J. Austral. Math. Soc. (Series A) 41 (1986), 411–420

PRIMITIVE IDEALS WITH BOUNDED APPROXIMATE UNITS IN L^1-ALGEBRAS OF EXPONENTIAL LIE GROUPS

MOHAMMED EL BACHIR BEKKA

(Received 20 December 1984)

Communicated by J. F. Price

Abstract

Let G be an exponential Lie group. We study primitive ideals (i.e. kernels of irreducible $*$-representations of $L^1(G)$), with bounded approximate units (b.a.u.). We prove a result relating the existence of b.a.u. in certain primitive ideals with the geometry of the corresponding Kirillov orbits. This yields for a solvable group of class 2, a characterization of the primitive ideals with b.a.u.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 43 A 20, 22 E 30.

Introduction

A Banach algebra B has *bounded approximate units* if there is a bounded net $\{u_\alpha\}_{\alpha\in A}$ in B such that $\lim_{\alpha\in A}\|u_\alpha x - x\| = \lim_{\alpha\in A}\|xu_\alpha - x\| = 0$ for every $x \in B$. Given a locally compact group G, one problem of interest in harmonic analysis is to describe the closed ideals in $L^1(G)$ which possess bounded approximate units. For abelian G these are exactly the kernels of closed sets in the coset ring $R(\hat{G})$, where $\hat{G}$ is the dual group of G. This is a result of Liu, van Rooij, Wang [10] and Reiter [16]. In this paper we are concerned with primitive ideals with bounded approximate units in group algebras of exponential Lie groups (an ideal in $L^1(G)$ for a locally compact group G is called *primitive* if it is the kernel of a topologically irreducible $*$-representation of $L^1(G)$ in a Hilbert space). For such groups G there is a parametrization of $\hat{G}$ (the set of equivalence classes of irreducible unitary representations of G) by the orbits of the coadjoint representation Ad* of G in the dual $\mathfrak{G}^*$ of the Lie algebra $\mathfrak{G}$ (Kirillov [9], Bernat

[2]). It was shown in [1, 2.3, Theorem] that L^1-kernels of closed $\mathrm{Ad}^*(G)$-invariant sets in the coset ring $R(\mathfrak{G}^*)$ have bounded approximate units.

This paper is organized as follows. Section 1 contains some notations and definitions. In Section 2 we study, for later use, some properties of the orbits under the coadjoint representation of an exponential Lie group. In Section 3 we prove our main result. Let π be an irreducible unitary representation of an exponential Lie group G and let Ω be the corresponding $\mathrm{Ad}^*(G)$-orbit. Suppose that π satisfies the following condition: $(*)\pi$ is quasi-equivalent to the induced representation $\mathrm{ind}(N, G, \chi)$ for some normal subgroup N of G and some character χ of N. Then we show that the primitive ideal L^1-$\ker\pi$ has bounded approximate units if and only if Ω is affine linear. Moreover, condition $(*)$ is discussed: if, for example, G is solvable of class 2, then the above result yields a complete description of the primitive ideals in $L^1(G)$ which have bounded approximate units.

1. Preliminaries

1.1. A normed algebra A has *bounded approximate units*, abbreviated *b.a.u.*, if there is a constant C such that for every $x \in A$ and $\varepsilon > 0$, there is an element $u \in A$ with $\|u\| \leqslant C$, $\|ux - x\| < \varepsilon$ and $\|xu - x\| < \varepsilon$. This is equivalent to the existence of a bounded net $\{u_\alpha\}$ in A such that $\|u_\alpha x - x\| \to 0$ and $\|xu_\alpha - x\| \to 0$ for every $x \in A$ [6, 9.3].

Let G be a locally compact group and N a closed normal subgroup of G. Let χ be a unitary representation of N and π a unitary representation of G. For $x \in G$ denote by χ^x the unitary representation of N defined by $\chi^x(n) = \chi(x^{-1}nx)$. Denote by $\mathrm{ind}(N, G, \chi)$ the unitary representation of G induced by χ.

Every unitary representation of a locally compact group can be integrated to a $*$-representation of the corresponding L^1-algebra. We always denote this representation by the same symbol. The following result is proved in [1, 1.3, Corollary 2]. Assume that L^1-$\ker\pi = L^1$-$\ker\mathrm{ind}(N, G, \chi)$. Then L^1-$\ker\pi$ has b.a.u. if and only if $\bigcap_{x\in G} L^1$-$\ker\chi^x$ has b.a.u.

