > 0$ for $x > 0$, then the real numbers

$$0.\overline{F(P(2))}\,\overline{F(P(3))}\ldots\overline{F(P(n))}\ldots$$

and

$$0.\overline{F(P(2+1))}\,\overline{F(P(3+1))}\ldots\overline{F(P(p+1))}\ldots,$$

where p runs through the sequence of primes, are q-normal numbers.

Here, we prove that the same result holds if $P(n)$ is replaced by $P_k(n)$, the kth largest prime factor of n. The case of $P_k(n)$ relies on the same basic tool we used to study the case of $P(n)$, namely a 1996 result of Bassily and Kátai [1]. However, the $P_k(n)$ case raises new technical challenges and therefore needs a special treatment. We thereby create a much larger family of normal numbers. To conclude, we raise an open problem.

2. Main results

Denote by $\omega(n)$ the number of distinct prime factors of the integer $n \ge 2$, with $\omega(1) = 0$. Given an integer $k \ge 1$, for each integer $n \ge 2$, we let $P_k(n)$ stand for the kth largest prime factor of n if $\omega(n) \ge k$, while we set $P_k(n) = 1$ if $\omega(n) \le k - 1$. Thus, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ stands for the prime factorisation of n, where $p_1 < p_2 < \cdots < p_s$, then

$$P_1(n) = P(n) = p_s, \quad P_2(n) = p_{s-1}, \quad P_3(n) = p_{s-2}, \ldots.$$

Let $F \in \mathbb{Z}[x]$ be a polynomial of positive degree satisfying $F(x) > 0$ for $x > 0$. Also, let $T \in \mathbb{Z}[x]$ be such that $T(x) \to \infty$ as $x \to \infty$ and assume that $\ell_0 = \deg T$. Fix an integer $k \ge \ell_0$. We then have the following results.

Theorem 2.1. *The number*

$$\theta = 0.\overline{F(P_k(T(2)))}\,\overline{F(P_k(T(3)))}\ldots\overline{F(P_k(T(n)))}\ldots$$

is a q-normal number.

Theorem 2.2. *Assuming that $k \ge \ell_0 + 1$, the number*

$$\rho = 0.\overline{F(P_k(T(2+1)))}\,\overline{F(P_k(T(3+1)))}\ldots\overline{F(P_k(T(p+1)))}\ldots$$

is a q-normal number.

3. Notation and preliminary lemmas

Let $\wp$ stand for the set of all prime numbers. For each integer $n \geq 2$, let $L(n) = \lfloor \log n / \log q \rfloor$. Let $\beta \in A_q^\ell$ and n be written as in (1.1). We then let $\nu_\beta(\overline{n})$ stand for the number of occurrences of the word β in the q-ary expansion of the positive integer n, that is, the number of times that $\varepsilon_j(n) \ldots \varepsilon_{j+\ell-1}(n) = \beta$ as j varies from 0 to $t - (\ell - 1)$.

The letters p and Q will always denote prime numbers. The letter c with or without subscript always denotes a positive constant but not necessarily the same at each occurrence.

We first state two key lemmas already proved in [2].

Lemma 3.1. *Let $F \in \mathbb{Z}[x]$ with $\deg(F) = r \geq 1$. Assume that κ_u is a function of u such that $\kappa_u > 1$ for all u. Given a word $\beta \in A_q^\ell$ and setting*

$$V_\beta(u) := \#\Big\{Q \in \wp : u \leq Q \leq 2u \text{ such that } \Big|\nu_\beta(\overline{F(Q)}) - \frac{L(u^r)}{q^\ell}\Big| > \kappa_u \sqrt{L(u^r)}\Big\},$$

there exists a positive constant c such that

$$V_\beta(u) \leq \frac{cu}{(\log u)\kappa_u^2}.$$

Lemma 3.2. *Let F and κ_u be as in Lemma 3.1. Given $\beta_1, \beta_2 \in A_q^\ell$ with $\beta_1 \neq \beta_2$, set*

$$\Delta_{\beta_1,\beta_2}(u) := \#\Big\{Q \in \wp : u \leq Q \leq 2u \text{ such that } |\nu_{\beta_1}(\overline{F(Q)}) - \nu_{\beta_2}(\overline{F(Q)})| > \kappa_u \sqrt{L(u^r)}\Big\}.$$

Then, for some positive constant c,

$$\Delta_{\beta_1,\beta_2}(u) \leq \frac{cu}{(\log u)\kappa_u^2}.$$

The following three lemmas will also be useful in the proofs of our theorems.

