Proceedings of the Edinburgh Mathematical Society (2003) **46**, 67–73 ©
DOI:10.1017/S0013091502000081 Printed in the United Kingdom

PACKING STRIPS IN THE HYPERBOLIC PLANE

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(Received 11 January 2002)

Abstract A *strip* of radius r in the hyperbolic plane is the set of points within distance r of a given geodesic. We define the density of a packing of strips of radius r and prove that this density cannot exceed

$$\mathcal{S}(r) = \frac{3}{\pi} \sinh r \operatorname{arccosh}\left(1 + \frac{1}{2\sinh^2 r}\right).$$

This bound is sharp for every value of r and provides sharp bounds on collaring theorems for simple geodesics on surfaces.

Keywords: strip; horodisc; packing; density; hyperbolic plane; Fuchsian group

AMS 2000 *Mathematics subject classification:* Primary 51M09; 52C15
Secondary 51M04

1. Introduction

A *strip packing* of the hyperbolic plane $\mathbb{H}^2$ is a set of isometric copies of a strip of radius r whose interiors are disjoint. It is well known that one cannot define the density of such a packing by considering its density within discs of fixed centre whose radii tend to ∞ [**2**]. To circumvent this problem, one defines a local density by partitioning $\mathbb{H}^2$ into finite-area regions $\{R_i\}$ and then defining, for each i, the local density $\rho_i = \operatorname{vol}(P \cap R_i)/\operatorname{vol}(R_i)$. As these densities may differ among regions, density is defined as the least upper bound of the ρ_i. In the next section we construct such a partition of $\mathbb{H}^2$ into convex polyhedra called *supporting polyhedra* and prove that the local packing density cannot exceed $\mathcal{S}(r)$, defined at (1.1). This bound is sharp for every r and is obtained by generalizing the method of [**2**] used to get the bound $3/\pi$ for horodisc packings (a horodisc may be thought of as a degenerate strip of infinite radius). As

$$\frac{1}{x}\operatorname{arccosh}(1 + \tfrac{1}{2}x^2) = 1 - \tfrac{1}{24}x^2 + \tfrac{3}{640}x^4 - \cdots$$

is an alternating series with decreasing terms for $x \leqslant 1$, we have

$$\frac{3}{\pi}\left(1 - \frac{1}{24\sinh^2 r}\right) < \mathcal{S}(r) < \frac{3}{\pi}, \quad \sinh(r) > 1,$$

and we observe exponential convergence to the horodisc packing bound as $r \to \infty$ in (1.1). We see this convergence geometrically by normalizing so that some finite point of the hyperbolic plane remains a fixed point of tangency of two different strips. On the other hand, for r small we have

$$\mathcal{S}(r) \approx \frac{6}{\pi} r \log \frac{1}{r} \to 0 \quad \text{as } r \to 0,$$

and so thin strips do not pack well in the hyperbolic plane.

In comparison, optimal disc packings are known only for the special values $\cosh r = \frac{1}{2}\sin(\pi/p)$, $p \geqslant 7$, where the symmetry group is the $(2,3,p)$-hyperbolic triangle group.

The *symmetry group* $\Gamma(\mathcal{P})$ of a strip packing $\mathcal{P}$ is the group of orientation-preserving isometries which permute the members of $\mathcal{P}$. This group is discrete whenever $|\mathcal{P}| \geqslant 2$. We are generally interested in the case where $\Gamma(\mathcal{P})$ also has co-finite area and acts transitively on $\mathcal{P}$, in which case we refer to $\mathcal{P}$ as a *lattice packing*. For these packings it is natural to let the $\{R_i\}$ be the translates of a fundamental domain for $\Gamma(\mathcal{P})$. The values of ρ_i are then clearly all the same, and independent of the fundamental domain chosen. We refer to the local density defined in this way as the *group density* of $\mathcal{P}$. When $\mathcal{P}$ is a lattice packing, the group density is also bounded above by (1.1) (Corollary 1.3 below).

Any co-finite-area group leaving a lattice packing $\mathcal{P}$ invariant is finite index in $\Gamma(\mathcal{P})$, thus if Γ is a co-finite-area group with fundamental domain D acting transitively on a lattice packing $\mathcal{P}$, then the group density of $\mathcal{P}$ is $\operatorname{vol}(P \cap D)/\operatorname{vol}(D)$, where P is the union of the strips in $\mathcal{P}$.