1.2. The *coset ring* of an abelian group X, denoted by $R(X)$, is the smallest Boolean algebra of subsets of X containing the cosets of all subgroups of X. Let G be a locally compact abelian group. We recall that a closed ideal in $L^1(G)$ has b.a.u. if and only if it is the kernel of a closed set in the coset ring $R(\hat{G})$ of the dual group $\hat{G}$ of G [10, Theorem 8; 16, Section 17, Theorem 12]. A description of the closed sets of $R(\hat{G})$ is given in [17, Theorem 1.7].

1.3. Let G be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{G}$, let $\mathfrak{G}^*$ be the dual space of $\mathfrak{G}$, and let Ad^* be the coadjoint representation of G in $\mathfrak{G}^*$. Then G is called *exponential* if the exponential map $\exp\colon \mathfrak{G} \to G$ is surjective (this is the case if, for example, G is nilpotent). The irreducible unitary representations of an exponential Lie group G (or, equivalently, the irreducible $*$-representations of $L^1(G)$) are, by the Kirillov theory, in a one-to-one correspondence with the orbits of Ad^*[2].

2. Coset ring, Kirillov orbits and quasiequivalence

We prove here some results which may be of independent interest and which will be needed in the next section.

2.1. The orbits of the coadjoint representation of an exponential Lie group $\exp \mathfrak{G}$ are connected embedded submanifolds of $\mathfrak{G}^*$ [3, Chapter I, Théorème 3.8]. We shall need the following simple lemma.

LEMMA. *Let Ω be a connected embedded submanifold of $\mathbb{R}^n$ such that the closure $\overline{\Omega}$ of Ω belongs to the coset ring $R(\mathbb{R}^n)$. Then $\overline{\Omega}$ is affine linear.*

PROOF. By [17, Theorem 1.7] $\overline{\Omega}$ has the form

$$\overline{\Omega} = \bigcup_{j=1}^{N} \left(K_0^j \setminus \bigcup_{i=1}^{n(j)} K_i^j \right),$$

where the K_i^j are (possibly void) closed cosets in $\mathbb{R}^n$ such that for each $j = 1, \ldots, N$, K_i^j is open in K_0^j, $i = 1, \ldots, n(j)$. Now it follows from the well-known structure of the closed subgroups of $\mathbb{R}^n$ that $\overline{\Omega}$ can be written in the form

$$\overline{\Omega} = \bigcup_{i \in I} A_i,$$

where I is at most countable, every A_i is an affine subspace of $\mathbb{R}^n$, $A_i \not\subseteq \bigcup_{j \neq i} A_j$ for every $i \in I$, and any union of the A_i is closed. Thus $(A_i \setminus \bigcup_{j \neq i} A_j) \cap \Omega$ is open in Ω and nonvoid for every $i \in I$. This shows that

$$(*) \qquad \dim A_i = \dim \Omega$$

for every $i \in I$. Now let $i, j \in I$ and $i \neq j$ (that is $A_i \neq A_j$). We claim that $A_i \cap A_j \cap \Omega = \varnothing$. If $\omega \in A_i \cap A_j \cap \Omega$, then the tangent space $T_\omega(\Omega)$ to Ω at ω contains A_i and A_j, since Ω is open in $\overline{\Omega}$ (Ω is locally compact). Thus, by $(*)$,

$A_i = A_j = T_\omega(\Omega)$, and we get a contradiction. Therefore

$$\Omega = \bigcup_{i \in I} (\Omega \cap A_i),$$

where $(\Omega \cap A_i) \cap (\Omega \cap A_j) = \varnothing$ for $i \neq j$, and $\Omega \cap A_i = \Omega \cap (A_i \setminus \bigcup_{j \neq i} A_j)$ is open in Ω and nonvoid for every $i \in I$. As Ω is connected it follows that I consists of one point, i.e. that $\overline{\Omega}$ is affine linear.

2.2. The notion of an exponential Lie algebra can be generalized as follows. Let $G = \exp \mathfrak{G}$ be an exponential Lie group and ρ a continuous representation of G on a finite-dimensional real vectorspace V. If the weights of ρ on V are of the form $\lambda(1 + i\alpha)$, where α is a real number and λ is a real linear form on $\mathfrak{G}$, then V is said to be an exponential G-module. The structure of the orbits of such representations is described in [15, Proposition 5]. Using this result we can prove the following lemma (which is more precise than Lemma 2.1 in case Ω is an orbit in an exponential G-module).

