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Sharp Norm Estimates for the Bergman Operator From Weighted Mixed-norm Spaces to Weighted Hardy Spaces

Carme Cascante, Joan Fàbrega, and Joaquín M. Ortega

Abstract. In this paper we give sharp norm estimates for the Bergman operator acting from weighted mixed-norm spaces to weighted Hardy spaces in the ball, endowed with natural norms.

1 Introduction

The study of weighted norm inequalities for the Hardy–Littlewood maximal operator and for singular operators in $\mathbb{R}^n$ and their relation with A_p-weights goes back to the works of Hunt, Muckenhoupt, Wheeden, Coifman, and Fefferman in the 70's (see [16, 18]). More recently, many authors have studied the sharp dependence of the constants in these weighted norm inequalities. For the Hardy–Littlewood maximal operator, the weighted $L^p(\omega)$-norm is bounded, up to a constant, by $[\omega]_{A_p}^{1/(p-1)}$ and the exponent is sharp, in the sense that it cannot be replaced by any smaller one (see [5]). For the more difficult case of singular integral operators in Euclidean spaces, the so-called A_2-conjecture for general Calderon–Zygmund operators was solved in [19], that is, the $L^2(\omega)$-norm of these operators are bounded, up to a constant, by $[\omega]_{A_2}$. Later, using a different approach, a simple and very elegant proof of the A_2-conjecture was given in [27]. The sharp dependence on the weight has also been studied for other operators like the Lusin square function on $\mathbb{R}^n$ (see [25] and the references therein). The proof of this result is based on the intrinsic square function of [36] and strongly relies on properties of convolution operators, among other key ingredients. Recently, an alternative proof has been given in [28], which also proves the sharp dependence on the fixed aperture of the square function. In the context of homogeneous spaces, some of these results have been extended for the Hardy–Littlewood maximal operator and Calderon–Zygmund operators (see, for instance, [2, 22], respectively).

A characterization of the measures on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ for which the Bergman operator is bounded from the weighted L^p-space to the weighted Bergman space has been proved in [3]. The condition is given in terms of the so-called B_p-class. Later, sharp estimates on the norm of the Bergman operator on these weighted spaces were

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obtained in [32]. On the other hand, the boundedness of the Bergman operator acting from a mixed-norm space to the Hardy space H^p was proved in [9].

The main objective of this paper is to obtain sharp estimates of the Bergman operator acting on weighted mixed-norm operators to weighted Hardy spaces endowed with different norms. The fact that we are in a homogeneous space framework means that usual techniques, as convolution, cannot be applied. In particular, it is necessary to use the specific properties of the kernels involved in the different problems we consider, as well as the dyadic decomposition for homogeneous spaces obtained in [22], which gives the existence of adjacent and sparse families of cubes.

Before we state our main results, we recall some definitions.

Let $H = H(\mathbb{B})$ be the space of holomorphic functions on the unit ball of $\mathbb{C}^n$, $\mathbb{B}$. We denote by H^* the space of functions $f \in H$ having boundary values $f(\zeta) = \lim_{r \nearrow 1} f(r\zeta)$ a.e. on the unit sphere $\mathbb{S}$.

For $0 < p < \infty$ and ω a weight on $\mathbb{S}$, *i.e.*, a function $\omega \in L^1$ satisfying $\omega > 0$ a.e., let $L^p(\omega) = L^p(\mathbb{S}, \omega d\sigma)$, where $d\sigma$ denotes the normalized Lebesgue measure on the unit sphere $\mathbb{S}$. If $\omega = 1$, we use L^p to denote $L^p(\mathbb{S}, d\sigma)$.

Denote by $H^p(\omega) = \{f \in H^* : \|f\|_{H^p(\omega)} = \|f\|_{L^p(\omega)} < \infty\}$.

If $1 < p < \infty$ ω is in the Muckenhoupt class $A_p = A_p(\mathbb{S})$, which will be defined in Section 2, there exist other characterizations of the space $H^p(\omega)$ (see, for instance, [29, Section 5], [6] and the references therein). In this paper, we consider the equivalent norm given in terms of the Littlewood–Paley function.

Let $L^{p,2}(\omega)$ be the mixed-norm space of measurable functions φ on $\mathbb{B}$ satisfying

$$\|\varphi\|^p_{L^{p,2}(\omega)} = \int_{\mathbb{S}} \Big(\int_0^1 |\varphi(r\zeta)|^2 \frac{2nr^{2n-1}\,dr}{1-r^2} \Big)^{p/2} \omega(\zeta)\,d\sigma(\zeta) < \infty,$$

We denote by $F_0^{p,2}(\omega)$ the weighted Triebel–Lizorkin space of holomorphic functions f on $\mathbb{B}$ satisfying

$$\|f\|_{F_0^{p,2}(\omega)} = \Big\| (1-r^2)\Big(I + \frac{R}{n}\Big) f(r\zeta) \Big\|_{L^{p,2}(\omega)} < \infty. \tag{1.1}$$

Here I denotes the identity operator and $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$. If $\omega = 1$, we simply write H^p and $F_0^{p,2}$. In this case, it is well known that for $0 < p < \infty$ these spaces are isomorphic and this isomorphism still holds if we replace the operator $I + \frac{R}{n}$ by $\alpha I + \beta R$ with $\alpha, \beta > 0$ (see, for instance, [1,31] and the references therein). If $1 < p < \infty$ and $\omega \in A_p$, these results are also true for the spaces $H^p(\omega)$ and $F_0^{p,2}(\omega)$ (see [6]). In particular, there exist constants $c, C > 0$ depending on p, n, and ω, such that

$$c\|f\|_{H^p(\omega)} \le \|f\|_{F_0^{p,2}(\omega)} \le C\|f\|_{H^p(\omega)}.$$

Denote by $\mathcal{C}$ the Cauchy integral operator on L^p and by $\mathcal{B}$ the Bergman integral operator on $L^p(dv) = L^p(\mathbb{B}, dv)$, $p \ge 1$, given, respectively, by

$$\mathcal{C}(\psi)(z) = \int_{\mathbb{S}} \psi(\zeta)\mathcal{C}(z,\zeta)\,d\sigma(\zeta), \quad \text{and} \quad \mathcal{B}(\varphi)(z) = \int_{\mathbb{B}} \varphi(w)\mathcal{B}(z,w)\,dv(w),$$

where

$$\mathcal{C}(z,\zeta) = \frac{1}{(1-z\overline{\zeta})^n} \quad \text{and} \quad \mathcal{B}(z,w) = \frac{1}{(1-z\overline{w})^{n+1}} = \Big(I + \frac{R}{n}\Big)\mathcal{C}(z,w).$$

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Here, dv denotes the normalized Lebesgue measure on $\mathbb{B}$.

In [18, 23], the authors proved that if $1 < p < \infty$, then the Cauchy operator is bounded on $L^p(\omega)$ if and only if $\omega \in A_p$. It is then natural to consider this problem for the norm $\|\mathcal{C}: L^p(\omega) \to F_0^{p,2}(\omega)\|$ with weights $\omega \in A_p$, $p > 1$, where we recall that $F_0^{p,2}(\omega)$ is normed by (1.1). Using adequate pairings, it is easy to check (see for instance [9] for the unweighted case and Section 2 in general) that the adjoint operator of $(1 - |z|^2)\big(I + \frac{R}{n}\big)\mathcal{C}: L^p(\omega) \to L^{p,2}(\omega)$ is the Bergman operator $\mathcal{B}: L^{p',2}(\omega') \to H^{p'}(\omega')$, where p' is the conjugate exponent of p and $\omega' = \omega^{1-p'}$.

The main result of this paper is the following theorem.

Theorem 1.1 *Let $1 < p < \infty$ and let ω be a weight on $\mathbb{S}$. Then, $\mathcal{B}$ is a bounded operator from $L^{p,2}(\omega)$ to $H^p(\omega)$ if and only if $\omega \in A_p$. In this case, we have*

$$\|\mathcal{B}: L^{p,2}(\omega) \to H^p(\omega)\| \le C(p,n)[\omega]_{A_p}^{\max\{1, 1/(2(p-1))\}} \tag{1.2}$$

and the estimate is sharp.

Throughout the paper, a sharp estimate will mean that the exponent of $[\omega]_{A_p}$ cannot be replaced by a smaller one.

For the unweighted case we have the following.

Theorem 1.2 *If $1 < p < \infty$, then $\|\mathcal{B}: L^{p,2} \to H^p\| \le C(n)\max\{p, \sqrt{p'}\}$. This estimate is also sharp.*

One natural question that arises from Theorem 1.1, is the study of the norm of the operator $\mathbb{B}: L^{p,2}(\omega) \to F_0^{p,2}(\omega)$.

Theorem 1.3 *Let $1 < p < \infty$ and ω a weight on $\mathbb{S}$. Then $\mathcal{B}$ is a bounded operator from $L^{p,2}(\omega)$ to $F_0^{p,2}(\omega)$ if and only if $\omega \in A_p$.*

In this case, we have that

$$\|\mathcal{B}: L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \le C(p,n)[\omega]_{A_p}^{\max\{1, 1/(p-1)\}}.$$

If the weight ω is in a more regular class, then it can be obtained a sharper estimate of the norm of the operator. Namely, we have the following result.

Theorem 1.4 *Let $1 < p < \infty$.*

(i) *If $p > 2$ and $\omega \in A_1$, then $\|\mathcal{B}: L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \le C(p,n)[\omega]_{A_1}^{1/2}$.*

(ii) *If $1 < p < 2$ and $\omega' \in A_1$, then $\|\mathcal{B}: L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \le C(p,n)[\omega']_{A_1}^{1/2}$. In both cases the estimate is sharp.*

For the unweighted case, we obtain the following.

Theorem 1.5 *If $1 < p < \infty$, then $\|\mathcal{B}: L^{p,2} \to F_0^{p,2}\| \le C(n)\max\{\sqrt{p}, \sqrt{p'}\}$. This estimate is also sharp.*

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The paper is organized as follows. In Section 2, we recall the results on Muckenhoupt weights, duality, and extrapolation theorems that will be needed in the proof of our main results. In Section 3, we prove the necessary condition $\omega \in A_p$ in Theorems 1.1 and 1.3. In Section 4, we prove the estimate (1.2) in Theorem 1.1. Its sharpness will be proved in Section 5. In this section we also prove Theorem 1.2. Finally, Sections 6 and 7 are devoted to the proof of Theorems 1.3, 1.4, and 1.5.

A final remark on notations. If X and Y are a couple of normed spaces and T is a bounded linear operator from X to Y, we denote its norm by $\|T\colon X \to Y\|$. If $X = Y$, we also denote this norm by $\|T\|_X$.

Throughout the paper $C(p_1,\dots,p_k)$ will denote a positive constant depending only on the parameters $p_1,\dots,p_k$, which may vary from place to place. If we do not need to track the dependence of the constant, we write $f \lesssim g$ to denote the existence on a constant C such that $f \le Cg$ and $f \approx g$ to denote $f \lesssim g \lesssim f$.

2 Preliminaries

2.1 The Muckenhoupt Class A_p

In this section we recall the definition and some properties of the weights in A_p.

Definition 2.1 We say that a nonnegative function $\omega \in L^1$ is in the Muckenhoupt class A_p, $1 < p < \infty$, if

$$[\omega]_{A_p} = \sup_B \frac{\omega(B)\big(\omega'(B)\big)^{p/p'}}{|B|^p} < \infty,$$

where the supremum is taken over all nonisotropic balls B

$$B = B(\zeta, r) = \{\eta \in \mathbb{S} : |1 - \zeta\overline{\eta}| < r\},$$

$\omega' = \omega^{-(p'-1)} = \omega^{-p'/p}$ and $\omega(B) = \int_B \omega\, d\sigma$. Here, if $E \subset \mathbb{S}$ is measurable, we write $|E| = \sigma(E)$.

Definition 2.2 A nonnegative function $\omega \in L^1$ is in A_1 if there exists $C > 0$ such that for a.e. $\zeta \in \mathbb{S}$, $M[\omega](\zeta) \le C\omega(\zeta)$, where if $\psi \in L^1$, we denote by $M(\psi)$ the nonisotropic Hardy–Littlewood maximal function defined by

$$M(\psi)(\zeta) = \sup_{\zeta \in B} \frac{1}{|B|} \int_B |\psi|\, d\sigma.$$

Here $[\omega]_{A_1} = \operatorname{ess\,sup}_{\zeta\in\mathbb{S}} \frac{M(\omega)(\zeta)}{\omega(\zeta)}$.

The following property of the weights is well known (see [14]).

Lemma 2.3 (i) *If $1 \le p \le q < +\infty$, then $A_p \subset A_q$ and $[\omega]_{A_q} \le [\omega]_{A_p}$.*
(ii) *If $\omega \in A_p$, then $\omega' \in A_{p'}$ and $[\omega']_{A_{p'}} = [\omega]_{A_p}^{p'-1} = [\omega]_{A_p}^{1/(p-1)}$.*

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2.2 The Spaces $L^{p,2}(\omega)$

It is well known that if μ is a positive measure on a set $X \subset \mathbb{C}^n$, then for $1 < p < \infty$, the dual of $L^p(\mu)$ can be identified with $L^{p'}(\mu)$, in the sense that for each $\Gamma \in (L^p(\mu))'$ there exists a unique $\psi \in L^{p'}(\mu)$ such that $\Gamma(\varphi) = \int_X \varphi\psi\, d\mu$ and $\|\Gamma\| = \|\psi\|_{L^{p'}(\mu)}$.

Using this fact, we have that if $1 < p < \infty$ and $\omega \in A_p$, then for any linear form $\Gamma \in (L^p(\omega))'$ there exists a unique $\psi \in L^{p'}(\omega')$ such that $\Gamma(\psi) = \langle \varphi, \psi \rangle_{\mathbb{S}} = \int_{\mathbb{S}} \varphi \overline{\psi} d\sigma$, and moreover, $\|\Gamma\| = \|\psi\|_{L^{p'}(\omega')}$. That is, the dual of $L^p(\omega)$ with respect to the pairing $\langle \cdot, \cdot \rangle_{\mathbb{S}}$ is $L^{p'}(\omega')$.

