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Isometry on Linear n-G-quasi Normed Spaces

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Abstract. This paper generalizes the Aleksandrov problem: the Mazur-Ulam theorem on n-G-quasi normed spaces. It proves that a one-n-distance preserving mapping is an n-isometry if and only if it has the zero-n-G-quasi preserving property, and two kinds of n-isometries on n-G-quasi normed space are equivalent; we generalize the Benz theorem to n-normed spaces with no restrictions on the dimension of spaces.

1 Introduction and Preliminaries

Let X and Y be metric spaces. A mapping $f: X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot, \cdot)$ and $d_Y(\cdot, \cdot)$ denote the metrics in the spaces X and Y, respectively. And for some fixed number $\rho > 0$, suppose that f preserves distance ρ, *i.e.*, for all $x, y \in X$ with $d_X(x, y) = \rho$, we have $d_Y(f(x), f(y)) = \rho$. Then ρ is called a conservative distance for the mapping f. Mazur and Ulam gave a theorem [9] stating that every isometry of a real normed space onto a real normed space is a linear mapping up to a translation.

Aleksandrov [1] asked whether the existence of a single conservative distance for some mapping f between two metric spaces implies that f is an isometry.

Benz [2] gave another relative result: let X and Y be real linear normed spaces such that $\dim X \geq 2$ and Y is strictly convex. Suppose that $\rho > 0$ is a fixed real number and that $N > 1$ is a fixed integer. Finally, if $f: X \to Y$ is a mapping such that for all $x, y \in X$ $\|x - y\| = \rho \Rightarrow \|f(x) - f(y)\| \leq \rho$ and $\|x - y\| = N\rho \Rightarrow \|f(x) - f(y)\| \geq N\rho$, then f is an affine isometry.

Rassias and Šemrl [17], Jing [8], and Ma [10] proved a series of results on the Aleksandrov problem on normed spaces. Initial research in linear n-normed spaces [3–6,14,16] defined the concept of a w-n-isometry and an n-isometry that are suitable to represent the notion of a volume-preserving mapping, and generalized the Mazur–Ulam theorem and Aleksandrov problem to n-normed spaces. Recently, Yumei Ma [11–13] generalized the above results and proved the following.

Theorem 1.1 *For every mapping f that preserves the unit distance between n-normed linear spaces X and Y, the following properties are equivalent:*

(i) *f preserves w-n-0-distance (n-collinear);*

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(ii) *f is a w-n-Lipschitz;*
(iii) *f preserves 2-collinearity;*
(iv) *f is affine;*
(v) *f is an n-isometry;*
(vi) *f is an n-Lipschitz;*
(vii) *f preserves n-0-distance;*
(viii) *f is a w-n-isometry.*

This paper contains two parts: in the first section, we generalize Theorem 1.1 to n-general quasi normed spaces; in the second section, we prove that the Benz theorem holds in n-normed spaces.

Henceforth, let $n \geq 2$.

Definition 1.2 Assume that X is a real linear space with $\dim X \geq n$ and $\|\cdot,\dots,\cdot\|: X^n \to \mathbb{R}$ is a function satisfying

- $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent,
- $\|x_1,\dots,x_n\| = \|x_{j_1},\dots,x_{j_n}\|$ for every permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$,
- $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$,

for any $\alpha \in \mathbb{R}$ and all $x_1,\dots,x_n \in X$. Then the function $\|\cdot,\dots,\cdot\|$ is called the n-generalized quasi norm (written as n-G-quasi norm) on X, and $(X,\|\cdot,\dots,\cdot\|)$ is called a linear n-generalized quasi normed space (written as n-G-quasi normed space).

The following two definitions [16,19] are n-G-quasi norms.

Definition 1.3 Assume that X is a real linear space with $\dim X \geq n$ and $\|\cdot,\dots,\cdot\|: X^n \to \mathbb{R}$ is a function satisfying

- $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent,
- $\|x_1,\dots,x_n\| = \|x_{j_1},\dots,x_{j_n}\|$ for every permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$,
- $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$,
- $\|tx + (1-t)y, x_2,\dots,x_n\| \leq \max\{\|x,x_2,\dots,x_n\|, \|y,x_2,\dots,x_n\|\}$,

for any $\alpha \in \mathbb{R}$, $t \in [0,1]$, and all $x_1,\dots,x_n \in X$. Then the function $\|\cdot,\dots,\cdot\|$ is called the n-quasi convex norm on X, and $(X,\|\cdot,\dots,\cdot\|)$ is called a linear n-quasi convex normed space.

Definition 1.4 Assume that X is a real linear space with $\dim X \geq n$ and $\|\cdot,\dots,\cdot\|: X^n \to \mathbb{R}$ is a function satisfying

- $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent,
- $\|x_1,\dots,x_n\| = \|x_{j_1},\dots,x_{j_n}\|$ for every permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$,
- $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$,
- $\|x + y, x_2,\dots,x_n\| \leq K(\|x,x_2,\dots,x_n\| + \|y,x_2,\dots,x_n\|)$,

for any $\alpha \in \mathbb{R}$, $K \geq 1$, and all $x_1,\dots,x_n \in X$. Then the function $\|\cdot,\dots,\cdot\|$ is called the n-quasi norm on X, and $(X,\|\cdot,\dots,\cdot\|)$ is called a linear n-quasi normed space.