LEMMA. *Let $G = \exp \mathfrak{G}$ be an exponential group and V an exponential G-module. Let Ω be an orbit of G in V and suppose that the closure $\overline{\Omega}$ of Ω is affine linear. Then Ω is closed.*

PROOF. Denote by ρ the representation of G on V, take $f \in \Omega$ and consider the map $\tau \colon\ G \to \Omega$, $x \to \rho(x)f$. For $X \in \mathfrak{G}$ put $X \cdot f = d\tau(e)X = d\tau/dt(\exp tX)|_{t=0}$, where $d\tau(e)$ is the differential of τ at the group unit e of G. Then for the tangent space $T_f(\Omega)$ of Ω at f we have $T_f(\Omega) = f + \mathfrak{G} \cdot f$. By hypothesis $\overline{\Omega}$ is affine linear, and by [15, 4, Corollary 1] Ω is open in $\overline{\Omega}$. Thus

$$\overline{\Omega} = T_f(\Omega) = f + \mathfrak{G} \cdot f.$$

This implies that $\mathfrak{G} \cdot f$ is G-invariant. Hence $\mathfrak{G} \cdot f$ is $\mathfrak{G}$-invariant, i.e. it is invariant under the action $\mathfrak{G} \times V \to V$, $(X, h) \to d\rho/dt(\exp tX)h|_{t=0}$ of $\mathfrak{G}$ on V. Now let $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_m = \{0\}$ be a Jordan-Hölder sequence of the $\mathfrak{G}$-module V, with $\mathfrak{G} \cdot f = V_i$ for some $i \in \{0, \ldots, m\}$. Denote by π_j the projection $V \to V/V_j$ and by $\mathfrak{G}_j$ the set $\{X \in \mathfrak{G} \mid X \cdot f \in V_j\}$ $(j = 0, 1, \ldots, m)$. Then $\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \cdots \supseteq \mathfrak{G}_m = \mathfrak{G}(f)$, where $\mathfrak{G}(f)$ is the Lie algebra of the stabilizer of f in G. Let $1 \leqslant j_1 < j_2 < \cdots < j_d \leqslant m$ be the indices for which $\mathfrak{G}_{j_k} \neq \mathfrak{G}_{j_k - 1}$. Observe that $j_1 > i$, since $\mathfrak{G} \cdot f = V_i$.

Now by [15, Proposition 5] one can choose a basis $\{v_j'\}$ (respectively $\{v_j', v_j''\}$) of V_{j-1}/V_j for each $j = 1, \ldots, m$ if $\dim V_{j-1}/V_j = 1$ (respectively if $\dim V_{j-1}/V_j = 2$), and a basis $\{X_{j_k}\}$ (respectively $\{X_{j_k}', X_{j_k}''\}$) of $\mathfrak{G}_{j_k - 1}/\mathfrak{G}_{j_k}$ for each $k = 1, \ldots, d$ if $\dim \mathfrak{G}_{j_k - 1}/\mathfrak{G}_{j_k} = 1$ (respectively if $\dim \mathfrak{G}_{j_k - 1}/\mathfrak{G}_{j_k} = 2$) with the following properties. (Write $T_k = t_k \in \mathbb{R}$ if $\dim \mathfrak{G}_{j_k - 1}/\mathfrak{G}_{j_k} = 1$, write $T_k = (t_k', t_k'')$

if $\dim \mathfrak{G}_{j_k-1}/\mathfrak{G}_{j_k} = 2$, let $g_k(T_k) = \rho(\exp t_k X_k)$ or $g_k(T_k) = \rho(\exp t'_k X'_k \exp t''_k X''_k)$, let $g(T) = g_1(T_1) \cdots g_d(T_d)$, and let $v''_j = 0$ if $\dim V_{j-1}/V_j = 1$). The map $T = (T_1, \dots, T_d) \to g(T)f = \Sigma_{j=1}^m (P'_j(T)v'_j + P''_j(T)v''_j)$ is a diffeomorphism between $\mathbb{R}^l$ and Ω ($l = \dim\Omega$), where for $P'_j(T)$ and $P''_j(T)$ the following holds.