Since the Cauchy projection maps $L^p(\omega)$ onto $H^p(\omega)$, with the same pairing we can identify the dual of $H^p(\omega)$ with $H^{p'}(\omega')$ for $p > 1$ (see [29]).

An analogous duality result for mixed-norm spaces was proved in [4], which, restricted to our case, states as follows.

Proposition 2.4 *Let $1 < p < \infty$ and $\omega \in A_p$. The dual of the mixed-norm space $L^{p,2}(\omega)$ with respect to the pairing*

$$\langle \varphi, \psi \rangle_{\mathbb{B}} = \int_{\mathbb{B}} \varphi(z)\overline{\psi(z)} \frac{dv(z)}{1 - |z|^2} = \int_{\mathbb{S}} \int_0^1 \varphi(r\zeta)\overline{\psi(r\zeta)} \frac{2nr^{2n-1} dr}{1 - r^2}\, d\sigma(\zeta)$$

is $L^{p',2}(\omega')$. That is, for any $\Gamma \in (L^{p,2}(\omega))'$ there exists $\psi \in L^{p',2}(\omega')$ such that $\Gamma(\varphi) = \langle \varphi, \psi \rangle_{\mathbb{B}}$ and $\|\Gamma\| = \|\psi\|_{L^{p',2}(\omega')}$.

2.3 The Estimate of the Hardy–Littlewood Maximal Operator

We recall that in [5] a norm-estimate was obtained for the Hardy–Littlewood maximal operator M on weighted Lebesgue spaces on $\mathbb{R}^n$. This result was extended to metric spaces with a doubling measure.

Theorem 2.5 ([20, Proposition 7.13]) *If $1 < p < \infty$ and $\omega \in A_p$, then*

$$\|M\|_{L^p(\omega)} \lesssim [w]_{A_p}^{1/(p-1)}.$$

2.4 An Extrapolation Theorem

A version of the extrapolation theorem of Rubio de Francia [15] will be used in the proof of our results.

Theorem 2.6 ([15]) *Assume that for some family of pairs of nonnegative functions, (φ, ψ), for some $p_0 \in [1, \infty)$, and for all $\omega \in A_{p_0}$, we have*

$$\Big(\int_{\mathbb{S}} \psi^{p_0} \omega\, d\sigma \Big)^{1/p_0} \leq CN([w]_{A_{p_0}}) \Big(\int_{\mathbb{S}} \varphi^{p_0} \omega\, d\sigma \Big)^{1/p_0},$$

where N is an increasing function and the constant C does not depend on ω. Then for all $1 < p < \infty$ and all $\omega \in A_p$ we have

$$\Big(\int_{\mathbb{S}} \psi^p \omega\, d\sigma \Big)^{1/p} \leq CK(w) \Big(\int_{\mathbb{S}} \varphi^p \omega\, d\sigma \Big)^{1/p},$$

where

$$K(\omega)=\begin{cases}N([\omega]_{A_p}(2\|M\|_{L^p(\omega)})^{p_0-p}) & \text{if } p<p_0,\\ N([\omega]_{A_p}^{(p_0-1)/(p-1)}(2\|M\|_{L^{p'}(\omega')})^{(p-p_0)/(p-1)}) & \text{if } p>p_0.\end{cases}$$

In particular, $K(w)\le C_1N((C_2[\omega]_{A_p}^{\max\{1,(p_0-1)/(p-1)\}})$.

Remark 2.7 This theorem is proved in [15] in $\mathbb{R}^n$, but it can be easily extended to the setting of homogeneous spaces using Theorem 2.5.

3 Proof of the Necessary Condition in Theorems 1.1 and 1.3

Since

$$\|\mathcal{B}:L^{p,2}(\omega)\to F_0^{p,2}(\omega)\|=\|\mathcal{Q}:L^{p,2}(\omega)\to L^{p,2}(\omega)\|,$$

where $\mathcal{Q}(\varphi)(z)=(1-|z|^2)(I+\frac{R}{n})\mathcal{B}(\varphi)(z)$, the necessary condition $\omega\in A_p$ in Theorems 1.1 and 1.3, follows from Proposition 3.1.

Proposition 3.1 *Let $1<p<\infty$ and let $0\le\omega\in L^1$. If either $\mathcal{B}$ is bounded from $L^{p,2}(\omega)$ to $L^p(\omega)$, or $\mathcal{Q}$ is bounded from $L^{p,2}(\omega)$ to itself, then $\omega\in A_p$.*

Proof The proof of this proposition follows using standard arguments (see, for instance, [3,10]) and, for a sake of completeness, we will give a sketch of it.

For $0\ne a\in\mathbb{B}$, let $a^*=a/|a|$, $B_a=\{\zeta\in\mathbb{S}:|1-\zeta\overline{a}^*|<1-|a|\}$ and let S_a be the nonisotropic square $S_a=\{w=s\eta\in\overline{\mathbb{B}}:1-s\le1-|a|,|1-\eta\overline{a}^*|\le1-|a|,\eta\in\mathbb{S}\}$.

Note that if $w=s\eta\in S_a$, then $1-|a|\le|1-w\overline{a}|\le1-|a|+1-s+|1-\eta\overline{a}^*|\le3(1-|a|)$. Since $d(z,w)=|1-z\overline{w}|^{1/2}$ satisfies the triangle inequality (see [33, Proposition 5.1.2]), for $\kappa>0$ large enough, there exists $0<r_\kappa<1$ such that for each $a\in\mathbb{B}$, $|a|>r_\kappa$, there exists $b\in\mathbb{B}$ satisfying $|a|=|b|$, $|1-b\overline{a}|=\kappa(1-|a|)$, and

$$|1-z\overline{w}|\approx|1-z\overline{a}|\approx\kappa(1-|a|)\approx\kappa|1-w\overline{a}|,\quad\text{for any } w\in S_a \text{ and } z\in S_b,$$

where the constants in the equivalences do not depend on z,w,a,b, and κ. Thus,

$$|\mathcal{B}(z,w)-\mathcal{B}(z,a)|\le c(n)\frac{|1-w\overline{a}|^{1/2}}{|1-z\overline{a}|^{n+3/2}}\le\frac{c'(n)}{\sqrt{\kappa}}\frac{1}{|1-z\overline{a}|^{n+1}}.$$

So, choosing $\kappa\ge(2c'(n))^2$, for any $0\le\varphi\in L^1(d\nu)$ we have

$$\mathcal{X}_b(z)|\mathcal{B}(\mathcal{X}_a\varphi)(z)|\ge\frac{1}{2}\frac{\mathcal{X}_b(z)}{|1-z\overline{a}|^{n+1}}\int_{S_a}\varphi\,d\nu\approx\frac{\mathcal{X}_b(z)}{(1-|a|^2)^{n+1}}\int_{S_a}\varphi\,d\nu. \tag{3.1}$$

where $\mathcal{X}_a$ and $\mathcal{X}_b$ denote the characteristic function of S_a and S_b, respectively, and the constants in the last equivalence depend only on n and κ.

Analogously we have

$$\mathcal{X}_b(z)\frac{1-|z|^2}{(1-|a|^2)^{n+2}}\int_{S_a}\varphi\,d\nu\lesssim\mathcal{X}_b(z)|\mathcal{Q}(\mathcal{X}_a\varphi)(z)|, \tag{3.2}$$

where, as above, the constants in the inequality depend only on n and κ.

Let $\psi\ge0$ be a continuous function on $\overline{\mathbb{B}}$ and for $s>0$ let

$$\varphi(s\eta)=(1-s)\mathcal{X}_a(\eta)\psi(\eta)\in L^{p,2}(\omega).$$

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Then by integration in polar coordinates and using that $1-|a| = 1-|b|$, we have

$$\int_{\mathbb{B}} \mathcal{X}_a(w)\varphi(w)\,dv(w) \approx (1-|a|^2)^2 \int_{\mathbb{S}} \mathcal{X}_a(\eta)\psi(\eta)\,d\sigma(\eta),$$
$$\|\mathcal{X}_a\varphi\|_{L^{p,2}(\omega)} \approx (1-|a|) \int_{\mathbb{S}} \mathcal{X}_a(\eta)\psi(\eta)\omega(\eta)d\sigma(\eta),$$
$$\int_{\mathbb{S}} \mathcal{X}_b(\eta)\omega(\eta)\,d\sigma(\eta) \approx \omega(B_b)$$
$$\|(1-|z|^2)\mathcal{X}_b(z)\|_{L^{p,2}(\omega)} \approx (1-|a|)\omega(B_b)^{1/p}.$$

Therefore, (3.1) and (3.2) give

$$\frac{\omega(B_b)^{1/p}}{(1-|a|^2)^n}\int_{B_a}\psi d\sigma \lesssim \|\mathcal{B}\colon L^{p,2}(\omega)\to H^p(\omega)\|\Big(\int_{B_a}\psi^p\omega\,d\sigma\Big)^{1/p},$$
$$\frac{\omega(B_b)^{1/p}}{(1-|a|^2)^n}\int_{B_a}\psi d\sigma \lesssim \|\mathcal{B}\colon L^{p,2}(\omega)\to F_0^{p,2}(\omega)\|\Big(\int_{B_a}\psi^p\omega\,d\sigma\Big)^{1/p}.$$

These inequalities applied to the function $\psi = 1$ give $\omega(B_b) \lesssim \omega(B_a)$. Interchanging a and b we also obtain $\omega(B_b) \approx \omega(B_a)$. Hence, in both cases for any $|a| > r_\kappa$ and ψ a continuous function on $\mathbb{S}$, we have

$$\Big(\frac{1}{\sigma(B_a)}\int_{B_a}\psi\,d\sigma\Big)^p \lesssim \frac{1}{\omega(B_a)}\int_{B_a}\psi^p\omega\,d\sigma.$$

Since $\mathbb{S}$ is the finite union of sets B_{a_j}, $|a_j| > r_\kappa$, and the space of continuous functions on $\mathbb{S}$ is dense in $L^p(\omega)$, the above inequality holds for any B_a and any $\psi \in L^p(\omega)$. This is equivalent to $\omega \in A_p$ (see, for instance, [34, p. 195]). ∎

Remark 3.2 It is well known that the boundedness of $\mathcal{B}$ on $L^p(\mathbb{B})$ is equivalent to the boundedness on $L^p(\mathbb{B})$ of the integral operator $|\mathcal{B}|$ associated to the kernel $|\mathcal{B}(z,w)|$. In our situation, even for $n = 1$, we have that the integral operator $|\mathcal{B}|$ is not bounded from $L^{p,2}$ to L^p. For instance, consider the function $\varphi(w) = (\log\frac{2}{1-|w|^2})^{-1} \in L^{p,2}$. We have

$$\int_{\mathbb{D}}\frac{\big(\log\frac{2}{1-|w|^2}\big)^{-1}}{|1-z\overline{w}|^2}\,dv(w) \gtrsim \int_0^1 r\Big(\log\frac{2}{1-r^2}\Big)^{-1}\int_0^1\frac{dt}{(1-|z|^2+1-r^2+t)^2}\,dr$$
$$\gtrsim \int_0^1\frac{r\big(\log\frac{2}{1-r^2}\big)^{-1}}{1-|z|^2+1-r^2}\,dr \gtrsim \log\log\frac{2}{1-|z|^2}.$$

Consequently, $|\mathcal{B}|(\varphi)$ does not have boundary values.

4 Proof of the Estimate (1.2) in Theorem 1.1

In Proposition 3.1, we proved that if the Bergman operator is bounded from $L^{p,2}(\omega)$ to $H^p(\omega)$, then $\omega \in A_p$. In order to finish the proof of Theorem 1.1, we first observe that condition (1.2) can be rewritten by duality as an estimate of the weighted Triebel–Lizorkin norm of the Cauchy operator. Namely, we have the following result.

Proposition 4.1 *If $1 < p < \infty$, we have that the following conditions are equivalent.*

(i) *For any* $\omega \in A_p$ *and any* $\varphi \in L^{p,2}(\omega)$,

$$\|\mathcal{B}(\varphi)\|_{H^p(\omega)} \leq C(p,n)[\omega]_{A_p}^{\max\{1,1/(2(p-1))\}}\|\varphi\|_{L^{p,2}(\omega)}. \tag{4.1}$$

(ii) *For* $\omega \in A_p$ *and any* $\psi \in L^p(\omega)$,

$$\begin{aligned}\|\mathcal{C}(\psi)\|_{F_0^{p,2}(\omega)} &= \|(1-|z|^2)(I+\frac{R}{n})\mathcal{C}(\psi)(z)\|_{L^{p,2}(\omega)}\\ &\leq C(p,n)[\omega]^{\max\{1/2,1/(p-1)\}}\|\psi\|_{L^p(\omega)}.\end{aligned}$$

Moreover, $\|\mathcal{B}\colon L^{p,2}(\omega)\to L^p(\omega)\| = \|\mathcal{C}\colon L^{p'}(\omega')\to F_0^{p',2}(\omega')\|$.

Proof Since

$$\Big(I+\frac{1}{n}R\Big)\frac{1}{(1-z\bar\zeta)^n} = \frac{1}{(1-z\bar\zeta)^{n+1}},$$

for any smooth functions φ and ψ on $\mathbb{B}$ and $\mathbb{S}$, respectively, Fubini's Theorem gives that

$$\langle\mathcal{B}(\varphi),\psi\rangle_{\mathbb{S}} = \Big\langle \varphi(z),(1-|z|^2)\Big(I+\frac{1}{n}R\Big)\mathcal{C}(\psi)(z)\Big\rangle_{\mathbb{B}}.$$

Hence, (4.1) is equivalent to

$$\begin{aligned}\|\mathcal{C}(\psi)\|_{F_0^{p',2}(\omega')} &= \Big\|(1-|z|^2)\Big(I+\frac{R}{n}\Big)\mathcal{C}(\psi)(z)\Big\|_{L^{p',2}(\omega')}\\ &\leq C(p,n)[\omega']^{\max\{1/2,1/(p'-1)\}}\|\psi\|_{L^{p'}(\omega')},\end{aligned} \tag{4.2}$$

which is also equivalent to (ii). ∎

Observe that the key estimate (4.2) is a non isotropic version of [25, Theorem 1.1], which is based in the intrinsic square function introduced in [36]. The original proof heavily relies on the convolution in $\mathbb{R}^n$. In our situation, there is no such convolution and we instead follow closely some of the main ideas in [28, Theorem 1.1].