Lemma 3.3. *Let $\varepsilon > 0$ be a small number. Given any integer $k \geq \ell_0 + 1$, there exists $x_0 = x_0(\varepsilon)$ such that, for all $x \geq x_0$,*

$$\#\{p \in I_x : P_k(T(p+1)) < x^\varepsilon\} \leq c\varepsilon \frac{x}{\log x}. \tag{3.1}$$

Moreover, for each integer $k \geq \ell_0$, there exists $x_0 = x_0(\varepsilon)$ such that, for all $x \geq x_0$,

$$\#\{n \in I_x : P_k(T(n)) < x^\varepsilon\} \leq c\varepsilon x. \tag{3.2}$$

Proof. For a proof of (3.1) in the case $k = 1$ and $T(n) = n$, see the proof of Theorem 1 in our paper [2]. The more general case $k \geq 2$ and $T \in \mathbb{Z}[x]$ can be handled along the same lines. The estimate (3.2) also follows easily. □

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Lemma 3.4 (Brun–Titchmarsh inequality). *Letting* $\pi(x; m, \nu) := \#\{p \le x : p \equiv \nu \pmod{m}\}$, *there exists a positive constant c such that*

$$\pi(x; m, \nu) < c\frac{x}{\varphi(m)\log(x/m)} \quad \textit{for all } m < x,$$

where φ stands for the Euler function.

Proof. For a proof, see Halberstam and Richert [4]. □

Lemma 3.5. *For $2 \le y \le x$, let $\Psi(x, y) = \#\{n \le x : P(n) \le y\}$. Then*

$$\Psi(x, y) \ll x\exp\Big(-\frac{1}{2}\frac{\log x}{\log y}\Big).$$

Proof. For a proof, see De Koninck and Luca [3]. □

4. The proof of Theorem 2.1

Let x be a fixed large number. Let $I_x = [x, 2x]$, $N_0 = \lceil x\rceil$, $N_1 = \lfloor 2x\rfloor$ and set

$$\theta^{(x)} := \overline{F(P_k(T(N_0)))}\;\overline{F(P_k(T(N_0+1)))}\;\ldots\;\overline{F(P_k(T(N_1)))}.$$

Given any prime p, we know that

$$\#\{n \in I_x : T(n) \equiv 0 \pmod{p}\} = \frac{\rho(p)}{p}x + O(1), \tag{4.1}$$

where $\rho(p)$ stands for the number of solutions n of the congruence $T(n) \equiv 0 \pmod{p}$.

On the other hand, since we have assumed that $k \ge \ell_0$, there exists a constant $c > 1$ such that $P_k(T(n)) < cx$ for all $n \in I_x$. We then have

$$\#\{n \in I_x : P_k(T(n)) \ge x\} \ll \pi([x, cx]) + x\sum_{x<p<cx}\frac{\rho(p)}{p} = O\Big(\frac{x}{\log x}\Big) = o(x). \tag{4.2}$$

Finally, given a fixed small positive number $\delta = \delta(k)$, setting

$$\omega_\delta(T(n)) := \sum_{\substack{p\mid T(n)\\ x^\delta<p<x^{1/2}}} 1,$$

we can show, using a type of Turán–Kubilius inequality, that a positive proportion of the integers $n \in I_x$ satisfy the inequality $\omega_\delta(T(n)) \ge k$. It follows from this observation and from (4.2) that

$$\nu_\beta(\theta^{(x)}) = \sum_{n\in I_x}\nu_\beta(\overline{F(P_k(T(n)))}) + O(x) \asymp x\log x, \tag{4.3}$$

where the constant implied by the $\asymp$ symbol may depend on k as well as on the degrees of T and F.