(i) $P'_j(T)$ and $P''_j(T)$ depend only on $T_1, \dots, T_k$, where $k = \sup\{s \mid j_s \leqslant j\}$;

(ii) if $T_k = t_k$, then

$$(*) \quad P'_{j_k}(T) + iP''_{j_k}(T) = \frac{e^{a_k(1+i\alpha_k)t_k} - 1}{a_k(1+i\alpha_k)} e^{L_k(T_1,\dots,T_{k-1})} + G_k(T_1, \dots, T_{k-1});$$

if $T_k = (t'_k, t''_k)$, then

$$(**) \quad P'_{j_k}(T) + P''_{j_k}(T) = (t'_k + it''_k)e^{L_k(T_1,\dots,T_{k-1})} + G_k(T_1, \dots, T_{k-1});$$

where $\{a_k, \alpha_k\}$ are real numbers, L_k and G_k are functions of $T_1, \dots, T_{k-1}$ and $i = \sqrt{-1}$. Now let k be the smallest index such that $P'_{j_k}(T) + iP''_{j_k}(T)$ has the form $(*)$ with $a_k \neq 0$. Fix $T^0 = (T_1^0, \dots, T_d^0) \in \mathbb{R}^l$ and take $v \in V$ such that

$$\pi_{j_k} v = \sum_{j=1}^{j_k-1} \left[P'_j(T^0)v'_j + P''_j(T^0)v''_j + \mathrm{Re}\left[\frac{-2}{a_k(1+i\alpha_k)} e^{L_k(T^0)} + G_k(T^0)\right] v'_{j_k} \right.$$
$$\left. + \mathrm{Im}\left[\frac{-2}{a_k(1+i\alpha_k)} e^{L_k(T^0)} + G_k(T^0)\right] v''_{j_k} \right].$$

As $j_k > i$, we see that $\pi_i v = \pi_{if}$, and hence $v \in \overline{\Omega} = f + V_i$. Let $\{T^n\}_{n \in \mathbb{N}}$ be a sequence of points in $\mathbb{R}^l$ such that $g(T^n)f$ converges to v. Hence $\lim_n \pi_{j_k} g(T^n)f = \pi_{j_k} v$. For each $s = 1, \dots, k-1$ we have $P'_{j_s}(T) + iP''_{j_s}(T)$ has either the form $(*)$ with $a_s = 0$, i.e.

$$(***) \qquad P'_{j_s}(T) + iP''_{j_s}(T) = t_s e^{L_s(T_1,\dots,T_{s-1})} + G_s(T_1, \dots, T_{s-1}),$$

or it has the form $(**)$. Thus we get (by an easy induction) that $\lim_n T_s^n = T_s^0$ for each $s = 1, \dots, k-1$. As

$$\lim_n P'_{j_k}(T^n) + iP''_{j_k}(T^n) = \frac{-2}{a_k(1+i\alpha_k)} e^{L_k(T_1^0,\dots,T_{k-1}^0)} + G_k(T_1^0, \dots, T_{k-1}^0),$$

it now follows that

$$\lim_n e^{a_k(1+i\alpha_k)t_k^n} = -1,$$

which is impossible. This implies that there is no k such that $P_{j_k}(T) + iP''_{j_k}(T)$ has the form $(*)$ with $a_k \neq 0$. Thus each $P'_{j_k}(T) + iP''_{j_k}(T)$ has the form $(**)$ or the form $(***)$, and this implies that if $\{g(T^n)f\}_n$ is a convergent sequence, then $\{T^n\}$ is bounded. Passing to a subsequence, we can assume that $\{T^n\}_n$ is convergent. Thus we see that Ω is closed, and the proof is finished.

REMARK. The above lemma cannot be generalized to coadjoint orbits of nonexponential Lie groups. This is shown by the following example.

Let $\mathfrak{G} = \mathfrak{G}_{4,2}$ (in the notation of [3]) be the Lie algebra with basis $X_1, \dots, X_4$ and nontrivial Lie products $[X_1, X_2] = X_3$, $[X_1, X_4] = X_4$, $[X_2, X_3] = -X_4$ and $[X_2, X_4] = X_3$. As $i = \sqrt{-1}$ is an eigenvalue of ad X_2, we see that $\mathfrak{G}$ is nonexponential. Let $\{X_1^*, \dots, X_4^*\}$ be the dual basis in $\mathfrak{G}^*$. Then $\Omega = \mathfrak{G}^* \setminus (\mathbb{R} X_1^* + \mathbb{R} X_2^*)$ is an orbit under the coadjoint representation of $\exp \mathfrak{G}$.