Let $\mathcal{P}$ be a packing of strips of radius r, in $\mathbb{H}^2$. A circle C_1 which touches $n \geqslant 3$ of these strips, but whose interior meets none of them, is said to be a *supporting circle*. The circle C with the same centre as C_1 and radius increased by r then touches the axes of these strips. Let C be such a circle and $B_1, B_2, \dots, B_n$ with axes $g_1, g_2, \dots, g_n$, respectively, be the strips touching it, taken anticlockwise around C. Let p_i be the arc joining g_i and g_{i+1}, perpendicular to both (here and subsequently indices are taken modulo n), and let a_i be the arc of g_i which lies between its intersections with p_{i-1} and with p_i. We define the *supporting polygon* P_C associated with the supporting circle C to be the right-angled $2n$-gon whose sides are $a_1, p_1, a_2, p_2, \dots, a_n, p_n$ (see Figure 1). This definition has a natural counterpart in the case $r = \infty$, i.e. when $\mathcal{P}$ is a packing of horoballs. In this case, supporting polygons are defined as polygons spanned by the boundary points of a collection of horoballs which are tangent to a common circle. This definition is used by Fejes Tóth [**2**] to define the local density of horoball packings. Also in [**2**, p. 248] it is proved that the interiors of any two supporting polygons are disjoint and that, when the packing is saturated (i.e. that it is not properly contained in any larger packing), they tessellate the plane. These arguments apply, essentially unchanged, to prove the same conclusion for supporting polygons in strip packings.

Associated with each supporting circle C and corresponding supporting polygon P_C we have the local density ρ_C defined by $\rho_C = \operatorname{vol}(P \cap P_C)/\operatorname{vol}(P_C)$. We can now state our main result.

Theorem 1.1. *For every packing of strips of radius r in $\mathbb{H}^2$ and every supporting circle C,*

$$\rho_C \leqslant \mathcal{S}(r) = \frac{3}{\pi} \sinh r \operatorname{arccosh}\biggl(1 + \frac{1}{2\sinh^2 r}\biggr). \tag{1.1}$$

This bound is sharp.

Before proving this result we apply it to hyperbolic surfaces $M = \mathbb{H}^2/\Gamma$, where Γ is a torsion-free Fuchsian group. If M contains a strip (an embedded neighbourhood of a simple closed geodesic, sometimes called a *collar*), then the area of M is, trivially at least, that of the strip. Indeed, in [**4**] it is shown that every simple closed geodesic on M admits a collar about it of area at least $8/\sqrt{5}$; refinements can be deduced from [**1**]. The bound (1.1) can be used to improve this.

Corollary 1.2. *If a hyperbolic surface $M = \mathbb{H}^2/\Gamma$ contains n geodesics of length ℓ_i, $i = 1, 2, \dots, n$, with pairwise-disjoint neighbourhoods of radius r, then the area of M is at least*

$$\tfrac{2}{3}\pi\ell\biggl[\operatorname{arccosh}\biggl(1 + \frac{1}{2\sinh^2 r}\biggr)\biggr]^{-1},$$

where $\ell = \ell_1 + \cdots + \ell_n$.

Proof. Lifting the n collars to $\mathbb{H}^2$ gives a packing $\mathcal{P}$ of the plane. If necessary, we extend this packing to a saturated packing which is still invariant under Γ. Clearly, Γ permutes the supporting polygons induced by this packing and the only isometry in Γ which maps a supporting polygon to itself is the identity (since such a map fixes the centre of the supporting circle). Thus some finite union of supporting polygons is a fundamental domain of Γ and so the group density of $\mathcal{P}$ is bounded by the right-hand side of (1.1). Mapping down to $\mathbb{H}^2/\Gamma$ again, this also bounds the density of the collar packing in M. Since these collars have total area $2\ell \sinh r$ the theorem follows. □

Corollary 1.3. *The group density of a lattice packing is at most the right-hand side of (1.1).*

Proof. By Selberg's Lemma [**7**], the symmetry group of $\mathcal{P}$ has a torsion-free finite-index subgroup Γ. Since the group density of $\mathcal{P}$ equals the proportion of the area of $\mathbb{H}^2/\Gamma$ which lies in the projection of the strips in $\mathcal{P}$ to $\mathbb{H}^2/\Gamma$, the result follows from Theorem 1.1 above. □

The existence of embedded collars about geodesics, particularly short geodesics, is an important geometric fact about surfaces and more general hyperbolic manifolds leading to the so-called thick and thin decomposition. Lower bounds for collar radii are found from such things as Jørgensen's inequality or the Margulis Lemma (see [**1**, **3**, **4**]). Corollary 1.2 provides sharp upper bounds for collaring theorems for surfaces.

Corollary 1.4. *Let M be a hyperbolic surface of genus $g > 1$ containing a simple closed geodesic γ of length ℓ. Let r be the radius of the largest embedded collar about γ. Then*

$$\sinh r \leqslant \left[2\sinh\left(\frac{\ell}{12(g-1)}\right)\right]^{-1}. \tag{1.2}$$

Proof. The area of M is $4\pi(g-1)$, which must exceed the area bound of Corollary 1.2. □

The examples used to show that our bounds are sharp (groups generated by reflections in certain hyperbolic hexagons (see § 2), also show that Corollary 1.2 is optimal for infinitely many ℓ and r. Indeed it was our attempt to find a higher-dimensional analogue of Corollary 1.2 (strips become cylinders) that led to the results herein. However, other than asymptotic results very little is known in higher dimensions (see [**5**]).