Although we state our main results for the operator $(I+\frac{R}{n})\mathcal{C}$, all the norm-operator estimates also hold for any operator $(\alpha I+\beta R)\mathcal{C}$, $\alpha,\beta\in\mathbb{R}$ (see Remark 4.12 below).

4.1 Preliminary Results

In the proof of our main results we will use the dyadic decomposition of a quasi-metric space of [8] (see also [20, 22]). We recall that ρ is a quasi-metric on a space X if it satisfies the axioms of a metric except for the triangle inequality, which is assumed in a weaker form: there exists $A_0\geq 1$ such that for any $x,y,z\in X$, $\rho(x,y)\leq A_0(\rho(x.z)+\rho(z,y))$. The quasi-metric space (X,ρ) is also assumed to satisfy the following geometric doubling property: there exists $N\in\mathbb{N}$ such that for every $x\in X$ and for every $r>0$, the ball $B(x,r)=\{y\in X;\rho(x,y)<r\}$ can be covered by at most N balls $B(x_i,r/2)$. We will state the decomposition for $\mathbb{S}$ and the quasi-metric $\rho(\zeta,\eta)=|1-\zeta\overline{\eta}|$. Observe that $A_0=2$.

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Proposition 4.2 *Given a fixed parameter $0 < \delta < 1$, small enough and a fixed point $x_0 \in \mathbb{S}$, there exists a finite collection of families of sets, $\mathcal{D}^j$, $j = 1, \dots, M$, called the* adjacent dyadic systems, *such that each $\mathcal{D}^j$ is a family of Borel sets Q^k_α, $k \in \mathbb{Z}$, $\alpha \in I_k$, called the* dyadic cubes, *that are associated with points ζ^k_α, which we will call the* center points *of the cubes Q^k_α, having the following properties.*

(i) $\mathbb{S} = \bigcup_{\alpha \in I_j} Q^k_\alpha$ *(disjoint union), for each $k \in \mathbb{Z}$.*
(ii) *if $k < l$, then either $Q^l_\beta \cap Q^k_\alpha = \varnothing$ or $Q^l_\beta \subset Q^k_\alpha$.*
(iii) *There exist $c_1, C_1 > 0$ such that $B(\zeta^k_\alpha, c_1\delta^k) \subset Q^k_\alpha \subset B(\zeta^k_\alpha, C_1\delta^k) = B(Q^k_\alpha)$.*
(iv) *If $k \le l$ and $Q^l_\beta \subset Q^k_\alpha$, then $B(Q^l_\beta) \subset B(Q^k_\alpha)$.*
(v) *For any $k \in \mathbb{Z}$, there exists α such that $x_0 = \zeta^k_\alpha$, the center point of Q^k_α.*
(vi) *There exists $C > 0$ (only depending on A_0 and δ) such that for any nonisotropic ball $B(\zeta, r) \subset \mathbb{S}$ with $\delta^{k+3} < r \le \delta^{k+2}$, there exists j and $Q^k_\alpha \in \mathcal{D}^j$ such that $B(\zeta, r) \subset Q^k_\alpha$ and* diam $Q^k_\alpha \le Cr$.

The family $\mathcal{D} = \bigcup_{j=1}^M \mathcal{D}^j$ is called a dyadic decomposition *of $\mathbb{S}$, and we say that the set Q^k_α is a* dyadic cube of generation k *centered at ζ^k_α with radius $l(Q^k_\alpha) = \delta^k$.*

Remark 4.3 It is immediate to check that from properties (iii), (i), and (ii) that there exists $\varepsilon > 0$ (only depending on the dimension n and on δ) and for any $Q^k_1 \in \mathcal{D}^j$ there exists at least one $Q^{k+1}_2 \in \mathcal{D}^j$ so that $Q^{k+1}_2 \subset Q^k_1$ and

$$|Q^{k+1}_2| \ge \varepsilon |Q^k_1|. \tag{4.3}$$

Before we go back to the proof of Proposition 4.1, we need to introduce some more notations and results. The non-increasing rearrangement of a measurable function ψ on $\mathbb{S}$ is defined by

$$\psi^*(t) = \inf\{\alpha > 0 ; |\{\zeta \in \mathbb{S}; |\psi(\zeta)| > \alpha\}| \le t\} = \sup_{C \subset \mathbb{S}; |C| = t} \inf_{\zeta \in C} |\psi(\zeta)|, \quad 0 < t < \infty.$$

It is immediate to check that

$$|\{\zeta \in \mathbb{S}; |\psi(\zeta)| > \lambda\}| = |\{t > 0 ; \psi^*(t) > \lambda\}|.$$

Let ψ be a measurable function on $\mathbb{S}$. If Q is a dyadic cube, the local mean oscillation of ψ on Q is given by

$$\omega_\lambda(\psi; Q) = \inf_{c \in \mathbb{R}} ((\psi - c)\mathcal{X}_Q)^*(\lambda |Q|), \quad 0 < \lambda < 1.$$

We will denote by $m_Q(\psi)$, the median value of ψ over Q, a (possibly non unique) real number such that

$$\max\{ |\{\zeta \in Q ; \psi(\zeta) > m_Q(\psi)\}|, |\{\zeta \in Q ; \psi(\zeta) < m_Q(\psi)\}| \} \le |Q|/2.$$

It is immediate to check that $|m_Q(\psi)| \le (\psi \mathcal{X}_Q)^*(|Q|/2)$. Next, given a dyadic cube $Q_0 \in \mathcal{D}^j$, let us denote $\mathcal{D}^j(Q_0)$ the dyadic cubes of $\mathcal{D}^j$ contained in Q_0. The dyadic local sharp maximal function $m^{\#,d}_{\lambda;Q_0}\psi$ is defined by

$$m^{\#}_{\lambda;Q_0}\psi(\zeta) = \sup_{\zeta \in Q' \in \mathcal{D}^j(Q_0)} \omega_\lambda(\psi; Q').$$

It is also well known (see, for instance, [21]) that a.e $\zeta \in \mathbb{S}$, $m^{\#}_{\lambda;Q_0}\psi(\zeta) \lesssim M[\psi](\zeta)$.

If $Q_0 \in \mathcal{D}^j$, a family of sets $\mathcal{S}(Q_0)$ is sparse in Q_0 with respect to the dyadic decomposition $\mathcal{D}$, if $\mathcal{S}(Q_0) = \bigcup_{m\geq 0} C_m$, where (1) each C_m is a family of sets in $\mathcal{D}^j$ which are subsets of Q_0; (2) $C_0 = \{Q_0\}$; (3) the elements of each family C_m are pairwise disjoint; (4) for any $m > 0$, every $Q \in C_m$ is a subset of an element of C_{m-1}; (5) for any $Q_1 \in C_m$ we have that $|\bigcup_{Q\in C_{m+1}} Q \cap Q_1| \leq \frac{|Q_1|}{2}$. We let

$$E_{Q_1} = Q_1 \setminus \bigcup_{Q\in C_{n+1}} Q \cap Q_1. \tag{4.4}$$

We then have that $|E_{Q_1}| \geq |Q_1|/2$.

The proof of Theorem 1.1 is based on a homogeneous version of the key estimate in [24], that it is proved in [2].

Theorem 4.4 *Let ψ a measurable function on $\mathbb{S}$ and $Q_0 \in \mathcal{D}^j$ a fixed cube and ε as in (4.3). Then there exists a (possibly empty) sparse family of cubes $\mathcal{S}(Q_0)$ such that for a.e. $\zeta \in Q_0$,*

$$|\psi(\zeta) - m_{Q_0}(\psi)| \leq m^{\#}_{\varepsilon/4,Q_0}(\psi)(\zeta) + \sum_{Q\in\mathcal{S}(Q_0)} \omega_{\varepsilon/4}(\psi, Q)\mathcal{X}_Q(\zeta).$$

4.2 Main Estimate

We begin recalling some technical lemmas. The first one is a version of a Whitney decomposition of an open set in $\mathbb{S}$ that can be found in [7].

Lemma 4.5 *Let $R > 1$ and let Ω be an open set in $\mathbb{S}$. Consider a dyadic adjacent system $\mathcal{D}^j$ in $\mathbb{S}$, $j \in \{1,\dots,M\}$. If j is fixed, let Λ^j be the family of cubes $Q^k_\alpha \in \mathcal{D}^j$, which are maximal with respect to the property $RB(Q^k_\alpha) \subset \Omega$. We then have:*

(i) *$\Omega = \bigcup_{Q^k_\alpha\in\Lambda^j} Q^k_\alpha$ and for the cubes in Λ^j, either $Q^k_\alpha \cap Q^{k_1}_{\alpha_1} = \varnothing$ or $Q^k_\alpha = Q^{k_1}_{\alpha_1}$.*

(ii) *There exists $K > 0$ only depending on the constants C_1 and δ of the definition of the dyadic adjacent system (see Proposition 4.2), such that for every $Q^k_\alpha \in \Lambda^j$, we have that $KRB(Q^k_\alpha) \cap \Omega^c \neq \varnothing$.*

(iii) *There exists $C(C_1,\delta) > 0$, only depending on the constants C_1 and δ of the definition of the dyadic adjacent system, such that*

$$\sum_{Q^k_\alpha\in\Lambda^j} \mathcal{X}_{RB(Q^k_\alpha)} \leq C(C_1,\delta)\mathcal{X}_\Omega.$$

Tchoundja [35] proved the following.

Lemma 4.6 *There exist $K_1, K_2 > 0$ such that for any $\zeta, \zeta', \xi \in \mathbb{S}$, $\rho < 1$, satisfying $|1-\zeta\overline{\xi}| \geq K_1|1-\zeta\overline{\zeta'}|$, we have*

$$\Big|\frac{1}{(1-\rho\zeta\overline{\xi})^{n+1}} - \frac{1}{(1-\rho\zeta'\overline{\xi})^{n+1}}\Big| \leq K_2\Big(\frac{|1-\zeta\overline{\zeta'}|}{|1-\zeta\overline{\xi}|}\Big)^{\frac12}\frac{1}{|1-\zeta\overline{\xi}|^{n+1}}.$$

The following lemma is based on the well-known technique of splitting functions of A. P. Calderon and A. Zygmund.

Lemma 4.7 *There exists $C > 0$ such that for any $\lambda > 0$, $\psi \in L^1$,*

$$\Big|\Big\{\eta \in \mathbb{S}\,;\Big(\int_0^1 (1-r^2)\Big|\int_{\mathbb{S}} \frac{\psi(\zeta)}{(1-r\eta\overline{\zeta})^{n+1}}\,d\sigma(\zeta)\Big|^2 dr\Big)^{1/2} > \lambda\Big\}\Big| \lesssim \frac{\|\psi\|_{L^1}}{\lambda}.$$

Proof We denote by $G(\psi)$ the function on $\mathbb{S}$ defined by

$$G(\psi)(\eta) = \Big(\int_0^1 \Big|\Big(I + \frac{R}{n}\Big)\mathcal{C}(\psi)(r\eta)\Big|^2 (1-r^2)\,dr\Big)^{\frac{1}{2}}.$$

If $\lambda > 0$ and $\psi \in L^1$, we denote $\Omega_\lambda = \{\eta \in \mathbb{S}\,;\, M(\psi)(\eta) > \lambda\}$.

Since the nonisotropic Hardy–Littlewood maximal operator is of weak type $(1,1)$, we have that $|\Omega_\lambda| \lesssim \frac{1}{\lambda}\int_{\mathbb{S}} |\psi(\zeta)|\,d\sigma(\zeta)$.

We must then estimate $|\{\eta \notin \Omega_\lambda\,;\, G(\psi)(\eta) > \lambda\}|$. By Lemma 4.5, there exists $(Q_k)_k$ a Whitney decomposition of the set Ω_λ. We split ψ into two pieces, $\psi = g + b$, where

$$g(\zeta) = \begin{cases} \psi(\zeta) & \zeta \notin \Omega_\lambda, \\ \frac{1}{|Q_k|}\int_{Q_k} \psi\,d\sigma & \zeta \in Q_k. \end{cases}$$

Property (ii) of the Whitney decomposition gives that $\|g\|_\infty \lesssim \lambda$. Put $b_k = b\mathcal{X}_{Q_k} = (\psi - \psi_{Q_k})\mathcal{X}_{Q_k}$, where $\psi_{Q_k} = \frac{1}{|Q_k|}\int_{Q_k} \psi\,d\sigma$. Then b_k is supported in Q_k, $\int_{Q_k} b_k = 0$ and $\|b_k\|_{L^1} \lesssim \int_{Q_k} |\psi|\,d\sigma$. We also have that $b = \sum_k b_k$.

We decompose:

$$\begin{aligned} |\{\eta \notin \Omega_\lambda; G(\psi)(\eta) > \lambda\}| &\le |\{\eta \notin \Omega_\lambda\,; G(g)(\eta) > \lambda/2\}| + |\{\eta \notin \Omega_\lambda\,; G(b)(\eta) > \lambda/2\}| \\ &= I + II. \end{aligned}$$

We will estimate each term separately.