2.3. Let G be a connected, simply connected exponential Lie group, let $\mathfrak{G}$ be its Lie algebra and $\mathfrak{G}^*$ the dual of $\mathfrak{G}$. Let $\mathbf{n}$ be an ideal of $\mathfrak{G}$ and $N = \exp \mathbf{n}$ the corresponding normal subgroup of G. Denote by $\hat{G}$ (resp. $\hat{N}$) the set of all unitary irreducible representations of G (respectively N) and by Θ the Kirillov map $\mathfrak{G}^* \to \hat{G}$ (respectively Θ_N: $\mathbf{n}^* \to \hat{N}$). Let $\pi \in \hat{G}$, $\chi \in \hat{N}$ and $f \in \mathfrak{G}^*$ with $\Theta_N(f\,|\,\mathbf{n}) = \chi$, and denote by Ω the coadjoint orbit corresponding to π. Now using Lemma 2.2 and [8, Proposition 2], we give a necessary and sufficient condition for π to be quasiequivalent to the induced representation $\rho = \operatorname{ind}(N, G, \chi)$.

LEMMA. *π is quasiequivalent to $\rho = \operatorname{ind}(N, G, \chi)$ (abbreviated $\pi \approx \rho$) if and only if the affine subspace $f + \mathbf{n}^\perp$ is contained in Ω.*

PROOF. Let μ be a probability measure on $\mathfrak{G}^*$ which is equivalent to the Lebesgue measure of $f + \mathbf{n}^\perp$ and let ν be the image of μ under the (continuous) map Θ. It is shown in [8, Proposition 2] that ρ is quasiequivalent to the direct integral

$$\int_{\hat{G}}^{\oplus} \xi \, d\nu(\xi).$$

Now, by a general fact (cf. [4, 8.4.4.]), $\pi \approx \rho$ if and only if the corresponding central decomposition measures are equivalent, i.e. if and only if ν and the Dirac measure δ_π at $\{\pi\}$ are equivalent. Thus $\pi \approx \rho$ if and only if $\mu(f + \mathbf{n}^\perp \setminus \Omega) = 0$. Assuming that $\pi \approx \rho$, we see that $(f + \mathbf{n}^\perp) \cap \Omega$ is dense in $f + \mathbf{n}^\perp$. Now consider $\mathbf{n}^*$ as an exponential G-module (the action of G on $\mathbf{n}^*$ is the natural one: $x \to \mathrm{Ad}^*(x)\,|\,\mathbf{n}^*$, $x \in G$). Let $g \in (f + \mathbf{n}^\perp) \cap \Omega$ and let K be the stabilizer of $g\,|\,\mathbf{n}$, that is $K = \{x \in G \,|\, \mathrm{Ad}^*(x) g\,|\,\mathbf{n} = g\,|\,\mathbf{n}\}$. K is connected by [3, Chapter I, Théorème 3.3]. Moreover, it is easily seen that

$$\mathrm{Ad}^*(K) g = (f + \mathbf{n}^\perp) \cap \Omega.$$

Thus we see that the orbit of g in the exponential K-module $\mathfrak{G}^*$ is dense in its linear hull $f + \mathbf{n}^\perp$. Now Lemma 2.2 applies and yields $(f + \mathbf{n}^\perp) \cap \Omega = f + \mathbf{n}^\perp$, i.e., $f + \mathbf{n}^\perp$ is contained in Ω.

If, conversely, $f + \mathbf{n}^{\perp} \subseteq \Omega$ then obviously ν is equivalent to δ_{π}, and $\pi \approx \rho$.

COROLLARY. *Let G, N, $\mathfrak{G}$, $\mathbf{n}$, π, χ, Ω and f be as above. Then π is quasiequivalent to* $\operatorname{ind}(N, G, \chi)$ *if and only if $f \in \Omega$ and $\Omega + \mathbf{n}^{\perp} = \Omega$.*

REMARK. If G is nilpotent, then π is quasiequivalent to ρ if and only if π is weakly equivalent to ρ, since nilpotent Lie groups are liminal [4, 13.11.12].

3. Primitive ideals with bounded approximate units

Let G be an exponential Lie group and let $\pi \in \hat{G}$. Suppose that π satisfies the following condition:

$(*)$ $\pi \approx \operatorname{ind}(N, G, \chi)$ for a connected normal subgroup N of G and a character χ of N.