2. Proof of the main theorem

Definition 2.1. For $R > 0$, $|\sin\theta| > \operatorname{sech} R$,

$$\ell(R,\theta) = \operatorname{arcsinh}\left(\frac{\cos\theta}{\sqrt{\cosh^2 R\sin^2\theta - 1}}\right). \tag{2.1}$$

If a Lambert quadrilateral has acute angle θ and an edge e of length R adjacent to θ, then $\ell(R,\theta)$ is the length of the edge adjacent to e and opposite θ (readily proved from [**1**, Theorem 7.17.1], for example).

Let $\theta_0 = \arcsin(\operatorname{sech} R)$. Then the function $\ell(R,\theta)$ is decreasing in $[\theta_0, \pi-\theta_0]$, strictly convex in $[\theta_0, \pi/2]$, and satisfies

$$\ell(R,\pi-\theta) = -\ell(\theta). \tag{2.2}$$

Proof of Theorem 1.1. Let C_1 be a supporting circle, touching n strips, and let C, g_i, a_i and p_i be as above. The supporting polygon $P = P_{C_1}$ has area $(n-2)\pi$ and the intersection of P with the strips has area $\sigma\sinh r$, where σ is the sum of the lengths of the a_i. Therefore,

$$\rho = \rho_{C_1} = \frac{\sigma\sinh r}{(n-2)\pi}. \tag{2.3}$$

This density depends on n, on the radius R of the circle C, and on the arrangement of the axes $g_1,\dots,g_n$ around it. We show that, for fixed R and n, ρ is maximized when the axes are crowded as closely together as possible around C. We then show that ρ is maximized when R is as small as possible and finally, given this, when $n = 3$. In this case the upper bound of the theorem is attained.

We first suppose that n and R are fixed and that C is centred at the origin (in the disc model). We introduce some more notation, illustrated above for $n = 3$. Let r_i be the radius of C which meets g_i perpendicularly, and let θ_i be the angle between r_i and

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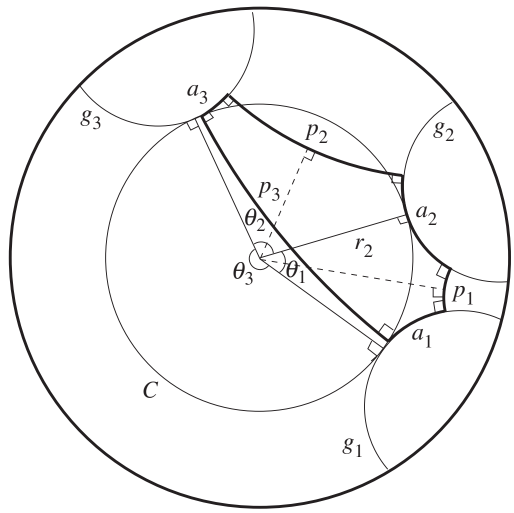

Figure 1. Supporting polygon.

r_{i+1}, measured anticlockwise from r_i to r_{i+1} (so that $\theta_i > \pi$ is possible). The sum σ of the lengths of the edges a_i is a function of $\theta_1, \dots, \theta_n$. Consider the n pentagons bounded by r_i, r_{i+1}, p_i and arcs from g_i and g_{i+1}. The sum of the lengths of edges of these n pentagons, which lie in the geodesics g_i, with these lengths being counted as negative if they come from a pentagon corresponding to angle $\theta_i > \pi$, is σ (Figure 1).

Each pentagon can be bisected into two Lambert quadrilaterals, and (2.1) and (2.2) give $\sigma = 2\sum_{i=1}^{n} \ell(R, \theta_i/2)$. We claim that σ is maximized when, for all but at most one i, $\theta_i/2 = A = A(R) := \arcsin(\cosh r/\cosh R)$. This is the minimum possible value of $\theta_i/2$ as it is attained when $\rho(g_i, g_{i+1}) = 2r$, i.e. when the corresponding strips touch. To prove that this gives the maximum we show that σ can be increased if $\theta_i > 2A$ for two or more values of i, which we may suppose to be $i = n$ and $i = j < n$. By the symmetry and convexity properties of $\ell(R, \theta)$, the sum $\ell(\phi_1) + \ell(\phi_2)$ is maximized, subject to $\phi_1 + \phi_2 = \phi$ $(2A < \phi < \pi)$ and $\phi_1, \phi_2 \geqslant A$ when either ϕ_1 or ϕ_2 is A. It follows that if we rotate g_j clockwise about the origin until θ_j is reduced to $2A$, σ is not decreased (and, unless $\theta_{j+1} = 2A$, is increased). Having rotated g_j in this way, θ_{j+1} is increased, and so we can then rotate θ_{j+1} clockwise until θ_{j+1} is reduced to $2A$. Continuing in this way we rotate each geodesic in turn clockwise so that it is as near as possible to its clockwise neighbour. Finally, we rotate g_n. By assumption $\theta_n > 2A$, this increases σ. Since $\sum \theta_i = \pi$ we have thus shown that the maximum value of σ, for given n and R, is $\sigma_{\max} = 2[(n-1)\ell(R, A) + \ell(R, \pi - (n-1)A)]$.