For the first one we use Chebyshev's inequality and the facts that both $\mathcal{C}$ and the Littlewood–Paley g-function are bounded on $L^2(\mathbb{S})$.

$$\begin{aligned} |\{\eta \notin \Omega_\lambda; G(g)(\eta) > \lambda/2\}| &\lesssim \frac{1}{\lambda^2}\int_{\mathbb{S}} G(g)(\eta)^2\,d\sigma(\eta) \lesssim \frac{1}{\lambda^2}\int_{\mathbb{S}} |g(\eta)|^2\,d\sigma(\eta) \\ &\lesssim \frac{1}{\lambda}\Big(\int_{\mathbb{S}\setminus\Omega_\lambda} |\psi(\zeta)|\,d\sigma(\zeta) + \int_{\Omega_\lambda} |g(\zeta)|\,d\sigma(\zeta)\Big) \\ &\lesssim \frac{1}{\lambda}\Big(\int_{\mathbb{S}\setminus\Omega_\lambda} |\psi(\zeta)|\,d\sigma(\zeta) + \sum_k \frac{1}{|Q_k|}\int_{Q_k} |\psi(\zeta)|\,d\sigma(\zeta)\Big) \\ &\lesssim \frac{1}{\lambda}\int_{\mathbb{S}} |\psi(\zeta)|\,d\sigma(\zeta), \end{aligned}$$

We now estimate II. Let $\eta \in \mathbb{S} \notin \Omega_\lambda$. Denote by ξ_k the "center" of Q_k, $k \ge 1$. Since for each $k \ge 1$, $\int_{\mathbb{S}} b_k = 0$, we have

$$\int_{\mathbb{S}} \frac{1}{(1-r\eta\overline{\zeta})^{n+1}} b_k(\zeta)\,d\sigma(\zeta) = \int_{\mathbb{S}} \Big(\frac{1}{(1-r\eta\overline{\zeta})^{n+1}} - \frac{1}{(1-r\eta\overline{\xi_k})^{n+1}}\Big) b_k(\zeta) d\sigma(\zeta).$$

Next, observe that if we choose R in Lemma 4.5 such that for any $\zeta \in Q_k$ and $\eta \in \mathbb{S} \notin \Omega_\lambda$, we have that $|1 - r\eta\overline{\zeta}| \ge K_1|1 - \zeta\overline{\xi}_k|$, where K_1 is as in Lemma 4.6. Thus, this lemma

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gives that the above integral is bounded by

$$\int_{\mathbb{S}} \frac{|1-\zeta\overline{\xi_k}|^{1/2}}{|1-r\eta\overline{\xi_k}|^{n+1+1/2}}|b_k(\zeta)|\,d\sigma(\zeta) \lesssim \int_{\mathbb{S}} \frac{l(Q_k)^{1/2}}{|1-r\eta\overline{\xi_k}|^{n+1+1/2}}|b_k(\zeta)|\,d\sigma(\zeta).$$

But

$$\Big(\int_0^1 (1-r^2)\frac{dr}{|1-r\eta\overline{\xi_k}|^{2n+3}}\Big)^{1/2} \lesssim \frac{1}{|1-\eta\overline{\xi_k}|^{n+1/2}},$$

and, consequently,

$$\begin{aligned}\int_{\mathbb{S}\setminus\Omega_\lambda} G(b_k)(\eta)\,d\sigma(\eta) &\lesssim l(Q_k)^{1/2}\int_{Q_k}|b_k(\zeta)|\,d\sigma(\zeta)\int_{\mathbb{S}\setminus\Omega_\lambda}\frac{d\sigma(\eta)}{|1-\eta\overline{\xi_k}|^{n+1/2}}\\ &\lesssim \int_{Q_k}|b_k(\zeta)|\,d\sigma(\zeta).\end{aligned}$$

Altogether,

$$\begin{aligned}\int_{\mathbb{S}\setminus\Omega_\lambda} G(b)(\eta)\,d\sigma(\eta) &\lesssim \sum_k \int_{Q_k}|b_k(\zeta)|\,d\sigma(\zeta)\\ &\lesssim \sum_k \int_{Q_k}|\psi(\zeta)|\,d\sigma(\zeta) \lesssim \int_{\mathbb{S}}|\psi(\zeta)|\,d\sigma(\zeta).\end{aligned}$$

From this estimate we deduce immediately that

$$|\{\eta\notin\Omega_\lambda\,;\,G(b)(\eta)>\lambda/2\}| \lesssim \frac{1}{\lambda}\int_{\mathbb{S}}|\psi(\zeta)\,d\sigma(\zeta)|.$$

And that finishes the proof. ∎

We now can prove the main lemma, which is a version for the sphere with the nonisotropic distance ρ of Lemma 3.1 in [25]. In our situation here, we skip the fact that we do not have convolution, using the estimate in Lemma 4.6.

Lemma 4.8 *Let $\mathcal{D}^j$, $j=1,\dots,M$ an adjacent dyadic system in $\mathbb{S}$ as in Proposition 4.2 and let $0<\lambda<1$ be fixed. Then for $\psi\in L^1$ and for any cube $Q\in\mathcal{D}^j$, we have the estimate*

$$\omega_\lambda(G(\psi)^2;Q) \lesssim \sum_{k\geq 0}\frac{1}{2^{k/2}}\Big(\frac{1}{|2^kB(Q)|}\int_{2^kB(Q)}|\psi(\eta)|\,d\sigma(\eta)\Big)^2.$$

Proof Let K_1, K_2 be as in Lemma 4.6. If $\zeta\in Q$, we decompose $G(\psi)^2(\zeta)$ in two terms given by

$$\begin{aligned}G(\psi)^2(\zeta) &= \int_{1-4K_1 l(Q)}^1 \Big|\Big(I+\frac{R}{n}\Big)\mathcal{C}(\psi)(r\zeta)\Big|^2(1-r^2)\,dr\\ &\quad+\int_0^{1-4K_1 l(Q)}\Big|\Big(I+\frac{R}{n}\Big)\mathcal{C}(\psi)(r\zeta)\Big|^2(1-r^2)\,dr\\ &= I_1(\psi)(\zeta)+I_2(\psi)(\zeta).\end{aligned}$$

We will first show that

$$(I_1(\psi)\mathcal{X}_Q)^*(\lambda|Q|) \lesssim \sum_{k\geq 0}\frac{1}{2^k}\Big(\frac{1}{|2^kB(Q)|}\int_{2^kB(Q)}|\psi(\eta)|\,d\sigma(\eta)\Big)^2. \tag{4.5}$$

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Since $(x+y)^2 \le 2(x^2+y^2)$, we have that for any $\zeta \in Q$,

$$I_1(\psi)(\zeta) \le 2\big(I_1(\psi \mathcal{X}_{4B(Q)})(\zeta) + I_1(\psi \mathcal{X}_{\mathbb{S}\smallsetminus 4B(Q)})(\zeta)\big),$$

and consequently,

$$(I_1(\psi)\mathcal{X}_Q)^*(\lambda|Q|) \lesssim \big(I_1(\psi \mathcal{X}_{4B(Q)})\big)^*(\lambda|Q|/2) + \big(I_1(\psi \mathcal{X}_{\mathbb{S}\smallsetminus 4B(Q)})\big)^*(\lambda|Q|/2).$$

By Lemma 4.7 we have that

$$\begin{aligned}\big(I_1(\psi \mathcal{X}_{4B(Q)})\big)^*(\lambda|Q|/2) &\le \big((G(\psi \mathcal{X}_{4B(Q)}))^2\big)^*(\lambda|Q|/2)\\ &\lesssim \Big(\frac{1}{|4B(Q)|}\int_{4B(Q)} |\psi(\eta)|\, d\sigma(\eta)\Big)^2.\end{aligned}$$

For any $\eta \in \mathbb{S} \smallsetminus 4B(Q)$, $|1-\zeta\overline{\eta}| > l(Q)$. Hence,

$$\begin{aligned}\Big|\Big(I + \frac{R}{n}\Big)\mathcal{C}(\psi \mathcal{X}_{\mathbb{S}\smallsetminus 4B(Q)})(r\zeta)\Big| &\lesssim \int_{|1-\zeta\overline{\eta}|>l(Q)} \frac{|\psi(\eta)|}{((1-r)+|1-\zeta\overline{\eta}|)^{n+1}}\, d\sigma(\eta)\\ &\lesssim \sum_{k\ge 0} \frac{1}{(2^k l(Q))^{n+1}} \int_{2^k B(Q)} |\psi(\eta)|\, d\sigma(\eta)\\ &\approx \frac{1}{l(Q)} \sum_{k\ge 0} \frac{1}{2^k} \frac{1}{|2^k B(Q)|}\int_{2^k B(Q)} |\psi(\eta)|\, d\sigma(\eta).\end{aligned}$$

Thus, for any $\zeta \in Q$

$$\begin{aligned}&I_1(\psi \mathcal{X}_{\mathbb{S}\smallsetminus 4B(Q)})(\zeta)\\ &\quad\lesssim \int_{1-4K_1 l(Q)}^1 \frac{1}{l(Q)^2}\Big(\sum_{k\ge 0}\frac{1}{2^k}\frac{1}{|2^k B(Q)|}\int_{2^k B(Q)} |\psi(\eta)|\, d\sigma(\eta)\Big)^2 (1-r^2)dr\\ &\quad\approx \Big(\sum_{k\ge 0}\frac{1}{2^k}\frac{1}{|2^k B(Q)|}\int_{2^k B(Q)} |\psi(\eta)|\, d\sigma(\eta)\Big)^2.\end{aligned}$$

By Chebyshev's inequality,

$$\begin{aligned}\big(I_1(\psi \mathcal{X}_{\mathbb{S}\smallsetminus 4B(Q)})\mathcal{X}_Q\big)^*(\lambda|Q|/2) &\lesssim \frac{\|I_1(\psi \mathcal{X}_{\mathbb{S}\smallsetminus 4B(Q)})\mathcal{X}_Q\|_{L^1}}{(\lambda|Q|)/2}\\ &\lesssim \frac{|Q|}{\lambda|Q|}\Big(\sum_{k\ge 0}\frac{1}{2^k}\frac{1}{|2^k B(Q)|}\int_{2^k B(Q)} |\psi(\eta)|\, d\sigma(\eta)\Big)^2,\end{aligned}$$

and consequently, applying Schwartz's inequality,

$$\begin{aligned}&\big(I_1(\psi \mathcal{X}_{\mathbb{S}\smallsetminus 4B(Q)})\mathcal{X}_Q\big)^*(\lambda|Q|/2)\\ &\qquad\lesssim \sum_{k\ge 0}\frac{1}{2^k}\Big(\frac{1}{|2^k B(Q)|}\int_{2^k B(Q)} |\psi(\eta)|\, d\sigma(\eta)\Big)^2,\end{aligned}\tag{4.6}$$

which finishes the proof of the estimate (4.5).

In order to estimate $\omega_\lambda(G(\psi)^2;Q)$, consider any $\zeta_2 \in \mathbb{S}$. Observe that

$$\begin{aligned}\omega_\lambda(G(\psi)^2;Q) &\le \big((G(\psi)^2 - I_2(\psi)(\zeta_2))\mathcal{X}_Q\big)^*(\lambda|Q|)\\ &\lesssim \big(I_1(\psi \mathcal{X}_Q)\big)^*((\lambda|Q|/2) + ((I_2(\psi) - I_2(\psi)(\zeta_2))\mathcal{X}_Q)^*((\lambda|Q|/2))\\ &\lesssim (I_1(\psi)\mathcal{X}_Q)^*((\lambda|Q|)/2) + \|I_2(\psi) - I_2(\psi)(\zeta_2)\|_{L^\infty(Q)}.\end{aligned}$$

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So we are left to estimate $\|I_2(\psi)-I_2(\psi)(\zeta_2)\|_{L^\infty(Q)}$. Let $\zeta_1,\zeta_2\in Q$. Then,

$$\begin{aligned}&|I_2(\psi)(\zeta_1)-I_2(\psi)(\zeta_2)|\\&\quad=\Big|\int_0^{1-4K_1l(Q)}\Big(\Big|\Big(I+\frac{R}{n}\Big)\mathcal{C}(\psi)(r\zeta_1)\Big|^2-\Big|\Big(I+\frac{R}{n}\Big)\mathcal{C}(\psi)(r\zeta_2)\Big|^2\Big)(1-r^2)\,dr\Big|.\end{aligned}$$

But

$$\begin{aligned}&\Big|\Big|\int_{\mathbb{S}}\frac{\psi(\eta)}{(1-r\zeta_1\overline{\eta})^{n+1}}\,d\sigma(\eta)\Big|^2-\Big|\int_{\mathbb{S}}\frac{\psi(\eta)}{(1-r\zeta_2\overline{\eta})^{n+1}}\,d\sigma(\eta)\Big|^2\Big|\\&\qquad\le\Big|\int_{\mathbb{S}}\Big|\frac{1}{(1-r\zeta_1\overline{\eta})^{n+1}}-\frac{1}{(1-r\zeta_2\overline{\eta})^{n+1}}\Big||\psi(\eta)|\,d\sigma(\eta)\Big|\\&\qquad\qquad\times\Big(\Big|\int_{\mathbb{S}}\frac{|\psi(\eta)|}{|1-r\zeta_1\overline{\eta}|^{n+1}}\,d\sigma(\eta)\Big|+\Big|\int_{\mathbb{S}}\frac{|\psi(\eta)|}{|1-r\zeta_2\overline{\eta}|^{n+1}}\,d\sigma(\eta)\Big|\Big).\end{aligned}$$

Now if $\zeta_1,\zeta_2\in Q$, $|1-\zeta_1\overline{\zeta_2}|\le 4l(Q)$, then for any $0<r<1-4K_1l(Q)$ and any $\eta\in\mathbb{S}$, we have $|1-r\overline{\eta}\zeta_1|\ge K_1|1-\zeta_1\overline{\zeta_2}|$. Consequently, applying Lemma 4.6,

$$\begin{aligned}\Big|\frac{1}{(1-r\zeta_1\overline{\eta})^{n+1}}-\frac{1}{(1-r\zeta_2\overline{\eta})^{n+1}}\Big|&\lesssim\Big(\frac{|1-\zeta_1\overline{\zeta_2}|}{|1-r\zeta_1\overline{\eta}|}\Big)^{\frac12}\frac{1}{|1-r\zeta_1\overline{\eta}|^{n+1}}\\&\lesssim\frac{l(Q)^{\frac12}}{(1-r)^{\frac12}|1-r\zeta_1\overline{\eta}|^{n+1}}.\end{aligned}$$