Remark 1.5 As $K = 1$, $\|\cdot,\dots,\cdot\|$ is called the n-norm on X and $(X, \|\cdot,\dots,\cdot\|)$ is called a linear n-normed space.

Let X and Y be two real n-G-quasi normed spaces.

Definition 1.6 A sequence $\{x_k\}$ is said to converge to an $x \in X$ (in the n-G-quasi norm) if $\lim_{k\to\infty} \|x_k - x, y_2, \dots, y_n\| = 0$ for every $y_2, \dots, y_n \in X$.

Definition 1.7 A mapping $f: X \to Y$ is said to be a w-n-isometry if

$$\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| = \|x_1 - x_0, \dots, x_n - x_0\|,$$

for all $x_0, x_1, \dots, x_n \in X$.

Definition 1.8 A mapping $f: X \to Y$ is said to have the w-n-distance one preserving property (w-n-DOPP) if $\|x_1 - x_0, \dots, x_n - x_0\| = 1$ implies

$$\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| = 1,$$

for all $x_0, x_1, \dots, x_n \in X$.

Definition 1.9 A mapping $f: X \to Y$ is said to be w-n-Lipschitz if

$$\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| \le \|x_1 - x_0, \dots, x_n - x_0\|,$$

for all $x_0, x_1, \dots, x_n \in X$.

Definition 1.10 A mapping $f: X \to Y$ is said to be an n-isometry if

$$\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = \|x_1 - y_1, \dots, x_n - y_n\|,$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$.

Definition 1.11 A mapping $f: X \to Y$ is said to have the n-distance one preserving property (n-DOPP) if $\|x_1 - y_1, \dots, x_n - y_n\| = 1$ implies

$$\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = 1,$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$.

Definition 1.12 A mapping $f: X \to Y$ is said to be n-Lipschitz if

$$\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| \le \|x_1 - y_1, \dots, x_n - y_n\|,$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$.

Definition 1.13 A mapping $f: X \to Y$ is said to preserve the 2-collinearity if for all $x, y, z \in X$, the existence of $t \in \mathbb{R}$ with $z - x = t(y - x)$ implies the existence of $s \in \mathbb{R}$ with $f(z) - f(x) = s(f(y) - f(x))$.

Definition 1.14 The points $x_0, x_1, \dots, x_n$ of X are called n-collinear if for every i, $\{x_j - x_i : 0 \le j \ne i \le n\}$ is linearly dependent.

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Definition 1.15 A mapping $f\colon X \to Y$ is said to preserve the n-collinearity if the n-collinearity of $f(x_0), f(x_1), \dots, f(x_n)$ follows from the n-collinearity of $x_0, x_1, \dots, x_n$.

Remark 1.16 $f\colon X \to Y$ preserves the n-collinearity means that f preserves w-0-distance, *i.e.*, $\|x_1 - x_0, \dots, x_n - x_0\| = 0$ implies $\|f(x_1) - f(x_0), \dots, f(x_n) - f(x_0)\| = 0$.

2 Main Results on n-G-quasi Normed Spaces

Lemma 2.1 *Let X and Y be two real n-G-quasi normed spaces. Suppose that f satisfies w-n-DOPP and $\|x_1 - x_0, x_2 - x_0, \dots, x_n - x_0\| = 0$ implies*

$$\|f(x_1) - f(x_0), f(x_2) - f(x_0), \dots, f(x_n) - f(x_0)\| = 0.$$

Then f preserves 2-collinearity.

Proof We first show that f is injective. For any distinct $x_0, x_1 \in X$, since $\dim X \geq n$, there are $x_2, \dots, x_n \in X$ such that $x_1 - x_0, \dots, x_n - x_0$ are linearly independent. Thus, $\|x_1 - x_0, \dots, x_n - x_0\| \neq 0$. Set $z_2 = x_0 + (x_2 - x_0)/\|x_1 - x_0, \dots, x_n - x_0\|$. Then we have $\|x_1 - x_0, z_2 - x_0, x_3 - x_0, \dots, x_n - x_0\| = 1$. Since f has w-n-DOPP, we get

$$\|f(x_1) - f(x_0), f(z_2) - f(x_0), f(x_3) - f(x_0), \dots, f(x_n) - f(x_0)\| = 1$$

and it follows that $f(x_0) \neq f(x_1)$. Hence, f is injective.

For $n = 2$, f is obviously 2-collinear because $\|x_1 - x_0, x_2 - x_0\| = 0$ implies

$$\|f(x_1) - f(x_0), f(x_2) - f(x_0)\| = 0.$$

Let $n > 2$. Assume that x_0, x_1, x_2 are distinct points of X which are 2-collinear. Then $x_1 - x_0, x_2 - x_0$ are linearly dependent and $f(x_0), f(x_1), f(x_2)$ are also distinct by the injectivity of f.