Then, using [1, 1.3, Theorem] and the result of Section 2, we give in 3.1 a necessary and sufficient condition for L^1-$\ker \pi$ to have b.a.u. The condition $(*)$ will be discussed in 3.2.

3.1. THEOREM. *Let G be a connected, simply connected exponential Lie group with Lie algebra $\mathfrak{G}$. Let $\pi \in \hat{G}$ and suppose that π satisfies the condition $(*)$. Let Ω be the coadjoint orbit corresponding to π. Then L^1-$\ker \pi$ has b.a.u. if and only if Ω is affine linear.*

PROOF. Let $\mathbf{n}$ be the subalgebra of $\mathfrak{G}$ corresponding to N and let $f \in \mathfrak{G}^*$ be such that $\Theta_N(f\,|\,\mathbf{n}) = \chi$. Let p: $\mathfrak{G}^* \to \mathbf{n}^*$ be the projection $h \to h\,|\,\mathbf{n}$. We have (Corollary 2.3): $f \in \Omega$ and $\Omega = p^{-1}(p(\Omega))$. Denote by J the ideal $\bigcap_{x \in G} L^1$-$\ker \chi^x$ of $L^1(N)$, where χ^x is defined by $\chi^x(n) = \chi(x^{-1}nx)$, $n \in N$. Assume that L^1-$\ker \pi$ has b.a.u. Then, by [1, 1.3, Corollary 2], J has also b.a.u. Thus $T(J)$ has b.a.u., where T: $L^1(N) \to L^1(N/N')$ is the canonical projection and N' the closure of the commutator subgroup of N. Hence $h(T(J))$ belongs to the coset ring $R((\mathbf{n}/[\mathbf{n}, \mathbf{n}])^*)$ of $(\mathbf{n}/[\mathbf{n}, \mathbf{n}])^*$, by the abelian result [16]. Clearly, $\Theta_N(p(\operatorname{Ad}^*(x)f)) = \chi^x$ for every $x \in G$, that is $h(J) = \overline{p(\Omega)}$. Thus $\overline{p(\Omega)} \in R(\mathbf{n}^*)$, and therefore $\overline{\Omega} = p^{-1}(\overline{p(\Omega)}) \in R(\mathfrak{G}^*)$. Now from Lemma 2.1 and Lemma 2.2 we get that Ω is affine linear. This completes the proof of one part of the theorem. The converse is contained in [1, Theorem, page 396].

3.2. Let G be a connected, simply connected exponential Lie group with Lie algebra $\mathfrak{G}$. Let $\pi \in \hat{G}$, and let Ω be the corresponding $\operatorname{Ad}^*(G)$-orbit. One can associate to Ω an ideal of $\mathfrak{G}$ in the following way, cf. [12]: Let $F = \{\lambda \in \mathfrak{G}^* \,|\, \Omega + t\lambda = \Omega$ for each $t \in \mathbb{R}\}$, and put $\mathbf{m}(\Omega) = F^{\perp}$. Then $\mathbf{m}(\Omega)$ is an ideal of $\mathfrak{G}$

and contains $\mathfrak{G}(f) = \{X \in \mathfrak{G} \mid (f, [X, \mathfrak{G}]) = 0\}$ for each $f \in \Omega$ [12, Theorem 1]. Using the results in 2.3, we can easily prove the following lemma (which is a generalization of [14, Remark 2.2]).

LEMMA. $\mathbf{m}(\Omega)$ *is the smallest ideal* $\mathbf{n}$ *of* $\mathfrak{G}$ *such that* $\pi \approx \operatorname{ind}(\exp \mathbf{n}, G, \chi)$ *for some* $\chi \in (\exp \mathbf{n})^{\wedge}$.

PROOF. If $\mathbf{n}$ is such that $\pi \approx \operatorname{ind}(\exp \mathbf{n}, G, \chi)$ for some $\chi \in (\exp \mathbf{n})^{\wedge}$, then by Corollary 2.3 $\Omega + \mathbf{n}^{\perp} = \Omega$, and hence $\mathbf{m}(\Omega) \subseteq \mathbf{n}$. If $f \in \Omega$, then by Lemma 2.3, $\pi \approx \operatorname{ind}(\exp \mathbf{m}(\Omega), G, \chi)$ for $\chi = \Theta_{\exp \mathbf{m}(\Omega)}(f \mid \mathbf{m}(\Omega))$.