As a consequence, since $|1-r\zeta_1\overline{\eta}|\approx|1-r\zeta_2\overline{\eta}|$,

$$\begin{aligned}&|I_2(\psi)(\zeta_1)-I_2(\psi)(\zeta_2)|\\&\quad\lesssim l(Q)^{\frac12}\int_0^{1-4K_1l(Q)}\Big(\int_{\mathbb{S}}\frac{|\psi(\eta)|}{|1-r\zeta_1\overline{\eta}|^{n+1}}\,d\sigma(\eta)\Big)^2(1-r^2)^{\frac12}\,dr\\&\quad\lesssim\sum_{k\ge2}{}'\int_{1-2^{k+1}K_1l(Q)}^{1-2^kK_1l(Q)}l(Q)^{\frac12}\Big(\int_{2^kB(Q)}\frac{|\psi(\eta)|}{|1-r\zeta_1\overline{\eta}|^{n+1}}]\,d\sigma(\eta)\Big)^2(1-r^2)^{\frac12}\,dr\\&\qquad+\sum_{k\ge2}{}'\int_{1-2^{k+1}K_1l(Q)}^{1-2^kK_1l(Q)}l(Q)^{\frac12}\Big(\int_{\mathbb{S}\setminus2^kB(Q)}\frac{|\psi(\eta)|}{|1-r\zeta_1\overline{\eta}|^{n+1}}\,d\sigma(\eta)\Big)^2(1-r^2)^{\frac12}dr\\&\quad=J_{21}+J_{22}.\end{aligned}$$

Here by $\sum'_{k\ge2}$ we mean that the summands are considered only for those $k\ge2$ such that $2^{k+1}K_1l(Q)<1$. We begin with the estimates of J_{21}.

$$\begin{aligned}J_{21}&\lesssim\sum_{k\ge2}{}'l(Q)^{\frac12}(2^kl(Q))^{\frac32}\Big(\frac{1}{(2^kl(Q))^{n+1}}\int_{2^kB(Q)}|\psi(\eta)|d\sigma(\eta)\Big)^2\\&\lesssim\sum_{k\ge2}{}'\frac{1}{2^{k\frac12}}\Big(\frac{1}{|2^kB(Q)|}\int_{2^kB(Q)}|\psi(\eta)|d\sigma(\eta)\Big)^2.\end{aligned}$$

Next

$$
\begin{aligned}
J_{22} &\lesssim {\sum_{k\geq 2}}' l(Q)^{\frac{1}{2}}(2^k l(Q))^{\frac{3}{2}}\Big(\sum_{i>k}\frac{1}{(2^i l(Q))^{n+1}}\int_{2^i B(Q)}|\psi(\eta)|\,d\sigma(\eta)\Big)^2\\
&\lesssim {\sum_{k\geq 2}}' 2^{\frac{k}{2}}\sum_{i\geq k}\frac{1}{2^i}\Big(\frac{1}{|2^i B(Q)|}\int_{2^i B(Q)}|\psi(\eta)|\,d\sigma(\eta)\Big)^2\\
&= {\sum_{i\geq 2}}'\sum_{k\leq i} 2^{\frac{k}{2}}\frac{1}{2^i}\Big(\frac{1}{|2^i B(Q)|}\int_{2^i B(Q)}|\psi(\eta)|\,d\sigma(\eta)\Big)^2\\
&\lesssim {\sum_{i\geq 2}}'\frac{1}{2^{\frac{i}{2}}}\Big(\frac{1}{|2^i B(Q)|}\int_{2^i B(Q)}|\psi(\eta)|\,d\sigma(\eta)\Big)^2.
\end{aligned}
$$

Finally, (4.6) and the above estimates finish the proof of the lemma. ■

4.3 Proof of the Estimate in Theorem 1.1

By Proposition 4.1, it is enough to show that for any $\omega \in A_p$,

$$\|\mathcal{C}(\psi)\|_{F_0^{p,2}(\omega)} \leq C(p,n)[\omega]_{A_p}^{\max\{1/2,1/(p-1)\}}\|\psi\|_{L^p(\omega)}.$$

As we have recalled at the beginning of this section, the proof of Theorem 1.1 follows closely the ideas in [26, 28]. We now sketch how to finish the proof. First by Lemma 4.8 we have that a.e. $\zeta \in Q$, $m^{\#}_{\lambda,Q}G(\psi)^2(\zeta) \lesssim M(\psi)(\zeta)^2$. Next,we have that for any $Q \in \mathcal{D}^i$, there exists a sparse family $\mathcal{S}(Q) = (Q_j^k)$, $Q_j^k \in \mathcal{D}^i$ so that if we let

$$\mathcal{T}_l^{\mathcal{S}}(\psi)(\zeta) = \Big(\sum_{Q_j^k \in \mathcal{S}(Q)}(\psi_{2^l B(Q_j^k)})^2 \mathcal{X}_{Q_j^k}(\zeta)\Big)^{1/2},$$

then by Theorem 4.4 and our previous observation, we have that for a.e $\zeta \in Q$,

$$|G(\psi)(\zeta)^2 - m_Q(G(\psi)^2)| \lesssim \Big(M(\psi)(\zeta)^2 + \sum_{l\geq 0}\frac{1}{2^{l/2}}(\mathcal{T}_l^{\mathcal{S}}(\psi))^2\Big).$$

Hence

$$|G(\psi)(\zeta)^2 - m_Q(G(\psi)^2)|^{1/2} \lesssim M(\psi)(\zeta) + \mathcal{T}^{\mathcal{S}}(\psi)(\zeta), \tag{4.7}$$

where $\mathcal{T}^{\mathcal{S}}(\psi)(\zeta) = \sum_{l\geq 0}\frac{1}{2^{l/4}}\mathcal{T}_l^{\mathcal{S}}(\psi)(\zeta)$.

The following lemma gives an estimate for the first term $\mathcal{T}_0^{\mathcal{S}}$. It was originally proved for $\mathbb{R}^n$ in [11]. For a sake of completeness, we give an alternative, much simpler, proof obtained in [26], adapted for our setting of homogeneous spaces.

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Lemma 4.9 *For each $\psi \in L^1(Q)$ and $\omega \in A_3$, $\|\mathcal{T}_0^{\mathcal{S}}(\psi)\|_{L^3(\omega)} \lesssim [\omega]_{A_3}^{1/2}\|\psi\|_{L^3(\omega)}$, with constant independent of the family $\mathcal{S}$.*

Proof Since $\|\mathcal{T}_0^{\mathcal{S}}(\psi)\|_{L^3(\omega)} = \|\mathcal{T}_0^{\mathcal{S}}(\psi)^2\|_{L^{3/2}(\omega)}^{1/2}$, using duality, it is enough that we show that for any $\varphi \geq 0$, with $\|\varphi\|_{L^3(\omega)} = 1$,

$$\begin{aligned}\int_Q (\mathcal{T}_0^{\mathcal{S}}(\psi)(\eta))^2\varphi(\eta)\omega(\eta)\,d\sigma(\eta)\\ &= \sum_{Q_j^k\in\mathcal{S}(Q)}\Big(\frac{1}{|B(Q_j^k)|}\int_{B(Q_j^k)}|\psi(\eta)|\,d\sigma(\eta)\Big)^2\int_{Q_j^k}\varphi(\eta)\omega(\eta)\,d\sigma(\eta)\\ &\lesssim [\omega]_{A_3}\|\psi\|_{L^3(\omega)}^2.\end{aligned}$$

We next let $T_3(E) = \omega(E)(\omega^{-1/2}(E))^2/|E|^3$. The sparsity of the family $(Q_j^k)_{j,k}$ gives that there exist sets $(E_{Q_j^k})_{j,k}$ that are pairwise disjoint and satisfying $|E_{Q_j^k}| \gtrsim |Q_j^k|$ (see (4.4)). Hence, using that $\omega \in A_3$, we have that there exists $A > 0$ such that

$$\begin{aligned}\Big(\frac{1}{|B(Q_j^k)|}\int_{B(Q_j^k)}|\psi(\eta)|d\sigma(\eta)\Big)^2\int_{Q_j^k}\varphi(\eta)\omega(\eta)d\sigma(\eta)\\ &\lesssim T_3(AB(Q_j^k))\Big(\frac{1}{\omega^{-1/2}(AB(Q_j^k))}\int_{B(Q_j^k)}|\psi|d\sigma\Big)^2\\ &\quad\times\Big(\frac{1}{\omega(AB(Q_j^k))}\int_{B(Q_j^k)}\varphi\omega d\sigma\Big)|E_{Q_j^k}|\\ &\lesssim [\omega]_{A_3}\int_{E_{Q_j^k}}(M_{\omega^{-1/2}}(\psi\omega^{1/2}))^2M_\omega(\varphi)d\sigma.\end{aligned}$$

Here M_ω denotes the weighted Hardy–Littlewood maximal function defined by

$$M_\omega(\varphi)(\zeta) = \sup_{\zeta\in B}\frac{1}{\omega(B)}\int_B|\varphi|\omega\,d\sigma.$$

Since $\omega \in A_3$, $w^{-1/2}$ is in $A_{3/2}$ and we have that both ω and $\omega^{-1/2}$ satisfy a doubling condition. Hence both weighted maximal functions are of strong type (see, for instance, [22]), and using Hölder's inequality, the sum of the above estimates can be bounded as follows:

$$\begin{aligned}\sum_{Q_j^k\in\mathcal{S}(Q)}\Big(\frac{1}{|B(Q_j^k)|}\int_{B(Q_j^k)}|\psi(\eta)|d\sigma(\eta)\Big)^2\int_{Q_j^k}\varphi(\eta)\omega(\eta)\,d\sigma(\eta)\\ &\lesssim [\omega]_{A_3}\int_{\mathbb{S}}(M_{\omega^{-1/2}}(\psi\omega^{1/2})(\eta))^2M_\omega(\varphi)(\eta)\,d\sigma(\eta)\\ &\lesssim [\omega]_{A_3}\|M_{\omega^{-1/2}}(\psi\omega^{1/2})^2\|_{L^{3/2}(\omega^{-1/2})}^2\|M_\omega(\varphi)\|_{L^3(\omega)}\\ &\lesssim [\omega]_{A_3}\|\psi\|_{L^3(\omega)}^2.\end{aligned}$$

∎

Lemma 4.10 *For each $l \geq 1$, $\|\mathcal{T}_l^{\mathcal{S}}(\psi)\|_{L^3(\omega)} \lesssim l^{1/2}[\omega]_{A_3}^{1/2}\|\psi\|_{L^3(\omega)}$.*

Proof If $l \geq 1$, we have that $\|\mathcal{T}_l^{\mathcal{S}}(\psi)\|_{L^3(\omega)} = \|(\mathcal{T}_l^{\mathcal{S}}(\psi))^2\|_{L^{3/2}(\omega)}^{1/2}$. Thus

$$
\begin{aligned}
\|(\mathcal{T}_l^{\mathcal{S}}(\psi))^2\|_{L^{3/2}(\omega)} &= \sup_{\|\varphi\|_{L^3(\omega^{-2})}\leq 1} \int_{\mathbb{S}} \mathcal{T}_l^{\mathcal{S}}(\psi)^2(\eta)\varphi(\eta)\,d\sigma(\eta) \\
&= \sup_{\|\varphi\|_{L^3(\omega^{-2})}\leq 1} \int_{\mathbb{S}} \mathcal{M}_l^{\mathcal{S}}(\psi,\varphi)(\eta)\psi(\eta)\,d\sigma(\eta),
\end{aligned} \tag{4.8}
$$

where

$$\mathcal{M}_l^{\mathcal{S}}(\psi,\varphi) = \sum_{Q_j^k\in\mathcal{S}(Q)} \psi_{2^l B(Q_j^k)}\Big(\frac{1}{|2^l B(Q_j^k)|}\int_{Q_j^k}\varphi\Big)\mathcal{X}_{2^l B(Q_j^k)}(\eta).$$

Using the existence of adjacent families of cubes $\mathcal{D}^i$, $i = 1,\dots,M$ as in Proposition 4.2, the cubes Q_j^k can be distributed in disjoint families $\mathcal{S}^i \in \mathcal{D}^i$ such that for any $Q_j^k \in \mathcal{S}^i$ there exists a dyadic cube $P_{j,k}^{l,i} \in \mathcal{D}^i$ with $2^l B(Q_j^k) \subset P_{j,k}^{l,i}$ and $l_{P_{j,k}^{l,i}} \lesssim 2^l l_{Q_j^k}$. Thus $\mathcal{M}_l^{\mathcal{S}}(\psi,\varphi)(\zeta) \lesssim \sum_{i=1}^{L} \mathcal{M}_{i,l}^{\mathcal{S}}(\psi,\varphi)(\zeta)$, where

$$\mathcal{M}_{l,i}^{\mathcal{S}}(\psi,\varphi)(\zeta) = \sum_{Q_j^k\in\mathcal{S}_i} \psi_{P_{j,k}^{l,i}}\Big(\frac{1}{|P_{j,k}^{l,i}|}\int_{Q_j^k}\varphi\Big)\mathcal{X}_{P_{j,k}^{l,i}}(\zeta).$$

The following lemma for $\mathbb{R}^n$ can be found in [12].