Since $\dim X \geq n$, there exist $y_1, y_2, \dots, y_n \in X$ such that $y_1 - x_0, y_2 - x_0, \dots, y_n - x_0$ are linearly independent. Hence, it holds that $\|y_1 - x_0, y_2 - x_0, \dots, y_n - x_0\| \neq 0$. Let $z_1 = x_0 + (y_1 - x_0)/\|y_1 - x_0, y_2 - x_0, \dots, y_n - x_0\|$. Then we have

$$\|z_1 - x_0, y_2 - x_0, \dots, y_n - x_0\| = 1.$$

Since f has w-n-DOPP,

$$\|f(z_1) - f(x_0), f(y_2) - f(x_0), \dots, f(y_n) - f(x_0)\| = 1.$$

Hence, the set $A = \{f(x) - f(x_0) : x \in X\}$ contains n linearly independent vectors. Then for any $x_3, \dots, x_n \in X$, $\|x_1 - x_0, x_2 - x_0, x_3 - x_0, \dots, x_n - x_0\| = 0$, and any f that preserves 0-distance, we have

$$\|f(x_1) - f(x_0), f(x_2) - f(x_0), f(x_3) - f(x_0), \dots, f(x_n) - f(x_0)\| = 0.$$

i.e., $f(x_1) - f(x_0), f(x_2) - f(x_0), f(x_3) - f(x_0), \dots, f(x_n) - f(x_0)$ are linearly dependent.

If there exist $x_3, \dots, x_{n-1}$ such that

$$f(x_1) - f(x_0), \quad f(x_2) - f(x_0), \quad f(x_3) - f(x_0), \dots, f(x_{n-1}) - f(x_0)$$

are linearly independent, then

$$A = \{f(x_n) - f(x_0) : x_n \in X\}$$
$$\subset \operatorname{span}\{f(x_1) - f(x_0), f(x_2) - f(x_0), f(x_3) - f(x_0), \ldots, f(x_{n-1}) - f(x_0)\},$$

which contradicts the fact that A contains n linearly independent vectors. Then for any $x_3, \ldots, x_{n-1}$, $f(x_1) - f(x_0), \ldots, f(x_{n-1}) - f(x_0)\}$ are linearly dependent.

If there exist $x_3, \ldots, x_{n-2}$ such that

$$f(x_1) - f(x_0), \quad f(x_2) - f(x_0), \quad f(x_3) - f(x_0), \ldots, f(x_{n-2}) - f(x_0)$$

are linearly independent, then

$$A = \{f(x_{n-1}) - f(x_0) : x_{n-1} \in X\}$$
$$\subset \operatorname{span}\{f(x_1) - f(x_0), f(x_2) - f(x_0), f(x_3) - f(x_0), \ldots, f(x_{n-2}) - f(x_0)\},$$

which contradicts the fact that A contains n linearly independent vectors. Therefore, $f(x_1) - f(x_0)$ and $f(x_2) - f(x_0)$ are linearly dependent, *i.e.*, $f(x_0), f(x_1), f(x_2)$ are 2-collinear. Therefore, f preserves 2-collinearity. ∎

Corollary 2.2 *Let X and Y be two real n-G-quasi normed spaces. If f is w-n-Lipschitz and satisfies w-n-DOPP, then f preserves 2-collinearity.*

Proof Because f is w-n-Lipschitz, $\|x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0\| = 0$ implies

$$\|f(x_1) - f(x_0), f(x_2) - f(x_0), \ldots, f(x_n) - f(x_0)\| = 0.$$

Hence f preserves 2-collinearity by Lemma 2.3. ∎

Lemma 2.3 *Let X and Y be two real n-G-quasi normed spaces. If $f\colon X \to Y$ satisfies w-n-DOPP and preserves 2-collinearity, then f is affine.*

Proof

Step 1: Let $x = \frac{y+z}{2}$ for distinct $x, y, z \in X$. Then $y - x = -(z - x)$. Since f is injective and preserves 2-collinearity, there exists an $s \neq 0$ such that

$$f(y) - f(x) = s(f(z) - f(x)). \tag{2.1}$$

Since $\dim X \geq n$, there exist $x_1, x_2, \ldots, x_{n-1} \in X$ with

$$\|y - x, x_1 - x, x_2 - x, \ldots, x_{n-1} - x\| \neq 0.$$

Set $w = x + (x_1 - x)/\|y - x, x_1 - x, x_2 - x, \ldots, x_{n-1} - x\|$. Thus

$$\|y - x, w - x, x_2 - x, \ldots, x_{n-1} - x\| = 1 \tag{2.2}$$

and $\|f(y) - f(x), f(w) - f(x), f(x_2) - f(x), \ldots, f(x_{n-1}) - f(x)\| = 1$. Clearly, it follows from (2.1) that

$$\|f(z) - f(x), f(w) - f(x), f(x_2) - f(x), \ldots, f(x_{n-1}) - f(x)\| = \frac{1}{|s|}. \tag{2.3}$$

Since $y - x = x - z$, (2.2) yields $\|z - x, w - x, x_2 - x, \ldots, x_{n-1} - x\| = 1$, and hence we have

$$\|f(z) - f(x), f(w) - f(x), f(x_2) - f(x), \ldots, f(x_{n-1}) - f(x)\| = 1. \tag{2.4}$$

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Because f is injective and comparing (2.3) with (2.4), we conclude that $s = -1$. Thus, $f(y) - f(x) = f(x) - f(z)$ and $f(\frac{y+z}{2}) = \frac{f(y)+f(z)}{2}$.