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In order to complete the proof of the theorem it will be sufficient, in light of (4.3), to prove that given any two words $\beta_1, \beta_2 \in A_q^\ell$,

$$|\nu_{\beta_1}(\theta^{(x)}) - \nu_{\beta_2}(\theta^{(x)})| = o(x \log x) \quad \text{as } x \to \infty. \tag{4.4}$$

Indeed, since A_q^ℓ contains exactly q^ℓ distinct words and since their respective occurrences are very close in the sense of (4.4), it will follow that

$$\frac{\nu_\beta(\theta^{(x)})}{x \log x} \to \frac{1}{q^\ell} \quad \text{as } x \to \infty, \tag{4.5}$$

thus establishing that θ is a q-normal number.

In the spirit of Lemma 3.1, we will say that the prime $Q \in I_u$ is a *bad prime* if

$$\max_{\beta \in A_q^\ell} \left| \nu_\beta(\overline{F(Q)}) - \frac{L(u^r)}{q^\ell} \right| > \kappa_u \sqrt{L(u^r)} \tag{4.6}$$

and a *good prime* if

$$\left| \nu_\beta(\overline{F(Q)}) - \frac{L(u^r)}{q^\ell} \right| \le \kappa_u \sqrt{L(u^r)}. \tag{4.7}$$

First observe that

$$|\nu_{\beta_1}(\theta^{(x)}) - \nu_{\beta_2}(\theta^{(x)})| \le \Sigma_1 + \Sigma_2 + \Sigma_3 + O(x), \tag{4.8}$$

where:

- in Σ_1 we sum the expression $m_n := |\nu_{\beta_1}(\overline{F(P_k(T(n)))}) - \nu_{\beta_2}(\overline{F(P_k(T(n)))})|$ over those integers $n \in I_x$ for which $P_k(T(n)) < x^\varepsilon$;
- in Σ_2 we sum the expression m_n over those integers $n \in I_x$ for which $p = P_k(T(n)) \ge x^\varepsilon$ with p being a good prime;
- in Σ_3 we sum the expression m_n over those integers $n \in I_x$ for which $p = P_k(T(n)) \ge x^\varepsilon$ with p being a bad prime.

It is clear that, in light of estimate (3.2) of Lemma 3.3,

$$\Sigma_1 \le c\varepsilon x \log x. \tag{4.9}$$

On the other hand, choosing $\kappa_u = \log\log u$ in the range $x^\varepsilon < u < x$,

$$\Sigma_2 \le cx\sqrt{\log x}\log\log x. \tag{4.10}$$

Finally,

$$\Sigma_3 = \sum_{\substack{n \in I_x \\ p = P_k(T(n)) \ge x^\varepsilon \\ p \text{ bad prime}}} m_n \le c \log x \sum_{\substack{n \in I_x \\ p = P_k(T(n)) \ge x^\varepsilon \\ p \text{ bad prime}}} 1 = c \log x \, \Sigma_4, \tag{4.11}$$

say.

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Subdivide the interval $[x^{\varepsilon}, \sqrt{x}]$ into disjoint intervals $[u, 2u)$ as follows. Let j_0 be the smallest positive integer such that $2^{j_0+1}x^{\varepsilon} \geq \sqrt{x}$, so that

$$[x^{\varepsilon}, \sqrt{x}] \subset \bigcup_{j=0}^{j_0} J_j,$$

where

$$J_j = [u_j, u_{j+1}) := [2^j x^{\varepsilon}, 2^{j+1} x^{\varepsilon}), \quad j = 0, 1, \ldots, j_0.$$

Using (4.1),

$$\begin{aligned}
\Sigma_4 &\leq \sum_{j=0}^{j_0} \sum_{\substack{p\in[u_j,2u_j) \\ p \text{ bad prime}}} \#\{n \in I_x : T(n) \equiv 0 \pmod{p}\} \\
&\leq cx \sum_{j=0}^{j_0} \sum_{\substack{p\in[u_j,2u_j) \\ p \text{ bad prime}}} \frac{\rho(p)}{p} \\
&\leq cx \sum_{j=0}^{j_0} \frac{1}{(\log\log u_j)^2 \log u_j} \\
&\ll \frac{1}{\varepsilon} \frac{x}{(\log\log x)^2}.
\end{aligned} \tag{4.12}$$

Substituting (4.12) in (4.11),

$$\Sigma_3 = O\Big(\frac{x \log x}{(\log\log x)^2}\Big). \tag{4.13}$$

Thus, gathering (4.9), (4.10) and (4.13) in (4.8), (4.4) follows immediately and therefore (4.5) as well, thereby completing the proof of Theorem 2.1.