COROLLARY. π *satisfies the condition* $(*)$ *if and only if* $\mathbf{m}(\Omega)$ *is subordinate to some* $f \in \Omega$ *(that is* $(f, [\mathbf{m}(\Omega), \mathbf{m}(\Omega)]) = 0$).

PROOF. Supose $\pi \approx \operatorname{ind}(N, G, \chi)$ for a connected normal subgroup N of G and a character χ of N. Then, by Lemma 2.3, $\chi = \Theta_N(f)$ for some $f \in \Omega$. It is clear that the Lie algebra $\mathbf{n}$ of N is subordinate to f. By the above lemma $\mathbf{m}(\Omega) \subseteq \mathbf{n}$, and $\mathbf{m}(\Omega)$ is thus subordinate to f. The converse is also clear from the above.

REMARKS. (i) The ideal of $\mathfrak{G}$ generated by $\mathfrak{G}(f)$ (for $f \in \Omega$) is, in general, strictly contained in $\mathbf{m}(\Omega)$: let $\mathfrak{G}$ be the "$ax + b$"-Lie algebra ($\mathfrak{G} = \mathbb{R} X_1 + \mathbb{R} X_2$, $[X_1, X_2] = X_2$). The orbit of X_2^* is $\Omega = \mathbb{R} X_1^* + \mathbb{R}^+ X_2^*$, where $\{X_1^*, X_2^*\}$ is the dual basis of $\mathfrak{G}^*$. Thus $\mathbf{m}(\Omega) = \mathbb{R} X_2$, but $\mathfrak{G}(f) = \{0\}$. (ii) Let $\mathfrak{G}^{\infty} = \bigcap_{k=0}^{\infty} D_{\mathfrak{G}}^k$, where $D_{\mathfrak{G}}^k$ is defined by $D_{\mathfrak{G}}^0 = \mathfrak{G}$, and $D_{\mathfrak{G}}^k = [\mathfrak{G}, D_{\mathfrak{G}}^{k-1}]$, $k \geqslant 1$. Take $f \in \Omega$ and let $\mathbf{n}$ be the ideal of $\mathfrak{G}$ generated by $\mathfrak{G}(f)$ and $\mathfrak{G}^{\infty}$. Then $\mathbf{m}(\Omega) \subseteq \mathbf{n}$ (see [7, Proposition 1.1] where this result is proved for arbitrary Lie groups).

EXAMPLES. (i) Let $\mathfrak{G}$ be an exponential Lie algebra which is solvable of class 2 (i.e. $[\mathfrak{G}, \mathfrak{G}]$ is abelian). Then

$(**)$ $\mathbf{m}(\Omega)$ is subordinate to each $f \in \Omega$ for every coadjoint orbit Ω in $\mathfrak{G}^*$.

Indeed, in this case $[\mathbf{n}, \mathbf{n}] \subseteq [\mathfrak{G}^{\infty}, \mathfrak{G}(f)] + [\mathfrak{G}(f), \mathfrak{G}]$. Theorem 3.1 yields in this case a characterization of the primitive ideals in $L^1(\exp \mathfrak{G})$ which have b.a.u.

(ii) Let G be the group of all upper triangular $n \times n$ real matrices with one's on the diagonal. Then for each orbit Ω in general position (in the sense of [13, II, Chapter I, Section 6]) $\mathbf{m}(\Omega)$ is subordinate to every $f \in \Omega$ (this follows from [9, Section 9.1]). (iii) Let $\mathfrak{G}$ be an exponential Lie algebra of dimension at most 4 (see the classification in [3, Chapter VIII, 1.1]), and let $\mathfrak{G} \neq \mathfrak{G}_{4,9}(0)$. It can be verified that the above property $(**)$ holds. The Lie algebra $\mathfrak{G}_{4,9}(0)$ with basis $\{X_1, X_2, X_3, X_4\}$ is defined by the nontrivial commutators $[X_2, X_3] = X_4$, $[X_1, X_2] = -X_2$, $[X_1, X_3] = X_4$. If $\{X_1^*, X_2^*, X_3^*, X_4^*\}$ is the dual basis in $\mathfrak{G}^*$, then the orbit of X_4^* is $\Omega = \{\sum_{i=1}^4 a_i X_i^* \mid a_1 = -a_2 a_3,\ a_4 = 1\}$ and $\mathbf{m}(\Omega) = \mathfrak{G}_{4,9}(0)$. Thus, for the representation π of $G = \exp \mathfrak{G}_{4,9}(0)$ corresponding to Ω, there is no normal subgroup N, $N \neq G$, such that $\pi \approx \operatorname{ind}(N, G, \pi \mid N)$. (iv) The condition (**) has been investigated in [14] (the notation there is $s(\Omega) \equiv 1$) for nilpotent Lie algebras of low dimension, and the following has been established: there are exactly 18 algebras among the nilpotent Lie algebras of dimension at most 7 for which the condition (**) is not satisfied, and they all are of dimension greater than 5 [14, Theorem 7, Proposition 5.7]. 3.3. We conclude with some remarks.