Step 2: Let $g(x) = f(x) - f(0)$. It is obvious that for any $x \in X$ and all rational numbers r, p, we have

$$g(rx) = rg(x), \qquad g(rx + py) = rg(x) + pg(y). \tag{2.5}$$

Step 3: Next we show that g preserves any rational number n-distance. Suppose that $\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{t}{m}$ for integers t and m. Then

$$\Big\| \frac{m}{t}(x_1 - y_1), x_2 - y_2, \dots, x_n - y_n \Big\| = 1.$$

According to (2.5), we have

$$\Big\| \frac{m}{t}(g(x_1) - g(y_1)), g(x_2) - g(y_2), \dots, g(x_n) - g(y_n) \Big\| = 1.$$

Thus $\|g(x_1) - g(y_1), g(x_2) - g(y_2), \dots, g(x_n) - g(y_n)\| = \frac{t}{m}$.

Step 4: For any $r \in \mathbb{R}$, since $g(0), g(x), g(rx)$ are also 2-collinear from $f(0)$, $f(x)$, $f(rx)$ are 2-collinear and $g(0) = 0$. There exists a real number s such that $g(rx) = sg(x)$. Let $\{r_k\}$ be rational number sequence with $\lim_{i\to\infty} r_k = s$. Then for any $y_2, \dots, y_n \in Y$,

$$\lim_{k\to\infty} \|r_k g(x) - sg(x), y_2, \dots, y_n\| = \lim_{k\to\infty} |r_k - s| \|g(x), y_2, \dots, y_n\| = 0.$$

So $g(rx) = \lim_{k\to\infty} r_k g(x)$. This implies $\lim_{k\to\infty} \|g(r_k x) - g(rx), y_2, \dots, y_n\| = 0$. Then for $x \neq 0$ and any k, we can find $x_2^k, \dots, x_n^k$ such that $\|x, x_2^k, \dots, x_n^k\| > 1$ and $|r - r_k| \|x, x_2^k, \dots, x_n^k\|$ is a rational number. This implies that

$$\begin{aligned} |r - r_k| \|x, x_2^k, \dots, x_n^k\| &= \|(r - r_k)x, x_2^k, \dots, x_n^k\| = \|rx - r_k x, x_2^k, \dots, x_n^k\| \\ &= \|g(rx) - g(r_k x), g(x_2^k), \dots, g(x_n^k)\|. \end{aligned}$$

Moreover, $\lim_{k\to\infty} \|g(r_k x) - g(rx), g(x_2^k), \dots, g(x_n^k)\| = 0$, and $\|x, x_2^k, \dots, x_n^k\| > 1$ implies that $\lim_{i\to\infty} r_k = r$. Thus $r = s$.

Step 5: For any $\alpha, \beta \in \mathbb{R}$,

$$g(\alpha x + \beta y) = \frac{1}{2}\big(g(2\alpha x) + g(2\beta y)\big) = g(\alpha x) + g(\beta y) = \alpha g(x) + \beta g(y),$$

which implies g is linear and f is affine. ∎

Lemma 2.4 *Let X and Y be two real n-G-quasi normed spaces. Suppose that $f\colon X \to Y$ satisfies w-n-DOPP and f is affine.*

(i) *f preserves n-0-distance.*
(ii) *f preserves n-1-distance (n-DOPP).*
(iii) *f is an n-isometry.*

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Proof Set $g(x) = f(x) - f(0)$. Then $g(x)$ is linear.

(i) Suppose that $\|y_1 - x_1, \dots, y_n - x_n\| = 0$. Thus $\{y_1 - x_1, \dots, y_n - x_n\}$ are linearly dependent. There are $a_1, a_2, \dots, a_n$ which are not all zero such that

$$a_1(y_1 - x_1) + a_2(y_2 - x_2, \dots, +a_n(y_n - x_n) = 0.$$

Then $a_1(g(y_1) - g(x_1)) + a_2(g(y_2) - g(x_2)), \dots, +a_n g((y_n) - g(x_n)) = 0$. Clearly, we have $\|g(y_1) - g(x_1), \dots, g(y_n) - g(x_n)\| = 0$. We deduce

$$\|f(y_1) - f(x_1), \dots, f(y_n) - f(x_n)\| = 0.$$

(ii) Suppose that for $x_1, \dots, x_n, y_1, \dots, y_n \in X$, $\|y_1 - x_1, \dots, y_n - x_n\| = 1$ for any $x_0 \in X$. Set $z_i = x_0 + y_i - x_i$. Then $\|z_1 - x_0, \dots, z_n - x_0\| = 1$. Since f satisfies w-n-DOPP, we have $\|f(z_1) - f(x_0), \dots, f(z_n) - f(x_0)\| = 1$. Clearly,

$$\|g(z_1) - g(x_0), \dots, g(z_n) - g(x_0)\| = 1,$$

and since g is linear, this means that $\|g(y_1) - g(x_1), \dots, g(y_n) - g(x_n)\| = 1$. This implies $\|f(y_1) - f(x_1), \dots, f(y_n) - f(x_n)\| = 1$.