5. The proof of Theorem 2.2

First observe that the additional condition $k \geq \ell_0 + 1$ guarantees that, for $p \leq x$, we have $Q = P_k(T(p+1)) < x^{\ell_0/k}$, with $\ell_0/k < 1$. Hence, it follows from the Brun–Titchmarsh inequality (Lemma 3.4) that

$$\sum_{\substack{p\in[x,2x] \\ T(p+1)\equiv 0 \,(\mathrm{mod}\, Q)}} 1 \ll \frac{\rho(Q)x}{\varphi(Q)\log(x/Q)} \ll \frac{\rho(Q)}{Q} \frac{x}{\log x}. \tag{5.1}$$

From this point on, the proof is somewhat similar to that of Theorem 2.1 but with various adjustments.

Let

$$\rho^{(x)} := \overline{F(P_k(T(\rho_1 + 1)))} \ldots \overline{F(P_k(T(\rho_S + 1)))},$$

where $\rho_1 < \cdots < \rho_S$ is the sequence of primes appearing in the interval I_x.

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Observe that, since $S = \pi([x, 2x]) \asymp x/\log x$, we may write

$$\nu_\beta(\rho^{(x)}) = \sum_{i=1}^{S} \nu_\beta(\overline{F(P_k(T(\rho_i + 1)))}) + O\Big(\frac{x}{\log x}\Big) \asymp x. \tag{5.2}$$

As in the proof of Theorem 2.1, in order to complete the proof of Theorem 2.2, it will be sufficient, in light of (5.2), to prove that, given any two arbitrary words $\beta_1, \beta_2 \in A_q^\ell$,

$$|\nu_{\beta_1}(\rho^{(x)}) - \nu_{\beta_2}(\rho^{(x)})| = o(x) \quad \text{as } x \to \infty. \tag{5.3}$$

Indeed, since A_q^ℓ contains exactly q^ℓ distinct words and since their respective occurrences are very close in the sense of (5.3), it will follow that

$$\frac{\nu_\beta(\rho^{(x)})}{x} \to \frac{1}{q^\ell} \quad \text{as } x \to \infty, \tag{5.4}$$

thus establishing that ρ is a q-normal number.

Hence, our main task will be to estimate the difference $|\nu_{\beta_1}(\rho^{(x)}) - \nu_{\beta_2}(\rho^{(x)})|$, where β_1 and β_2 are arbitrary words belonging to A_q^ℓ. To do so, we once more use the concepts of bad prime and good prime defined in (4.6) and (4.7), respectively. We first write

$$\begin{aligned} |\nu_{\beta_1}(\rho^{(x)}) - \nu_{\beta_2}(\rho^{(x)})| &\le \sum_{i=1}^{S} |\nu_{\beta_1}(\overline{F(P_k(T(\rho_i + 1)))}) - \nu_{\beta_2}(\overline{F(P_k(T(\rho_i + 1)))})| + O(S) \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 + O\Big(\frac{x}{\log x}\Big), \end{aligned} \tag{5.5}$$

where, letting $m_j := |\nu_{\beta_1}(F(P_k(T(\rho_j + 1)))) - \nu_{\beta_2}(F(P_k(T(\rho_j + 1))))|$:

- in Σ_1 we sum m_j over those j for which $p = P_k(T(\rho_j + 1)) < x^\varepsilon$;
- in Σ_2 we sum m_j over those j for which $p = P_k(T(\rho_j + 1)) \ge x^\varepsilon$, when p is a good prime;
- in Σ_3 we sum m_j over those j for which $p = P_k(T(\rho_j + 1)) \ge x^\varepsilon$, when p is a bad prime.