Lemma 4.11 *If the sum $\mathcal{M}_{l,i}^{\mathcal{S}}(\psi,\varphi)$ is finite, there exist a finite number of cubes $Q_\nu \in \mathcal{D}^i$ covering its support and such that for any cube Q_ν, there exist two families of sparse cubes $\mathcal{S}^{i,1}$, $\mathcal{S}^{i,2}$ of $\mathcal{D}^i$, $i = 1,\dots,M$ satisfying that for $\zeta \in Q_\nu$,*

$$\mathcal{M}_{l,i}^{\mathcal{S}}(\psi,\varphi)(\zeta) \lesssim l\sum_{k=1}^{2}\sum_{Q_j^k\in\mathcal{S}_{i,k}} \psi_{Q_j^k}\varphi_{Q_j^k}\mathcal{X}_{Q_j^k}(\zeta).$$

Proof The proof of this lemma is basically an application of Lerner's decomposition and the estimate $\|\mathcal{F}_l(\Psi)\|_{L^{1,\infty}} \lesssim l\|\Psi\|_{L^1}$, where

$$\mathcal{F}_l(\Psi) = \sum_{j,k}\Big(\frac{1}{|P_{j,k}^{l,i}|}\int_{Q_j^k}\Psi(\eta)d\sigma(\eta)\Big)\mathcal{X}_{P_{j,k}^{l,i}},$$

which can be found in [27, Lemma 3.2]. We remark that both constructions can be adapted to the framework of homogeneous spaces (see [2, Remark 4.22 and Lemma 6.5]). In consequence, the nonisotropic version of Lemma 4.11 for the unit sphere holds. ∎

Now we can finish the proof of Lemma 4.10, *i.e.*, the estimate of $\|\mathcal{T}_l^{\mathcal{S}}(\psi)\|_{L^3(\omega)}$.

Using the duality expression obtained in (4.8), Lemma 4.11, and Hölder's inequality, we have that

$$
\begin{aligned}
\int_{Q_\nu} \mathcal{M}_{l,i}^{\mathcal{S}}(\psi,\varphi)(\eta)\varphi(\eta)\,d\sigma(\eta) &\lesssim l\sum_{k=1,2}\sum_{Q_j^k\in\mathcal{S}_{i,k}} (\psi_{Q_j^k})^2\int_{Q_j^k}\varphi(\eta)\,d\sigma(\eta) \\
&\lesssim l\sum_{k=1,2}\int_{\mathbb{S}} (\mathcal{T}_0^{\mathcal{S}_{i,k}}(\psi))^2(\eta)\varphi(\eta)d\sigma(\eta).
\end{aligned}
$$

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Summing up over Q_v, and using (4.8), and Lemma 4.9, we deduce that

$$\|\mathcal{T}_l^{\mathcal{S}}(\psi)\|_{L^3(\omega)} = \|\mathcal{T}_l^{\mathcal{S}}(\psi)^2\|_{L^{3/2}(\omega)}^{1/2} \lesssim \Big(l \max_{1\le i\le M} \sup_{\mathcal{S}\in\mathcal{D}^i} \|\mathcal{T}_0^{\mathcal{S}}(\psi)^2\|_{L^3(\omega)} \Big)^{1/2} \lesssim l^{1/2}[\omega]_{A_3}^{1/2}\|\psi\|_{L^3(\omega)}. \qquad \blacksquare$$

We can now finish the proof of Theorem 1.1. By the last lemmas we have that

$$\|\mathcal{T}(\psi)\|_{L^3(\omega)} \le \sum_{l\ge 0} \frac{1}{2^{l/4}} \|\mathcal{T}_l^{\mathcal{S}}(\psi)\|_{L^3(\omega)} \lesssim \sum_{m\ge 0} \frac{l^{1/2}}{2^{l/4}} [\omega]_{A_3}^{1/2}\|\psi\|_{L^3(\omega)} \lesssim [\omega]_{A_3}^{1/2}\|\psi\|_{L^3(\omega)}.$$

Thus, using the estimate (4.7) and the continuity of $M(\psi)$, we obtain

$$\|(G(\psi)^2 - m_Q(G(\psi)^2))^{1/2}\|_{L^3(\omega)} \lesssim [\omega]_{A_3}^{\frac{1}{2}}\|\psi\|_{L^3(\omega)}.$$

Hence

$$\begin{aligned}\|G(\psi)\|_{L^3(\omega)} &= \|G(\psi)^2\|_{L^{3/2}(\omega)}^{1/2}\\ &\lesssim \|G(\psi)^2 - m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2} + \|m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2}\\ &\lesssim [\omega]_{A_3}^{1/2}\|\psi\|_{L^3(\omega)} + \|m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2}.\end{aligned}$$

Let us check that we also have that $\|m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2} \lesssim [\omega]_{A_3}^{1/2}\|\psi\|_{L^3(\omega)}$. Indeed,

$$\begin{aligned} m_Q((G(\psi))^2)^{1/2} &\le \Big((G(\psi)^2\mathcal{X}_Q)^*(|Q|/2)\Big)^{1/2} = (G(\psi)\mathcal{X}_Q)^*(|Q|/2)\\ &\lesssim \frac{1}{|Q|}\int_Q |\psi(\zeta)| d\sigma(\zeta).\end{aligned}$$

Consequently, $\Big(\int_Q m_{\mathbb{S}}((G(\psi))^2)^{3/2}\omega(\zeta)\,d\sigma(\zeta)\Big)^{1/3} \lesssim \omega(Q)^{1/3}\frac{1}{|Q|}\int_Q |\psi(\zeta)|\,d\sigma(\zeta)$. But

$$\begin{aligned}\int_Q |\psi(\zeta)|\,d\sigma(\zeta) &\le \Big(\int_Q |\psi(\zeta)|^3\omega(\zeta)\,d\sigma(\zeta)\Big)^{1/3}\Big(\int_Q \omega^{-1/2}(\zeta)\,d\sigma(\zeta)\Big)^{2/3}\\ &= \|\psi\|_{L^3(\omega)}\Big(\int_Q \omega^{-1/2}\,d\sigma\Big)^{2/3}.\end{aligned}$$

Thus,

$$\begin{aligned}&\Big(\int_Q m_Q((G(\psi))^2)^{3/2}(\zeta)\omega(\zeta)\,d\sigma(\zeta)\Big)^{1/3}\\ &\qquad\lesssim \|\psi\|_{L^3(\omega)}\frac{\omega(Q)^{1/3}}{|Q|}\Big(\int_Q \omega^{-1/2}(\zeta)d\sigma(\zeta)\Big)^{2/3}\\ &\lesssim \qquad \|\psi\|_{L^3(\omega)}[\omega]_{A_3}^{1/3} \le \|\psi\|_{L^3(\omega)}[\omega]_{A_3}^{1/2}.\end{aligned}$$

Finally, applying Theorem 2.6, we obtain

$$\|\mathcal{C}(\psi)\|_{F_0^{p,2}(\omega)} = \|G(\psi)\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{\max\{1/2,1/(p-1)\}}\|\psi\|_{L^p(\omega)},$$

which ends the proof of the theorem. $\blacksquare$

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Remark 4.12 In Section 4 we showed that

$$\left\| (1-|z|^2)\Big(I+\frac{R}{n}\Big)\mathcal{C}(\psi)\right\|_{L^{p,2}(\omega)} \lesssim [\omega]_{A_p}^{\max\{1/2,1/(p-1)\}}\|\psi\|_{L^p(\omega)}.$$

However, from this estimate we can obtain the analogous estimate for the operator $(1-|z|^2)(\alpha I+\beta R)\mathcal{C}$ with $\alpha,\beta\in\mathbb{R}$. Indeed, assuming the above estimate, it is enough to show that

$$\|(1-|z|^2)\mathcal{C}(\psi)\|_{L^{p,2}(\omega)} \lesssim [\omega]_{A_p}^{\max\{1/2,1/(p-1)\}}\|\psi\|_{L^p(\omega)}.$$

And this is an immediate consequence of the relation

$$\begin{aligned}\mathcal{C}(\psi)(z) &= \int_{\mathbb{S}}\frac{\psi(\zeta)}{(1-z\bar{\zeta})^{n+1}}\,d\sigma(\zeta)+\sum_{j=1}^n z_j\int_{\mathbb{S}}\frac{\bar{\zeta}_j\psi(\zeta)}{(1-z\bar{\zeta})^{n+1}}\,d\sigma(\zeta)\\ &= \Big(I+\frac{R}{n}\Big)\mathcal{C}(\psi)(z)+\sum_{j=1}^n z_j\Big(I+\frac{R}{n}\Big)\mathcal{C}(\bar{\zeta}_j\psi)(z),\end{aligned}$$

and the fact that $\|\overline{\zeta_j}\psi\|_{L^p(\omega)}\le\|\psi\|_{L^p(\omega)}$.

5 Proof of Theorem 1.2 and of the Sharpness in Theorem 1.1

In order to prove the sharpness of the estimate

$$\begin{aligned}\|\mathcal{B}\colon L^{p,2}(\omega)\to H^p(\omega)\| &\le C(p,n)[\omega]_p^{\max\{1/(2(p-1)),1\}}\\ &= C(p,n)\max\{[\omega]_{A_p},[\omega']_{A_{p'}}^{1/2}\},\end{aligned}$$

we use the techniques in [16] (see also [30]). They are based on the following lemma, whose proof follows from the Rubio de Francia algorithm.

Lemma 5.1 *Let $1\le p_0<\infty$ and let $C,\beta>0$. If the pair (φ,ψ) of nonnegative functions satisfies $\big(\int_{\mathbb{S}}\psi^{p_0}\omega\,d\sigma\big)^{1/p_0}\le C[\omega]_{A_1}^{\beta}\big(\int_{\mathbb{S}}\varphi^{p_0}\omega\,d\sigma\big)^{1/p_0}$ for all $\omega\in A_1$, then for any $p>p_0$ there exists a constant $C'=C'(n,\beta,p_0,C)$, such that*

$$\|\psi\|_{L^p}\le C'p^{\beta}\|\varphi\|_{L^p}.$$

Consequently, if the power β of p is sharp, then the power β of $[\omega]_{A_1}$ is also sharp.

Proof Let $q=p/p_0>1$. By duality,

$$\|\psi\|_{L^p}^{p_0}=\sup_{\|\phi\|_{L^{q'}}=1}\Big|\int_{\mathbb{S}}|\psi|^{p_0}\phi\,d\sigma\Big|.$$

Assume that $\phi\ge0$. By the Rubio de Francia algorithm, the function

$$\omega(\zeta)=\sum_{k=0}^{\infty}\frac{M^k(\phi)(\zeta)}{(2\|M\|_{L^{q'}})^k}$$

(M^k denotes the k-th iterate of M) satisfies $\phi(\zeta) \le \omega(\zeta)$ a.e., $\|\omega\|_{L^{q'}} \le 2\|\phi\|_{L^{q'}} = 2$ and $[\omega]_{A_1} \le 2\|M\|_{L^{q'}} \le cq \le cp$ for some $c > 0$. Thus,

$$\int_{\mathbb{S}} |\psi|^{p_0} \phi \, d\sigma \le \int_{S} |\psi|^{p_0} \omega \, d\sigma \le C^{p_0} [\omega]_{A_1}^{\beta p_0} \int_{\mathbb{S}} |\varphi|^{p_0} \omega \, d\sigma$$
$$\le C^{p_0} [\omega]_{A_1}^{\beta p_0} \|\varphi\|_{L^p}^{p_0} \|\omega\|_{L^{q'}} \le 2C^{p_0} c^{\beta p_0} p^{\beta p_0} \|\varphi\|_{L^p}^{p_0}. \qquad \blacksquare$$

Corollary 5.2 *Let $1 < p_0 < \infty$. If there exist positive constants C and β such that for any $\omega \in A_{p_0}$, $\|\mathcal{B}: L^{p_0,2}(\omega) \to H^{p_0}(\omega)\| \le C[\omega]_{A_{p_0}}^{\beta}$, then for any $1 < p < \infty$, there exists $C' > 0$, which does not depend on p, such that*

$$\|\mathcal{B}: L^{p,2} \to H^p\| \le C' \max\{p^\beta, (p')^{\beta(p_0-1)}\}.$$

Proof First we prove the case $p > p_0$, that is, $\|\mathcal{B} : L^{p,2} \to H^p\| \le C' p^\beta$. Let $C_c(\mathbb{B})$ be the space of continuous functions with compact support on $\mathbb{B}$. This space is in $L^{p_0,2}(\omega)$ for any $\omega \in A_{p_0}$ and it is dense in $L^{p,2}$ for every $1 < p < \infty$.

For each $\omega \in A_1$, since $[\omega]_{A_{p_0}} \le [\omega]_{A_1}$, we obtain

$$\|\mathcal{B}: L^{p_0,2}(\omega) \to H^{p_0}(\omega)\| \le C[\omega]_{A_{p_0}}^{\beta} \le C[\omega]_{A_1}^{\beta}.$$

Hence, Lemma 5.1 applied to the functions $\varphi(\zeta) = \left(\int_0^1 |\vartheta(r\zeta)|^2 \frac{2nr^{2n-1}}{1-r^2} \, dr\right)^{1/2}$ and $\psi(\zeta) = \mathcal{B}(\vartheta)(\zeta)$, $\vartheta \in C_c(\mathbb{B})$, gives $\|\mathcal{B}: L^{p,2} \to H^p\| \le C' p^\beta$ for any $p > p_0$.

Now we consider the case $1 < p < p_0$. By Proposition 4.1

$$\|\mathcal{B}: L^{p_0,2}(\omega) \to H^{p_0}(\omega)\| = \|\mathcal{C} : L^{p_0'}(\omega') \to F_0^{p_0',2}(\omega')\| \le C[\omega']_{A_{p_0'}}^{\gamma}$$

with $\gamma = \beta(p_0' - 1)$. Then for any $\omega' \in A_1$ we have $\|\mathcal{C}: L^{p_0'}(\omega') \to F_0^{p_0',2}(\omega')\| \le C[\omega']_{A_1}^{\gamma}$.

Note that $C(\mathbb{S})$, the space of continuous functions on $\mathbb{S}$, is in $L^{p_0'}(\omega')$ for any $\omega' \in A_1$ and it is dense in $L^{p'}$ for any $1 < p' < \infty$. Hence, the above estimate and Lemma 5.1 applied to the functions $\varphi \in C(\mathbb{S})$ and

$$\psi(\zeta) = \left(\int_0^1 (1-r^2) \left| \left(I + \frac{R}{n}\right) \mathcal{C}(\varphi)(r\zeta) \right|^2 r^{2n-1} dr \right)^{1/2},$$

gives $\|\mathcal{C}: L^{p'} \to F_0^{p',2}\| \le C'(p')^\gamma$.