(iii) Suppose that for $x_1, \dots, x_n, y_1, \dots, y_n \in X$, $\|y_1 - x_1, \dots, y_n - x_n\| \neq 0$, and we set

$$y = x_1 + \frac{y_1 - x_1}{\|y_1 - x_1, \dots, y_n - x_n\|}. \tag{2.6}$$

This implies that $\|y - x_1, \dots, y_n - x_n\| = 1$ and $\|f(y) - f(x_1), \dots, f(y_n) - f(x_n)\| = 1$. Hence, it holds that

$$\|g(y) - g(x_1), g(y_2) - g(x_2), \dots, g(y_n) - g(x_n)\| = 1. \tag{2.7}$$

Since g is a linear, it follows from (2.6) and (2.7) that

$$\left\| \frac{g(y_1) - g(x_1)}{\|y_1 - x_1, \dots, y_n - x_n\|}, g(y_2) - g(x_2), \dots, g(y_n) - g(x_n) \right\| = 1.$$

This implies that

$$\left\| \frac{f(y_1) - f(x_1)}{\|y_1 - x_1, \dots, y_n - x_n\|}, f(y_2) - f(x_2), \dots, f(y_n) - f(x_n) \right\| = 1.$$

Hence, $\|f(y_1) - f(x_1), \dots, f(y_n) - f(x_n)\| = \|y_1 - x_1, \dots, y_n - x_n\|$, which shows that f is an n-isometry. ∎

Theorem 2.5 *Let X and Y be two real n-G-quasi normed spaces. Suppose that f satisfies w-n-DOPP. Then the following properties are equivalent for f:*

w-n-Lipschitz, n-collinear (w-n-0-distance), 2-collinear, affine,
n-isometry, n-Lipschitz, n-0-distance, w-n-isometry.

Proof $\{w\text{-}n\text{-DOPP and } w\text{-}n\text{-Lipschitz}\} \Rightarrow \{w\text{-}n\text{-DOPP and } n\text{-collinear}\} \Rightarrow \{w\text{-}n\text{-DOPP and 2-collinear}\} \Rightarrow \{w\text{-}n\text{-DOPP and affine}\} \Rightarrow \{n\text{-DOPP and } n\text{-0-distance}\} \Rightarrow$

$$n\text{-isometry} \Rightarrow \left\{ \begin{array}{c} n\text{-DOPP} + n\text{-Lipschitz} \\ \textit{or} \\ w\text{-}n\text{-isometry} \end{array} \right\} \Rightarrow \{w\text{-}n\text{-DOPP and } w\text{-}n\text{-Lipschitz}\}.$$ ∎

Corollary 2.6 *Let X and Y be two real n-G-quasi normed spaces. A mapping $f: X \to Y$ is a w-n-isometry if and only if f is an n-isometry.*

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Proof If f is a w-n-isometry, then f preserves w-n-DOPP. ∎

Remark 2.7 Let X and Y be two real n-G-quasi normed spaces. Suppose that f satisfies n-DOPP. Then the following properties are equivalent for f: w-n-Lipschitz, n-collinear (w-n-0-distance), 2-collinear, affine, n-isometry, n-Lipschitz, n-0-distance, w-n-isometry.

3 Main Results on the Benz Theorem in n-normed Spaces

An n-normed space is a special n-G-quasi normed space. We give the definitions and property of n-normed space for the reader's convenience.

Definition 3.1 ([4]) Let X be a real linear space with $\dim X \geq n$ and $\|\cdot,\dots,\cdot\|: X^n \to R$, a function. Then $(X, \|\cdot,\dots,\cdot\|)$ is called a linear n-normed space if for any $\alpha \in R$ and all $x, y, x_1, \dots, x_n \in X$

nN_1: $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent,
nN_2: $\|x_1,\dots,x_n\| = \|x_{j1},\dots,x_{jn}\|$ for every permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$,
nN_3: $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$,
nN_4: $\|x+y, x_2,\dots,x_n\| \leq \|x, x_2,\dots,x_n\| + \|y, x_2,\dots,x_n\|$ The function $\|\cdot,\dots,\cdot\|$ is called the n-norm on X.

Remark 3.2 ([4]) Let X and Y be real n-normed spaces. Then

$$\|x_1,\dots,x_i,\dots,x_j,\dots,x_n\| = \|x_1,\dots,x_i + x_j,\dots,x_j,\dots,x_n\|$$

for $x_1,\dots,x_i,\dots,x_j,\dots,x_n \in X$.

Definition 3.3 X is said to be an n-strictly convex normed space provided that for any $x_0, x_1, x_2, \dots, x_n \in X$, if $x_2,\dots,x_n \notin \operatorname{span}\{x_0, x_1\}$ and $\|x_0 + x_1, x_2,\dots,x_n\| = \|x_0 x_2,\dots,x_n\| + \|x_1 x_2,\dots,x_n\|$, then x_0 and x_1 are linearly dependent.

Remark 3.4 Gehér [7] gave an example of a non-strictly convex n-normed space.

Lemma 3.5 *Let X and Y be real n-normed spaces. If a mapping $f: X \to Y$ preserves the distance $\frac{\rho}{k}$ for each $k \in \mathbb{N}$, then f preserves the distance zero.*

Proof Choose $x_1,\dots,x_n, y_1,\dots,y_n \in X$ such that $\|x_1 - y_1,\dots,x_n - y_n\| = 0$, *i.e.*, $x_1 - y_1,\dots,x_n - y_n$ are linearly dependent. Assume that $\{x_{m+1} - y_{m+1},\dots,x_n - y_n\}$ is a maximum linearly independent group of $\{x_1 - y_1,\dots,x_n - y_n\}$ $(m < n)$. As $\dim X \geq n$, we can find a finite sequence of vectors $\omega_1, \omega_2, \dots, \omega_m \in X$ such that $x_1 - \omega_1,\dots,x_m - \omega_m, x_{m+1} - y_{m+1},\dots,x_n - y_n$ are linearly independent. Hence, it holds that $\|x_1 - \omega_1,\dots,x_m - \omega_m, x_{m+1} - y_{m+1},\dots,x_n - y_n\| \neq 0$.