Now observe that, for any prime Q,

$$\nu_\beta(\overline{F(Q)}) \le cL(u^r) \le c_1 \log u \quad \text{for all } Q \in I_u. \tag{5.6}$$

Thus, using Lemma 3.3, we have, in light of (5.6), that

$$\Sigma_1 \ll \log x \cdot \frac{\varepsilon x}{\log x} = \varepsilon x. \tag{5.7}$$

Using Lemma 3.2 and estimate (5.6), we also have that

$$\Sigma_2 \le c\frac{u}{\log u} \cdot \frac{1}{(\log\log u)^2} \cdot \log u = o\Big(\frac{x}{\log x} \cdot \log x\Big) = o(x). \tag{5.8}$$

Finally, it is clear, using (5.6), that

$$\Sigma_3 = \sum_{\substack{p=P_k(T(\rho_j+1))\geq x^{\varepsilon} \\ p \text{ bad prime}}} m_j \leq c \log x \sum_{\substack{p=P_k(T(\rho_j+1))\geq x^{\varepsilon} \\ p \text{ bad prime}}} 1 = c \log x \Sigma_4, \tag{5.9}$$

say. Since

$$\Sigma_4 \leq \sum_{j=0}^{j_0} \sum_{\substack{p\in[u_j,2u_j) \\ p \text{ bad prime}}} \#\{j : T(\rho_j + 1) \equiv 0 \pmod p\},$$

it follows, by (5.1) and by adopting essentially the same approach used to establish (4.12), that

$$\begin{aligned}\Sigma_4 &\leq c \sum_{j=0}^{j_0} \frac{u_j}{\log u_j} \sum_{\substack{p\in[u_j,2u_j) \\ p \text{ bad prime}}} \frac{\rho(p)}{p} \\ &\leq c \frac{x}{\log x} \sum_{j=0}^{j_0} \frac{1}{(\log \log u_j)^2 \log u_j} \\ &\ll \frac{x}{\log x(\log \log x)^2}.\end{aligned} \tag{5.10}$$

Substituting (5.10) in (5.9),

$$\Sigma_3 = O\Big(\frac{x}{(\log \log x)^2}\Big). \tag{5.11}$$

Substituting (5.7), (5.8) and (5.11) in (5.5), we get that, given arbitrary words $\beta_1, \beta_2 \in A_q^{\ell}$,

$$|\nu_{\beta_1}(\rho^{(x)}) - \nu_{\beta_2}(\rho^{(x)})| < \varepsilon x,$$

which proves (5.3) and in consequence (5.4), thus completing the proof of Theorem 2.2.

6. A related open problem

Let q be a fixed prime number. Let n be a positive integer such that $(n, q) = 1$ and consider its sequence of divisors $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$, where $\tau(n)$ stands for the number of divisors of n. Given any positive integer m, we associate with it its congruence class modulo q, thus introducing the function $f_q(m) = \ell$, that is, $m \equiv \ell \pmod q$. Let us now introduce the arithmetical function ξ defined by

$$\xi(n) = f_q(d_1) \ldots f_q(d_{\tau(n)}) \in A_q^{\tau(n)}.$$

Given $\beta \in A_q^k$ and $\alpha \in A_q^*$, let $M(\alpha|\beta)$ stand for the number of occurrences of the word β in the word α.

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Is it true that the quantity

$$Q_k(n) := \max_{\beta\in A_q^k}\left|\frac{M(\xi(n)|\beta)(q-1)^k}{\tau(n)} - 1\right|$$

tends to 0 for almost all positive integers n for which $(n, q) = 1$?

This seems to be a difficult problem. Even proving the particular case $Q_2(n) \to 0$ appears to be quite a challenge. But observe that the case $k = 1$ is easy to establish. Indeed, let χ stand for a Dirichlet character and let

$$S_\chi(n) = \sum_{d|n} \chi(d) = \prod_{p^\alpha\|n} (1 + \chi(p) + \cdots + \chi(p^\alpha)).$$

Then, letting χ_0 stand for the principal character,

$$\begin{aligned}
\#\{d \mid n : d \equiv \ell \ (\mathrm{mod}\ q)\} &= \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(\ell) S_\chi(n) \\
&= \frac{1}{\varphi(q)} \overline{\chi_0}(\ell) S_{\chi_0}(n) + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \overline{\chi}(\ell) S_\chi(n) \qquad (6.1)\\
&= \frac{1}{q-1}\tau(n) + \frac{1}{q-1} \sum_{\chi\neq\chi_0} \overline{\chi}(\ell) S_\chi(n).
\end{aligned}$$