The relation $\|\mathcal{B}: L^{p,2} \to H^p\| = \|\mathcal{C}: L^{p'} \to F_0^{p',2}\|$ finishes the proof. $\blacksquare$

By Theorem 1.1, the hypotheses in the above corollary are true for $p_0 = 3/2$ and $\beta = 1$. Thus, we have the following corollary.

Corollary 5.3 *There exists $C > 0$ such for any $1 < p < \infty$ we have*

$$\|\mathcal{B}: L^{p,2} \to H^p\| \approx \|\mathcal{C}: L^{p'} \to F_0^{p',2}\| \le C \max\{p, \sqrt{p'}\}.$$

In order to prove that the exponents $\beta = 1$ and $\gamma = 1/2$ cannot be replaced by any smaller one, we consider the function $f(z) = (\frac{1+z}{1-z})^\alpha$, with $0 < \alpha < 1$. This function was used by several authors to estimate the norms of some classical operators. For instance, in [17], the authors used this function to prove that

$$\|\mathcal{C} : L^p \to H^p\| = \frac{1}{\sin(\pi/p)} \approx \max\{p, p'\}. \tag{5.1}$$

The next lemma states the properties of these functions that we will need.

Lemma 5.4 *Let $1 < p < \infty$ and $0 < \delta < 1$; let $f_{\delta,p}(z) = \left(\frac{1+z}{1-z}\right)^{\delta/p}$. Denote by $u_{\delta,p}$ and $v_{\delta,p}$ its real and imaginary parts, respectively. Then for each p there exists $\delta_p > 0$ such that for any $\delta_p < \delta < 1$ the following hold.*

(i) *For $0 < \theta < 2\pi$, $|v_{\delta,p}(e^{i\theta})| = \tan\frac{\delta\pi}{2p}\, u_{\delta,p}(e^{i\theta})$.*

(ii) *$\|f_{\delta,p}\|_{H^p} \approx \sqrt{p}\|f_{\delta,p}\|_{F_0^{p,2}} \approx \frac{1}{(1-\delta)^{1/p}}$.*

(iii) *If $1 < p \le 2$, then $\|\mathcal{C}(u_{\delta,p})\|_{H^p} \approx p'\|u_{\delta,p}\|_{L^p}$.*

(iv) *If $p \ge 2$, then $\|\mathcal{C}(v_{\delta,p})\|_{H^p} \approx p\|v_{\delta,p}\|_{L^p}$.*

Furthermore, the constants in the above equivalences do not depend on p and δ.

Proof In order to simplify the notations we will write f, u and v instead of $f_{\delta,p}$, $u_{\delta,p}$, and $v_{\delta,p}$, respectively.

Assertion (i) follows easily from the fact that for $0 < \theta < 2\pi$, $Re\,\frac{1+e^{i\theta}}{1-e^{i\theta}} = 0$.

Let us prove (ii). Since $|1 - e^{i\theta}| \approx |\theta|$, we have

$$\|f\|_{H^p} \approx 1 + \Big(\int_0^{\pi/2} \theta^{-\delta}\, d\theta\Big)^{1/p} \approx \frac{1}{(1-\delta)^{1/p}}.$$

Now we estimate the norm of f in $F_0^{p,2}$, that is, the norm of $(1-|z|^2)(I+R)f(z)$ on $L^{p,2}$. In order to obtain this estimate we prove that for δ near to 1, the functions $g(z) = (1-|z|^2)Rf(z)$ and $h(z) = (1-|z|^2)f(z)$ satisfy

$$\|g\|_{L^{p,2}} \approx \frac{1}{\sqrt{p}}\frac{1}{(1-\delta)^{1/p}} \quad \text{and} \quad \|h\|_{L^{p,2}} \lesssim 1,$$

with constants that do not depend on p and δ. Combining these results with

$$\|g\|_{L^{p,2}} - \|h\|_{L^{p,2}} \le \|f\|_{F_0^{p,2}} \le \|g\|_{L^{p,2}} + \|h\|_{L^{p,2}}$$

we obtain (ii).

Let us prove these norm estimates of the functions g and h.

$$\|g\|_{L^{p,2}} \approx \frac{\delta}{p}\Big(\int_{-\pi}^{\pi}\Big(\int_0^1 (1-r^2)\frac{|1+re^{i\theta}|^{2\delta/p-2}}{|1-re^{i\theta}|^{2\delta/p+2}} r^2\, dr\Big)^{p/2} d\theta\Big)^{1/p}.$$

From the equivalences

$$|1-re^{i\theta}|^2 = (1-r)^2 + 2r^2(1-\cos\theta) \approx (1-r)^2 + \theta^2,$$

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for $0 \le \theta \le \pi/2$, and analogously for $\pi/2 \le \theta \le \pi$, $|1 + re^{i\theta}|^2 \approx (1-r)^2 + (\pi-\theta)^2$, it is easy to check that

$$\|g\|_{L^{p,2}} \approx 1 + \frac{1}{p}\Big(\int_0^1 \Big(\int_0^1 \frac{2t}{(t^2+s^2)^{\delta/p+1}}\,dt\Big)^{p/2} ds\Big)^{1/p} \tag{5.2}$$
$$\approx 1 + \frac{1}{\sqrt{p}}\Big(\int_0^1 \Big(\frac{1}{s^{2\delta/p}} - \frac{1}{(1+s^2)^{\delta/p}}\Big)^{p/2} ds\Big)^{1/p}.$$

Thus, for $\delta > \delta_p = 1 - p^{-p/2}$,

$$\|g\|_{L^{p,2}} \lesssim 1 + \frac{1}{\sqrt{p}}\Big(\int_0^1 \frac{ds}{s^\delta}\Big)^{1/p} \approx \frac{1}{\sqrt{p}}\frac{1}{(1-\delta)^{1/p}}.$$

Conversely, since $\sqrt{a-b} \ge \sqrt{a} - \sqrt{b}$ for any $0 < b < a$, (5.2) and the triangular inequality give

$$\|g\|_{L^{p,2}} \gtrsim 1 + \frac{1}{\sqrt{p}}\Big(\int_0^1 \frac{ds}{s^\delta}\Big)^{1/p} - \frac{1}{\sqrt{p}}\Big(\int_0^1 \frac{ds}{(1+s)^\delta}\Big)^{1/p} \approx \frac{1}{\sqrt{p}}\frac{1}{(1-\delta)^{1/p}}.$$

The proof of $\|h\|_{L^{p,2}} \lesssim 1$ is easier. For $p \ge 2$ follows from the fact that $|h(z)|^2 \lesssim 1 - |z|^2$ and for $1 < p < 2$ from $|h(z)|^2 \lesssim (1-|z|^2)^2/|1-z|^2$.

In order to prove assertion (iii), note that if $1-\delta < p-1 \le 1$, then $\tan\frac{\delta\pi}{2p} \approx (1-\frac{\delta}{p})^{-1} \approx p'$. Hence, assertions (ii) and (i) give

$$\|2\mathcal{C}(u)(z)\|_{H^p} = \|f(z) + \overline{f(0)}\|_{H^p} \approx \|u\|_{L^p} + \|v\|_{L^p} \approx p'\|u\|_{L^p}.$$

Analogously, (iv) follows from the fact that for $p \ge 2$, $\tan\frac{\delta\pi}{2p} \approx \frac{1}{p}$, and

$$\|2\mathcal{C}(v)(z)\|_{H^p} = \|f(z) - \overline{f(0)}\|_{H^p} \approx \|u\|_{L^p} + \|v\|_{L^p} \approx p\|v\|_{L^p}.$$

This concludes the proof. ∎

5.1 Proof of Theorem 1.2

Proof By Corollary 5.3 we have that

$$\|\mathcal{B}\colon L^{p,2} \to H^p\| = \|\mathcal{C}\colon L^{p'} \to F_0^{p',2}\| \le c(n)\max\{p, \sqrt{p'}\}.$$

In order to prove that this estimate is sharp, we consider the case $n = 1$. Assume $1 < p \le 3/2$. Let $f = f_{\delta,p'}$ as in Lemma 5.4 and $v = v_{\delta,p'}$ its imaginary part. Then we have $\|\mathcal{C}(v)\|_{F_0^{p',2}} \approx \frac{1}{\sqrt{p'}}\|f\|_{F_0^{p,2}} \approx \sqrt{p'}\|v\|_{L^{p'}}$. Thus, $\|\mathcal{C}\colon L^{p'} \to F_0^{p',2}\| \gtrsim C\sqrt{p'}$.

Now we consider the case $p > 3/2$. Since $1 < p' < 3$, for $\varphi \in L^{p'}$, the norms of $\mathcal{C}(\varphi)$ on $H^{p'}$ and on $F_0^{p',2}$ are equivalent with constants that do not depend on p. Since by (5.1), the norm of $\mathcal{C}\colon L^{p'} \to H^{p'}$ is equivalent to p, we conclude the proof. ∎

5.2 Proof of the Sharpness in Theorem 1.1

Proof We prove that there is no $\lambda < 1$ such that

$$\|\mathcal{B}\colon L^{p,2}(\omega) \to H^p(\omega)\| \le C(p,n)[\omega]_{A_p}^{\lambda\max\{1,1/(2(p-1))\}}. \tag{5.3}$$

Assume that (5.3) is satisfied for some p_0 and some $\lambda < 1$. Then by Corollary 5.2, for $1 < p < \infty$, we have

$$\|\mathcal{B}\colon L^{p,2} \to H^p\| \le C' \max\{p^\beta, (p')^{\beta(p_0-1)}\}, \quad \beta = \lambda \max\{1, 1/(2(p_0-1))\}.$$

If $p_0 > 3/2$, then $\beta = \lambda$ and thus $\|\mathcal{B}\colon L^{p,2} \to H^p\| \le C' p^\lambda$ for any $p > 3/2$. This is not possible by Theorem 1.2.

If $p_0 \le 3/2$, then $\beta(p_0-1) = \lambda/2$ and thus $\|\mathcal{B}\colon L^{p,2} \to H^p\| \le C'(p')^{\lambda/2}$ for any $1 < p < 3/2$. As above, Theorem 1.2 gives that this is not possible. ■

6 Proof of Theorems 1.3 and 1.5

In Section 3 it was proved that if $\mathcal{B}$ is bounded from $L^{p,2}(\omega)$ to $F_0^{p,2}(\omega)$, then $\omega \in A_p$. Conversely, if $\omega \in A_p$, $p > 1$, then $H^p(\omega)$ and $F_0^{p,2}(\omega)$ are isomorphic. Hence, Theorem 1.1 ensures then that $\|\mathcal{B}\colon L^{p,2}(\omega) \to F_0^{p,2}(\omega)\|$ is finite.

In order to obtain norm-estimates for this operator, let

$$\mathcal{Q}(\varphi)(z) = (1-|z|^2)\Big(I + \frac{R}{n}\Big)\mathcal{B}(\varphi)(z).$$

If φ and ψ are smooth functions on $\mathbb{B}$, from

$$\Big(I + \frac{R_z}{n}\Big)\mathcal{B}(z,w) = \overline{\Big(I + \frac{R_w}{n}\Big)\mathcal{B}(w,z)}$$

and Fubini's theorem, we have $\langle \mathcal{Q}(\varphi), \psi\rangle_{\mathbb{B}} = \langle \varphi, \mathcal{Q}\psi\rangle_{\mathbb{B}}$, where $\langle\cdot,\cdot\rangle_{\mathbb{B}}$ denotes the pairing given in Proposition 2.4. Thus,

$$\begin{aligned}\|\mathcal{B}\colon L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| &= \|\mathcal{Q}\colon L^{p,2}(\omega) \to L^{p,2}(\omega)\| \\ &= \|\mathcal{Q}\colon L^{p',2}(\omega') \to L^{p',2}(\omega')\| \\ &= \|\mathcal{B}\colon L^{p',2}(\omega') \to F_0^{p',2}(\omega')\|.\end{aligned} \tag{6.1}$$

Consider the homogeneous space $(\mathbb{B}', d, \nu)$ where $\mathbb{B}' = \overline{\mathbb{B}} \setminus \{0\}$, d denotes the quasimetric $d(z,w) = \max\{||z|-|w||, |1 - z^*\overline{w}^*|\}$, $z^* = z/|z|$, $w^* = w/|w|$, and ν is the volume measure on $\mathbb{B}'$. Denote by $\Delta(z,r)$ the balls with respect to the metric d. Observe that if $\zeta \in \mathbb{S}$, then the ball $\Delta(\zeta, r)$ coincides with the square $S_{r\zeta}$ introduced in Proposition 3.1.

A weight $\Omega \in L^1(\mathbb{B}')$ is in the Muckenhoupt class $A_2(\mathbb{B}')$ with respect to the homogeneous space $\mathbb{B}'$ if $[\Omega]_{A^2(\mathbb{B}')} = \sup_\Delta \frac{1}{\nu(\Delta)^2}\int_\Delta \Omega\, d\nu \int_\Delta \Omega^{-1}\, d\nu < \infty$.