We will prove that $\|f(x_1) - f(y_1), f(x_2) - f(y_2),\dots,f(x_n) - f(y_n)\| \leq \frac{\rho}{k}$ for every $k \in \mathbb{N}$. Let $m = 1$. We can find a vector $\omega_1 \in X$ such that $x_1 - \omega_1, x_2 - y_2,\dots,x_n - y_n$ are linearly independent. Set

$$v_1 = x_1 + \frac{(x_1 - \omega_1)\rho}{2k\|x_1 - \omega_1, x_2 - y_2,\dots,x_n - y_n\|}$$

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for arbitrarily fixed $k \in \mathbb{N}$. Then $\|x_1 - v_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{\rho}{2k}$ and

$$\begin{aligned} &\|v_1 - x_1, x_2 - y_2, \dots, x_n - y_n\| - \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\le \|(v_1 - x_1) + (x_1 - y_1), x_2 - y_2, \dots, x_n - y_n\| \\ &\le \|v_1 - x_1, x_2 - y_2, \dots, x_n - y_n\| + \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\|. \end{aligned}$$

Since $\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = 0$, we get $\|v_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{\rho}{2k}$. Since f preserves the distance $\rho/(2k)$, we see that

$$\begin{aligned} &\|f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\quad \le \|f(x_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\qquad + \|f(v_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\quad = \frac{\rho}{2k} \cdot 2 = \frac{\rho}{k}. \end{aligned}$$

For $m \ge 2$, we set

$$v_1 = x_1 + \frac{(x_1 - \omega_1)\rho}{2^m k \|x_1 - \omega_1, \dots, x_m - \omega_m, x_{m+1} - y_{m+1}, \dots, x_n - y_n\|} \tag{3.1}$$

and $v_i = 2x_i - \omega_i$ for any $i \in \{2, 3, \dots, m\}$. Then we have

$$x_i - v_i = \omega_i - x_i \quad \text{and} \quad v_i - y_i = (x_i - \omega_i) + (x_i - y_i)$$

for each $i \in \{2, 3, \dots, m\}$. Since $x_i - y_i, x_{m+1} - y_{m+1}, \dots, x_n - y_n$ are linearly dependent, we get

$$\|\dots, x_i - y_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| = 0, \tag{3.2}$$

and hence

$$\begin{aligned} &\|\dots, x_i - \omega_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \\ &\qquad - \|\dots, x_i - y_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \\ &\le \|\dots, (x_i - \omega_i) + (x_i - y_i), \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \\ &\le \|\dots, x_i - \omega_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \\ &\qquad + \|\dots, x_i - y_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\|, \end{aligned}$$

which together with (3.2) imply that

$$\begin{aligned} &\|\dots, v_i - y_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \\ &\qquad = \|\dots, x_i - \omega_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\|, \end{aligned} \tag{3.3}$$

for all $i \in \{2, 3, \dots, m\}$. By a similar argument, we further obtain

$$\begin{aligned} &\|v_1 - y_1, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \\ &\qquad = \|v_1 - x_1, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\|. \end{aligned} \tag{3.4}$$

In view of (3.1), (3.3), and (3.4), we conclude that

$$\begin{aligned} &\|v_1 - y_1, \mu_2, \dots, \mu_m, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \\ &\qquad = \|x_1 - v_1, x_2 - \omega_2, \dots, x_m - \omega_m, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| = \frac{\rho}{2^m k}, \end{aligned} \tag{3.5}$$

where μ_i denotes either $v_i - y_i$ or $x_i - v_i$ for $i \in \{2, 3, \dots, m\}$.

Since f preserves the distance $\rho/(2^m k)$ for any $k \in \mathbb{N}$, it follows from (3.5) that

$$\begin{aligned}
&\|f(x_1) - f(y_1), f(x_2) - f(y_2), f(x_3) - f(y_3), \dots, f(x_n) - f(y_n)\| \\
&\le \big\| f(x_1) - f(v_1), f(x_2) - f(v_2), \dots, f(x_{m-1}) - f(v_{m-1}), f(x_m) - f(v_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \dots, f(x_n) - f(y_n)\big\| \\
&\quad + \big\| f(x_1) - f(v_1), f(x_2) - f(v_2), \dots, f(x_{m-1}) - f(v_{m-1}), f(v_m) - f(y_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \dots, f(x_n) - f(y_n)\big\| \\
&\quad + \big\| f(x_1) - f(v_1), f(x_2) - f(v_2), \dots, f(v_{m-1}) - f(y_{m-1}), f(x_m) - f(v_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \dots, f(x_n) - f(y_n)\big\| \\
&\quad + \big\| f(x_1) - f(v_1), f(x_2) - f(v_2), \dots, f(v_{m-1}) - f(y_{m-1}), f(v_m) - f(y_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \dots, f(x_n) - f(y_n)\big\| \\
&\quad \cdots \\
&\quad + \big\| f(v_1) - f(y_1), f(v_2) - f(y_2), \dots, f(v_{m-1}) - f(y_{m-1}), f(v_m) - f(y_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \dots, f(x_n) - f(y_n)\big\| \\
&= \frac{\rho}{2^m k} \cdot 2^m = \frac{\rho}{k},
\end{aligned}$$

where k is an arbitrary positive integer. Hence, we conclude that

$$\|f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| = 0,$$

which implies that f preserves the distance zero. ■

Corollary 3.6 *Let X and Y be two real n-normed spaces. Suppose that f preserves $\frac{\rho}{k}$ distance for any $k \in N$ if and and only if f is an n-isometry.*