Now, set $f(n) := |(S_\chi(n))/\tau(n)|$ and observe that $|f(p^\alpha)| \le 1$ for all prime powers p^α. For each real $Y > 0$, let f_Y be the multiplicative function defined on prime powers p^α by

$$f_Y(p^\alpha) = \begin{cases} f(p^\alpha) & \text{if } p \le Y, \\ 1 & \text{if } p > Y. \end{cases}$$

With this definition, it is clear that $f_Y(p^\alpha) \ge f(p^\alpha)$ and therefore that $f_Y(n) \ge f(n)$ for all $n \in \mathbb{N}$. Let us also define the multiplicative function $g_Y(n)$ implicitly by the relation $f_Y(n) = \sum_{d|n} g_Y(d)$, so that in particular $g_Y(p) = f_Y(p) - 1$ for all primes p and $g_Y(p^\alpha) = f_Y(p^\alpha) - f_Y(p^{\alpha-1})$ for all primes p and integers $\alpha \ge 2$. Finally, note that $|g_Y(p^\alpha)| \le 1$ for all p^α. In light of these facts, we may thus write that, for any given $Y > 0$,

$$\sum_{n\le x} f(n) \le \sum_{n\le x} f_Y(n) = \sum_{\substack{d\le x\\ P(d)\le Y}} g_Y(d)\left\lfloor\frac{x}{d}\right\rfloor = x \sum_{\substack{d\le x\\ P(d)\le Y}} \frac{g_Y(d)}{d} + O(\Psi(x, Y)). \qquad (6.2)$$

Since, for each fixed $Y > 0$, it follows from Lemma 3.5 that $\lim_{x\to\infty}(1/x)\Psi(x, Y) = 0$, we may conclude from (6.2) that

$$\begin{aligned}\limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n) &\le \limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} f_Y(n)\\ &= \limsup_{x\to\infty} \sum_{\substack{d\le x\\ P(d)\le Y}} \frac{g_Y(d)}{d}\\ &= \prod_{p\le Y}\Big(1 + \frac{f(p)-1}{p} + \frac{f(p^2)-f(p)}{p^2} + \cdots\Big) \qquad (6.3)\\ &= \prod_{p\le Y}\Big(1-\frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\Big)\\ &= \prod_{p\le Y} L_p,\end{aligned}$$

say. Observe that

$$0 \le L_p \le \exp\Big(-\frac{1}{p} + \frac{f(p)}{p} + O\Big(\frac{1}{p^2}\Big)\Big). \qquad (6.4)$$

Thus, using (6.4) in (6.3), we get that, for some constants $c_1 > 0$,

$$\limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n) \le \exp\Big(\sum_{p\le Y} \frac{f(p)-1}{p} + c_1\Big). \qquad (6.5)$$

Now, since χ is not the principal character, there must exist at least one nonzero residue class modulo $\ell \pmod q$ such that

$$f(p) = \Big|\frac{\chi(p)+1}{2}\Big| = \beta < 1 \qquad \text{for all primes } p \equiv \ell \pmod q.$$

Using this in (6.5), we get that, for some positive constants c_2 and c_3,

$$\begin{aligned}\limsup_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n) &\le \exp\Big(\sum_{\substack{p\le Y\\ p\equiv \ell \pmod q}} \frac{\beta-1}{p} + c_1\Big)\\ &= \exp\Big(\frac{\beta-1}{\varphi(q)}\log\log Y + c_2\Big) = \frac{c_3}{\log^{(1-\beta)/(q-1)} Y}.\end{aligned}$$

Since $1-\beta > 0$ and since Y was chosen arbitrarily, it follows that

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n) = 0,$$

thereby implying that $f(n) = o(1)$ for almost all n.

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Using this observation, it follows from (6.1) that

$$\#\{d \mid n : d \equiv \ell \pmod q\} = \frac{1}{q-1}\tau(n) + o(\tau(n)),$$

for almost all n, thus establishing the case $Q_1(n) \to 0$ for almost all positive integers n such that $(n, q) = 1$, as claimed.

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