Lemma 6.1 *If $\omega \in A_2(\mathbb{S})$, then the weight $\Omega(z) = \omega(z/|z|)$, $z \ne 0$, is in $A_2(\mathbb{B}')$ and $[\Omega]_{A^2(\mathbb{B}')} \lesssim [\omega]_{A_2}$.*

Proof By integration in polar coordinates

$$\int_{\Delta(a,r)} \Omega(z)\, d\nu(z) \lesssim r \int_{\{\zeta\in\mathbb{S}:\, d(a^*,\zeta)<r\}} \omega(\zeta)\, d\sigma(\zeta).$$

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Analogously

$$\int_{\Delta(a,r)} \Omega^{-1}(z)dv(z) \lesssim r \int_{\{\zeta\in\mathbb{S}:\, d(a^*,\zeta)<r\}} \omega^{-1}(\zeta)d\sigma(\zeta).$$

Since $\sigma\{\zeta \in \mathbb{S}: d(a^*,\zeta) < r\} \approx r^n$ and $v(\Delta(a,r)) \approx r^{n+1}$, we obtain $[\Omega]_{A^2(\mathbb{B})} \lesssim [\omega]_{A_2}$, which concludes the proof. ∎

Corollary 6.2 *If $\omega \in A_2$, then $\|\mathcal{B}: L^{2,2}(\omega) \to F_0^{2,2}(\omega)\| \le C[\omega]_{A_2}$.*

Proof In [2], it was proved that if T is a Calderon–Zygmund operator on a homogeneous space X, then for any $\Omega \in A_2(X)$ we have $\|T\|_{L^2(X,\Omega)} \le C(T,X)[\Omega]_{A_2(X)}$.

Observe that $L^{2,2}(\omega) = L^2\big(\mathbb{B}', \frac{\Omega(z)}{1-|z|^2}dv(z)\big)$. Thus, the boundedness of the operator $\mathcal{Q}$ on $L^{2,2}(\omega)$ is equivalent to the boundedness of the Calderon–Zygmund operator $T: L^2(\mathbb{B},\Omega) \to L^2(\mathbb{B},\Omega)$ defined by

$$T(\varphi)(z) = \int_{\mathbb{B}} \varphi(w)(1-|w|^2)^{1/2}(1-|z|^2)^{1/2}\Big(I + \frac{R}{n}\Big)\frac{1}{(1-z\overline{w})^{n+1}}\,dv(w).$$

Applying the above mentioned result to T and $X = \mathbb{B}'$ and using Lemma 6.1, we obtain the estimate. ∎

Using this estimate and the extrapolation Theorem 2.6 we obtain the following theorem.

Theorem 6.3 *Let $1 < p < \infty$. There exists a positive constant $C(p,n)$ such that*

$$\|\mathcal{B}: L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \le C(p,n)[\omega]_{A_p}^{\max\{1,1/(p-1)\}} = C(p,n)\max\{[\omega]_{A_p},[\omega']_{A_{p'}}\}$$

for any $\omega \in A_p$.

Remark 6.4 Note that the same arguments used to prove

$$\|\mathcal{Q}: L^{p,2}(\omega) \to L^{p,2}(\omega)\| \le C(p,n)[\omega]_{A_p}^{\max\{1,1/(p-1)\}}$$

show that for any real numbers α and β,

$$\|(1-|z|^2)(\alpha I + \beta R)\mathcal{B}: L^{p,2}(\omega) \to L^{p,2}(\omega)\| \le C(p,n,\alpha,\beta)[\omega]_{A_p}^{\max\{1,1/(p-1)\}}.$$

That is, if in the space $F_0^{p,2}(\omega)$ we consider the norm

$$\|f\|_{F_0^{p,2}(\omega)} = \|(1-|z|^2)(\alpha I + \beta R)f\|_{L^{p,2}(\omega)},$$

with $\alpha > 0$ and $\beta > 0$, we also obtain the same estimate

$$\|\mathcal{B}: L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \lesssim [\omega]_{A_p}^{\max\{1,1/(p-1)\}}.$$

Proof of Theorem 1.3 In Section 3 it was proved that if $\mathcal{B}$ is bounded from $L^{p,2}(\omega)$ to $F_0^{p,2}(\omega)$, then $\omega \in A_p$. The estimate $\|\mathcal{B}: L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \lesssim [\omega]_{A_p}^{\max\{1,1/(p-1)\}}$ follows from Theorem 6.3. ∎

The following couple of lemmas show that it cannot be obtained as an upper bound of the norm of $\mathcal{B}$ in terms of $[\omega]_{A_p}^{\lambda}$ with $\lambda < 1/p$.

Lemma 6.5 *For $1 < p < \infty$, $0 < \rho < 1/2$ and $0 < \delta < 1$, let $\omega_\delta(e^{i\theta}) = |1-e^{i\theta}|^{(p-1)(1-\delta)}$ and $\varphi_\delta(re^{i\theta}) = |1-e^{i\theta}|^{\delta-1}(1-r)\mathcal{X}_{\rho,1}$, where $\mathcal{X}_{\rho,e^{i\eta}}$ denotes the characteristic function of the square $S_{\rho,e^{i\eta}} = \{z = re^{i\theta} \in \mathbb{D} : 1-r < \rho, |1-e^{i(\theta-\eta)}| < \rho\}$. We then have the following.*

(i) $[\omega_\delta]_{A_p} \approx \delta^{1-p}$.

(ii) $\|\varphi_\delta\|_{L^{p,2}(\omega_\delta)} \approx \delta^{-1/p}$.

(iii) $\|\varphi_\delta\|_{L^1} \approx \delta^{-1}$.

(iv) $\|(1-|z|^2)\mathcal{X}_{\rho,-1}(z)\|_{L^{p,2}(\omega_\delta)} \approx C$.

Proof From $\omega_\delta(e^{i\theta}) \approx |\theta|^{(p-1)(1-\delta)}$ it is easy to check that $[\omega_\delta]_{A_p} \approx \delta^{1-p}$

The remaining estimates follow from

$$\|\varphi_\delta\|_{L^{p,2}(\omega_\delta)} \approx \Big(\int_{-\rho}^{\rho} |\theta|^{\delta-1}\,d\theta\Big)^{1/p}\Big(\int_{1-\rho}^{1}(1-r)\,dr\Big)^{1/2} \approx \delta^{-1/p}.$$

$$\|\varphi_\delta\|_{L^1} \approx \int_{-\rho}^{\rho}|\theta|^{\delta-1}d\theta \int_{1-\rho}^{1}(1-r)\,dr \approx \delta^{-1}.$$

$$\|(1-|z|^2)\mathcal{X}_{\rho,-1}(z)\|_{L^{p,2}(\omega_\delta)} \approx C.$$

The constant in the last equivalence depends of ρ. ∎

Lemma 6.6 *Let φ_δ and ω_δ be as in Lemma 6.5. Then*

$$\|\mathcal{B}\colon L^{p,2}(\omega_\delta) \to F_0^{p,2}(\omega_\delta)\| \gtrsim [\omega_\delta]_{A_p}^{1/p}.$$

Proof For any $z \in S_{\rho,-1}$ and any $w \in S_\rho$, $|1-z\overline{w}| > 1/2$ and consequently

$$\|\mathcal{B}(\varphi_\delta)\|_{L^{p,2}(\omega_\delta)} \gtrsim \|\mathcal{B}(\varphi_\delta)\mathcal{X}_{S_{\rho,-1}}\|_{L^{p,2}(\omega_\delta)} \gtrsim \|(1-|z|^2)\mathcal{X}_{-\rho}(z)\|_{L^{p,2}(\omega_\delta)}\|\varphi_\delta\|_{L^1(dv)}.$$

By Lemma 6.5, the last term is equivalent to $\delta^{-1} \approx \|\varphi_\delta\|_{L^{p,2}(\omega_\delta)}[\omega_\delta]_{A_p}^{1/p}$. Hence,

$$\|\mathcal{B}\colon L^{p,2}(\omega_\delta) \to L^{p,2}(\omega_\delta)\| \gtrsim [\omega_\delta]_{A_p}^{1/p}$$

which concludes the proof. ∎

6.1 Proof of Theorem 1.5

Proof As we have already said in the introduction, for $p > 0$ the norms on the spaces H^p and $F_0^{p,2}$ are equivalent. From this fact it is easy to check that for $1 \le p \le 2$ this equivalence can be established by constants that do not depend on p. Thus, by Theorem 1.2 we have $\|\mathcal{B}\colon L^{p,2} \to F_0^{p,2}\| \approx \|\mathcal{B} : L^{p,2} \to H^p\| \lesssim \sqrt{p'}$, and this estimate is sharp.

The case $p > 2$ follows from (6.1) and the above result. ∎

7 Proof of Theorem 1.4

In order to prove the estimate in Theorem 1.4, we need the following.

Proposition 7.1 *For $2 < p < \infty$ and $\omega \in A_{p/2}$,*

$$\|\mathcal{Q}\colon L^{p,2}(\omega) \to L^{p,2}(\omega)\| \le C(p,n)[\omega]_{A_{p/2}}^{1/2},$$

where, as in the above section, $\mathcal{Q} = (1-|z|^2)(I + \frac{R}{n})\mathcal{B}$.

Proof Assume $0 \le \varphi \in L^{p,2}(\omega)$ and denote by $|\mathcal{Q}|$ the integral operator with kernel $|\mathcal{Q}(z,w)|$. Since $|\mathcal{Q}|(1) \approx 1$, by Hölder's inequality, we have $|\mathcal{Q}(\varphi)(z)|^2 \lesssim |\mathcal{Q}|(\varphi^2)(z)$, Using this fact, Fubini's theorem, and $|1 - r\zeta\overline{w}| \approx 1 - r + |1 - \zeta\overline{w}|$, we obtain

$$\int_0^1 |\mathcal{Q}|(\varphi^2)(r\zeta)\frac{dr}{1-r^2} \lesssim \int_{\mathbb{B}} \frac{\varphi^2(w)}{|1-\zeta\overline{w}|^{n+1}}\, d\nu(w).$$

Hence, duality $(L^{p/2}(\omega))' = L^{(p/2)'}(\omega)$ and Fubini's theorem give

$$\|\mathcal{Q}(\varphi)\|_{L^{p,2}(\omega)} \lesssim \sup_{\|\psi\|_{L^{(p/2)'}(\omega)}=1} \Big(\int_{\mathbb{B}} \varphi^2(w) \int_{\mathbb{S}} \frac{|\psi(\zeta)|\omega(\zeta)}{|1-\zeta\overline{w}|^{n+1}}\, d\sigma(\zeta) d\nu(w)\Big)^{1/2}.$$

By [13, Lemma 3], there exists a function $v \in L^{(p/2)'}(\omega)$ such that

$$v \ge |\psi|, \quad \|v\|_{L^{(p/2)'}(\omega)} \le 2\|\psi\|_{L^{(p/2)'}(\omega)}, \quad [v\omega]_{A_1} \le 2C(p/2)[\omega]_{A_{p/2}}.$$

Thus, if $w = t\eta$, then a.e $\eta \in \mathbb{S}$, we have that

$$\begin{aligned}\int_{\mathbb{S}} \frac{|\psi(\zeta)|\omega(\zeta)}{|1-\zeta\overline{w}|^{n+1}}\, d\sigma(\zeta) &\le \frac{1}{1-t^2} M(v\omega)(\eta) \le \frac{2}{1-t^2}[v\omega]_{A_1} v(\eta)\omega(\eta)\\ &\le C((p/2)')[\omega]_{A_{p/2}} \frac{v(\eta)\omega(\eta)}{1-t^2}.\end{aligned}$$

By Hölder's inequality and $\|v\|_{L^{(p/2)'}(\omega)} \le 2\|\psi\|_{L^{(p/2)'}(\omega)}$, we obtain

$$\int_{\mathbb{B}} \varphi^2(w) \int_{S} \frac{|\psi(\zeta)|\omega(\zeta)}{|1-\zeta\overline{w}|^{n+1}}\, d\sigma(\zeta) d\nu(w) \lesssim C((p/2)')[\omega]_{A_{p/2}} \|\varphi\|^2_{L^{p,2}(\omega)},$$

which concludes the proof. ∎

Proof of Theorem 1.4 We want to prove the following.

- If $2 < p < \infty$ and $\omega \in A_1$, then $\|\mathcal{B}\colon L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \le C(p,n)[\omega]_{A_1}^{1/2}$.
- If $1 < p < 2$ and $\omega' \in A_1$, then $\|\mathcal{B}\colon L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| \le C(p,n)[\omega']_{A_1}^{1/2}$.

By (6.1), if $\omega \in A_p$, then

$$\begin{aligned}\|\mathcal{B}\colon L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| &= \|\mathcal{Q} : L^{p,2}(\omega) \to L^{p,2}(\omega)\|\\ &= \|\mathcal{B} : L^{p',2}(\omega') \to F_0^{p',2}(\omega')\|.\end{aligned}$$

Therefore, assertion (7) follows from Proposition 7.1 and the fact that $[\omega]_{A_{p/2}} \le [\omega]_{A_1}$, $p > 2$. Part (7) follows from identity (6.1) and part (7). Indeed, if $\omega' \in A_1$, then $\omega \in A_p$ and

$$\|\mathcal{B}\colon L^{p,2}(\omega) \to F_0^{p,2}(\omega)\| = \|\mathcal{B}\colon L^{p',2}(\omega') \to F_0^{p',2}(\omega')\| \le C(p,n)[\omega']_{A_1}^{1/2}.$$

The sharpness of the above estimates follows from Lemma 5.1 and Theorem 1.5. Indeed, for $p_0 > 2$, following the same arguments used to prove Corollary 5.2, we obtain that if $\|\mathcal{B}\colon L^{p_0,2}(\omega) \to F_0^{p_0,2}(\omega)\| \le C(p,n)[\omega]_{A_1}^{\beta}$, then Lemma 5.1 applied to the functions

$$\varphi(\zeta) = \Big(\int_0^1 |\vartheta(r\zeta)|^2 \frac{2nr^{2n-1}}{1-r^2}\, dr \Big)^{1/2},$$
$$\psi(\zeta) = \Big(\int_0^1 (1-r^2)\big(I + \frac{R}{n}\big) \mathcal{B}(\vartheta)(r\zeta) \frac{2nr^{2n-1}}{1-r^2}\, dr \Big)^{1/2},$$

for $\vartheta \in C_c(\mathbb{B})$, gives $\|\mathcal{B}\colon L^{p,2} \to F_0^{p,2}\| \le C(n)p^{\beta}$ for $p > p_0$. By Theorem 1.5 we have $\beta \ge 1/2$, which proves that the estimate is sharp. ∎

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Dept. Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain
e-mail: cascante@ub.edu joan_fabrega@ub.edu ortega@ub.edu