Proof This is obvious by Lemma 3.5 and Theorem 1.1. ■

Lemma 3.7 *Let X and Y be two real n-normed spaces. Suppose that f preserves ρ distance and $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ for any $x, y \in X$. Then f is an affine n–isometry.*

Proof Since $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ for any $x, y \in X$, we have $f(x+m(y-x)) = f(x) + m(f(y) - f(x))$ for all $m \in N$. Indeed, it is clear for $m = 0$ and $m = 1$. Suppose that it is also true for any $m \ge 1$. Set $p_m = x + m(y - x)$. Then $f(p_{m+1}) = 2f(p_m) - f(p_{m-1})$, which implies that

$$\begin{aligned}
f\big(x + (m+1)(y-x)\big) &= 2f\big(x + m(y-x)\big) - f\big(x + (m-1)(y-x)\big) \\
&= f(x) + (m+1)\big(f(y) - f(x)\big).
\end{aligned}$$

Let $g(x) = f(x) - f(0)$. Clearly, for any $x \in X$ and all rational numbers r, p, we have

$$g(rx) = rg(x), \quad g(rx + py) = rg(x) + pg(y). \tag{3.6}$$

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Next we show that g preserves $\frac{\rho}{m}$-n-distance for any integer $m \in N$. Suppose that $\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{\rho}{m}$ for integer m. Then

$$\|m(x_1 - y_1), x_2 - y_2, \dots, x_n - y_n\| = \rho$$

and by (3.6), we have

$$\|g(mx_1) - g(my_1)), g(x_2) - g(y_2), \dots, g(x_n) - g(y_n)\| = \rho.$$

Thus $\|g(x_1) - g(y_1), g(x_2) - g(y_2), \dots, g(x_n) - g(y_n)\| = \frac{\rho}{m}$. Thus f is an affine n-isometry from Corollary 3.6. ∎

Lemma 3.8 *Let X and Y be real n-normed spaces such that* $\dim X \geq n$ *and Y is n-strictly convex. If a mapping $f: X \to Y$ preserves ρ and 2ρ for some $\rho > 0$, then f is an affine n-isometry.*

Proof Since f preserves any integer ρ and 2ρ for some $\rho > 0$, let

$$\|y - x, y_2 - x \dots, y_n - x\| = 2\rho.$$

Set $p_i = x + \frac{i}{2}(y - x)$, for $i = 0, 1, 2$. Then $\|p_2 - x, y_2 - x, \dots, y_n - x\| = 2\rho$ and $\|p_i - p_{i-1}, y_2 - x, \dots, y_n - x\| = \rho$ for $i = 1, 2$, so we have

$$\|f(p_i) - f(p_{i-1}), f(y_2) - f(x), \dots, f(y_n) - f(x)\| = \rho,$$
$$\|f(p_2) - f(x), f(y_2) - f(x), \dots, f(y_n) - f(x)\| = 2\rho,$$

and

$$\begin{aligned}\|f(p_2) - f(p_0), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \\ = \|f(p_2) - f(p_1), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \\ + \|f(p_1) - f(p_0), f(y_2) - f(x), \dots, f(y_n) - f(x)\|.\end{aligned}$$

Since Y is n-strictly convex, then there exists t with $f(p_2) - f(p_1) = t(f(p_1) - f(p_0))$ and f is an injective mapping, so $t = 1$. We show that $f(y) - f(\frac{x+y}{2}) = f(\frac{x+y}{2}) - f(x)$. Thus we have $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ and f is an affine n-isometry from Lemma 3.7. ∎

We generalized Benz's theorem to n-normed spaces under the condition of $\dim X > n$ [13, Theorem 11]. In the following we have the same result without the condition. The proofs of (a)-(e) in [[13], Theorem 11] do not involve the dimension, so they are viable; however (f) in [[13] Theorem 11] is inviable here. For the convenience of the reader, some proofs in [[13], Theorem 11] are repeated.

Theorem 3.9 *Let X and Y be two real linear n-normed spaces and let Y be n-strictly convex. If a mapping $f: X \to Y$ is a function satisfying the conditions that*

(i) $\|y_1 - x_1, \dots, y_n - x_n\| = \rho$ *implies* $\|f(y_1) - f(x_1), \dots, f(y_n) - f(x_n)\| \leq \rho$,
(ii) $\|y_1 - x_1, \dots, y_n - x_n\| = N\rho$ *implies* $\|f(y_1) - f(x_1), \dots, f(y_n) - f(x_n)\| \geq N\rho$, *for any $x_1, \dots, x_n, y_1, \dots, y_n \in X$, some $\rho > 0$ and a integer N,*

then f is an affine n-isometry.