Remark 1. Let G be a connected nilpotent Lie group and let $\pi \in \hat{G}$. If L^1-ker π contains (bounded or unbounded) approximate units, then $\{\pi\}$ is a spectral set in $\hat{G}$. This follows from [11, Theorem 7]: it is proved there that L^1-ker $\pi/j(\pi)$ is a nilpotent algebra, $j(\pi)$ being the smallest ideal in $L^1(G)$ with hull $\{\pi\}$.

Remark 2. Theorem 1.3 in [1] can be used to characterize all closed two-sided ideals with b.a.u. in L^1-algebras of low dimensional nilpotent Lie groups. We mention without proof the following result. Let G be a connected, simply connected nilpotent Lie group with $\dim G \leqslant 5$ and $G \neq \Gamma_{5,4}$ (in the notation of [5]) and let $\mathfrak{G}$ be the corresponding Lie algebra. A closed two-sided ideal in $L^1(G)$ has b.a.u. if and only if it is the kernel of a closed $\mathrm{Ad}^*(G)$-invariant set in the coset ring $R(\mathfrak{G}^*)$.

References

[1] M. B. Bekka, 'On bounded approximate units in ideals of group algebras', *Math. Ann.* **266** (1984), 391–396.

[2] P. Bernat, 'Sur les représentations unitaires des groupes de Lie résolubles', *Ann. École Norm. Sup.* **82** (1965), 37–99.

[3] P. Bernat and N. Conze, *Représentations des groupes de Lie résolubles* (Dunod, Paris, 1972).

[4] J. Dixmier, *Les C*-algèbres et leurs représentations* (Gauthier-Villars, Paris, 1969).

[5] J. Dixmier, 'Sur les représentations des groupes de Lie nilpotents III', *Canad. J. Math.* **10** (1958), 321–348.

[6] R. S. Doran and J. Wichmann, *Approximate identities and factorization in Banach modules* (Lecture Notes in Mathematics 768, Springer-Verlag, Berlin, 1979).

[7] M. Duflo, 'Caractères des groupes et des algèbres de Lie résolubles', *Ann. Ecole Norm. Sup.* **3**, *Sér.* 4 (1970), 23–74.

[8] G. Grélaud, 'Désintégration des représentations induites d'un groupe de Lie résoluble exponentiel', *C. R. Acad. Sci. Paris* **277**, *Série* A (1973), 327–330.

[9] A. A. Kirillov, 'Unitary representations of nilpotent Lie groups', *Uspehi Mat. Nauk* **17** (1962), 57–110.

[10] T. S. Liu, A. van Rooij and J. K. Wang, 'Projections and approximate identities for ideals in group algebras', *Trans. Amer. Math. Soc.* **175** (1973), 469–482.

[11] J. Ludwig, 'On primary ideals in the group algebra of a nilpotent Lie group', *Math. Ann.* **262** (1983), 287–304.

[12] R. C. Penney, 'Canonical objects in the Kirillov theory of nilpotent Lie groups', *Proc. Amer. Soc.* **66** (1977), 175–178.

[13] L. Pukanszky, *Lecons sur les représentations des groupes* (Dunod, Paris, 1966).

[14] L. Pukanszky, 'On Kirillov's character formula', *J. Reine Angew. Math.* **311/312** (1979), 408–440.

[15] L. Pukanszky, 'On the unitary representations of exponential groups', *J. Funct. Anal.* **2** (1968), 73–113.

[16] H. Reiter, *L^1-Algebras and Segal Algebras* (Lecture Notes in Mathematics 231, Springer-Verlag, Berlin, 1971).

[17] B. M. Schreiber, 'On the coset ring and strong Ditkin sets', *Pacific J. Math.* **32** (1970), 805–812.

Fachbereich Mathematik/Informatik
der Universität-Gesamthochschule Paderborn
Warburgerstr. 100
D-4790 Paderborn
Federal Republic of Germany