We now maximize this expression by varying R. In fact A is the more convenient variable to use. Clearly, $A \in (0, \pi/n]$, the maximum value of A being taken when the strips are symmetrically spread around C. We show that this value of A also maximizes σ.

We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}A}\sigma_{\max} = \frac{\cosh^2 r \cos((n-1)A)[(n-1)\sin A \cos((n-1)A) - \sin((n-1)A)\cos A]}{(\cosh^2 r \sin^2((n-1)A) - \sin^2 A)\sqrt{\cosh^2 r - \sin^2 A}}.$$

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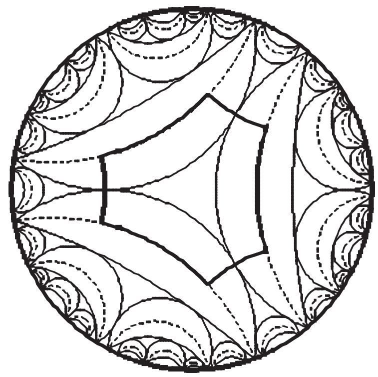

Figure 2. Densest packing of plane strips ($r = 0.5$).

The term in square brackets is zero at $A = 0$ and has derivative

$$[1 - (n-1)^2] \sin A \sin(n-1)A,$$

which is negative throughout $(0, \pi/n]$, $\cos(n-1)A$ changes sign once from positive to negative in $(0, \pi/n]$ and the other factors are positive. Thus, to show that $\sigma_{\max}$ is maximized at $A = \pi/n$, we need that only show that $\sigma_{\max}|_{A=\pi/n} \geqslant \lim_{A\to 0} \sigma_{\max}$. That is we must show $\forall r > 0$, $n \geqslant 3$,

$$n \operatorname{arcsinh}\left(\frac{\cos \pi/n}{\sinh r}\right) - (n-1) \operatorname{arcsinh}\left(\frac{1}{\sinh r}\right) + \operatorname{arcsinh}\left(\frac{1}{\sqrt{(n-1)^2 \cosh^2 r - 1}}\right) \geqslant 0.$$

This is readily verified in the limit as $r \to \infty$, and the derivative of the left-hand side of the above inequality is

$$\frac{-1}{\sinh r}\left[\frac{n \cos(\pi/n) \cosh r}{\sqrt{\cos^2(\pi/n) + \sinh^2 r}} - (n-1) + \frac{(n-1)\sinh^2 r}{(n-1)^2 \cosh^2 r - 1}\right] \leqslant -\frac{n \cos(\pi/n) - n + 1}{\sinh r}. \tag{2.4}$$

The expression on the right-hand side of (2.4) is negative for $n \geqslant 5$ and an easy calculation shows that the same is true of the expression on the left-hand side for $n \leqslant 4$. Consequently, for each $n \geqslant 3$ and $r > 0$, σ attains its maximum value at $A = \pi/n$ and this value is

$$2n \operatorname{arcsinh}\left(\frac{\cos(\pi/n)}{\sinh r}\right).$$

Thus, by (2.3),

$$\rho = \rho_{C_1} \leqslant \frac{2n \sinh(r)}{(n-2)\pi} \operatorname{arcsinh}\left(\frac{\cos(\pi/n)}{\sinh r}\right). \tag{2.5}$$

It is elementary to show that this expression is maximized at $n = 3$ and this completes the proof of the inequality (1.1).

Finally, we show by example that the density bound in (1.1) is sharp. For any $r > 0$ let P be a right-angled hexagon with side lengths alternating between $2r$ and s, where

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s is determined by the relation $\cosh s = (\cosh 2r/(\cosh 2r - 1))$ (such a hexagon exists (see, for example, [**6**]). Let Γ be the group generated by reflections through the sides of P, and let $\mathcal{P}$ be the set of strips of radius r whose axes are the geodesics containing the edges of length s of the images of P under Γ (see Figure 2). In the notation of the foregoing proof, $A = \pi/3$ and $n = 3$, or by direct calculation, the bound (1.1) is attained by this packing. □

Acknowledgements. Research supported in part from grants from the New Zealand Marsden Fund and the Royal Society of New Zealand.

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