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Proof (a) We first prove that f preserves the w-ρ-distance. Let

$$\|y_1 - x_0, \dots, y_n - x_0\| = \rho.$$

Set $p_i = x_0 + i(y_1 - x_0)$, $i = 0, 1, \dots, N$. Clearly, we have $p_1 = y_1$, $p_0 = x_0$, $p_i - x_0 = i(y_1 - x_0)$, $p_i - p_{i-1} = y_1 - x_0 = p_1 - x_0$, and

$$\|p_i - p_{i-1}, y_2 - x_0, \dots, y_n - x_0\| = \|y_1 - x_0, \dots, y_n - x_0\| = \rho.$$

By (i), $\|f(p_i) - f(p_{i-1}), f(y_2) - f(x_0), \dots, f(y_n) - f(x_0)\| \le \rho$. By (ii),

$$\begin{aligned} N\rho &\le \|f(p_N) - f(x_0), \dots, f(y_n) - f(x_0)\| \\ &\le \sum_1^N \|f(p_i) - f(p_{i-1}), f(y_2) - f(x_0), \dots, f(y_n) - f(x_0)\| \le N\rho. \end{aligned}$$

Thus $\|f(p_i) - f(p_{i-1}), f(y_2) - f(x_0), \dots, f(y_n) - f(x_0)\| = \rho$. This implies

$$\|f(y_1) - f(x_0), f(y_2) - f(x_0), \dots, f(y_n) - f(x_0)\| = \rho.$$

(b) We then prove that f preserves w-$N\rho$. Let $\|y_1 - x_0, \dots, y_n - x_0\| = N\rho$. Set $p_i = x_0 + \frac{i}{N}(y_1 - x_0)$, $i = 0, 1, \dots, N$. Clearly, we have $p_N = y_1$, $p_0 = x_0$, $p_i - x_0 = \frac{i}{N}(y_1 - x_0)$, $p_i - p_{i-1} = \frac{1}{N}(y_1 - x_0) = \frac{1}{i}(p_i - x_0)$,

$$\|p_i - p_{i-1}, y_2 - x_0, \dots, y_n - x_0\| = \|y_1 - x_0, y_2 - x_0, \dots, y_n - x_0\| = \rho,$$

and $\|p_N - x_0, \dots, y_n - x_0\| = N\rho$. By (i),

$$\|f(p_i) - f(p_{i-1}), f(y_2) - f(x_0), \dots, f(y_n) - f(x_0)\| \le \rho$$

By (ii),

$$\begin{aligned} N\rho &\le \|f(p_N) - f(x_0), \dots, f(y_n) - f(x_0)\| \\ &\le \sum_1^N \|f(p_i) - f(p_{i-1}), f(x_2) - f(x_0), \dots, f(x_n) - f(x_0)\| \le N\rho. \end{aligned}$$

This implies $\|f(y_1) - f(x_0), f(y_2) - f(x_0), \dots, f(y_n) - f(x_0)\| = N\rho$.

(c) Finally we prove that f preserves w-2ρ. Next, we discuss (a) and (b) for $N \ge 3$. Let $\|y - x, y_2 - x \dots, y_n - x\| = 2\rho$. Set $p_i = x + \frac{i}{2}(y - x)$, for $i = 0, 1, 2, \dots, N$. Then $\|p_N - x, y_2 - x, \dots, y_n - x\| = N\rho$ and $\|p_i - p_{i-1}, y_2 - x, \dots, y_n - x\| = \rho$ for $i = 1, 2, \dots, N$. And by (a) and (b) for $i = 1, 2, \dots, N$, we have

$$\begin{aligned} &\|f(p_i) - f(p_{i-1}), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \le \rho, \\ &\|f(p_N) - f(x), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \ge N\rho, \end{aligned}$$

and

$$(*) \qquad \begin{aligned} N\rho &\le \|f(p_N) - f(x), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \\ &\le \sum_{i=1}^N \|f(p_{N+1-i}) - f(p_{N-i}), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \le N\rho. \end{aligned}$$

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We obtain

$$\begin{aligned}\|f(p_2) - f(p_0), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \\ = \|f(p_2) - f(p_1), f(y_2) - f(x), \dots, f(y_n) - f(x)\| \\ + \|f(p_1) - f(p_0), f(y_2) - f(x), \dots, f(y_n) - f(x)\| = 2\rho.\end{aligned}$$

Thus, $\|f(y) - f(x), f(y_2) - f(x), \dots, f(y_n) - f(x)\| = 2\rho$.

(d) Since Y is n-strictly convex, there exists t with

$$f(p_2) - f(p_1) = t(f(p_1) - f(p_0)),$$

because $\|p_i - p_{i-1}, y_2 - x, \dots, y_n - x\| = \rho$ for $i = 1, 2, \dots, N$. Thus ($*$) implies that

$$\|f(p_i) - f(p_{i-1}), f(y_2) - f(x), \dots, f(y_n) - f(x)\| = \rho$$

and f is an injective mapping, so $t = 1$. We show that

$$f(y) - f\Big(\frac{x+y}{2}\Big) = f\Big(\frac{x+y}{2}\Big) - f(x).$$

Thus we have $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$, which implies f is an affine n-isometry from Lemma 3.7. ∎

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