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# BOOK REVIEWS

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ARVESON, W. *Noncommutative dynamics and E-semigroups* (Springer, 2003), 0 387 00151 4 (hardback), £67.

The author of this book is one of the great names in functional analysis (of the operator-theoretic or operator-algebraic variety). His work is always characterized by the brilliant opening up of important new areas, the uncovering of deep and crucial examples and connections, the seemingly magical introduction of wonderful new tools, the bringing to bear of important ideas from other fields of mathematics, and a vast technical strength. Arveson's work has so often been labelled as 'seminal' that I am sure he must hate the term. However, there is good reason for the adjective: anybody studying any part of his opus cannot fail to be struck by his awesome powers of creation. This is certainly abundantly clear in the book under review.

The '$E$' in '$E$-semigroup' stands for 'endomorphism'; $E$-semigroups are one-parameter semigroups $\{\alpha_t : t \geqslant 0\}$ of $*$-endomorphisms of a von Neumann algebra. They were introduced in papers of R. T. Powers in the second half of the 1980s (see, for example, [**4**]), and have been studied intensively since then, in large part by our author, Alevras, Bhat, Powers and Price, and in recent years by people like Muhly and Solel, Tsirelson, and a new generation of strong researchers. They have applications in many directions: for example, they arise from very basic considerations in the noncommutative dynamics of quantum systems, and in noncommutative probability are associated with 'white noise'. Also, as Powers often puts it, the study of $E_0$-semigroups is really the study of the differential operator $\mathrm{d}/\mathrm{d}t$ in a $C^*$-algebraic framework (note that $\mathrm{d}/\mathrm{d}t$ is the generator of the semigroup of isometries given by the shift on $L^2(0,\infty)$, copies of which are isomorphic to the pure shift part of the Wold decomposition of an arbitrary semigroup of isometries). Arveson's interest in the subject began within a year or two of its inception, and he has devoted most of his astonishing energies to their study since then. Thus the text under review contains the fruits of over 15 profound and prolific years of labour. It also represents the state of the art of a large part of this exciting and evolving research area, and indeed it is the only book on the subject to date, except for the proceedings volume for a conference on the topic of quantum dynamics held recently at Mount Holyoke College [**5**]. The reader might refer to the latter volume for a sample of some other directions that the field is taking.

The main focus of the text is the problem of classifying $E_0$-semigroups of endomorphisms of $B(H)$ for a Hilbert space $H$. An $E_0$-semigroup is an $E$-semigroup $\alpha = \{\alpha_t : t \geqslant 0\}$ with $\alpha_t(1) = 1$ for all $t$; the map $t \mapsto \alpha_t(A)$ is assumed to be continuous with respect to the weak operator topology for each $A \in B(H)$. Already this setting presents formidable technical challenges. One usually thinks of two $E_0$-semigroups $\alpha$ and $\beta$ as being 'the same' if they are *cocycle conjugate* in the sense of Connes, and hence we wish to classify $E_0$-semigroups up to cocycle conjugacy. The definition of $\alpha$ and $\beta$ being cocycle conjugate is that there exists a unitary $V$, and a strongly continuous semigroup $\{U_t : t \geqslant 0\}$ of unitary operators satisfying the cocycle equation $U_{s+t} = U_s\alpha_t(U_t)$, such that $\beta_t = VU_t\alpha_t(\cdot)U_t^*V^*$. This is a subtle equivalence relation;

some of the big surprises in the theory occur when two apparently unrelated semigroups turn out to be cocycle conjugate.

The 'symmetries' in basic quantum theory correspond to automorphisms of $B(H)$; thus a 'time flow' of symmetries is a one-parameter group of $*$-isomorphisms of $B(H)$. Mathematically such structures are easily classified and understood. One-parameter semigroups of $*$-endomorphisms (we are now no longer assuming surjectivity) are much harder to understand. They also arise naturally in quantum field theory, for example when one has a certain kind of *causal structure* around. In this case the 'past' and the 'future' pieces of the appropriate compressions of a flow of symmetries give interesting examples of $E_0$-semigroups. However, the basic examples of an $E_0$-semigroup are the *CAR/CCR flows*. These can be seen to arise very naturally in either bosonic or fermionic second quantization. For example, let $d \in 1, 2, \dots, \aleph_0$, let $H$ be the Hilbert space direct sum of $d$ copies of $L^2([0,\infty))$, and let $S = \{S_t : t \geqslant 0\}$ be the associated shift semigroup. Namely, $S_t$ takes $(\xi_n) \in H$ to the $n$-tuple whose entries are the functions that are zero on $[0, t)$, and which are $\xi_n(\cdot - t)$ on $[t, \infty)$. One then takes a Hilbert space $K$, and a suitable function $\xi \in H \mapsto W_\xi \in B(K)$ satisfying the Weyl commutation relations $W_\xi W_\eta = e^{i \operatorname{Im}(\xi,\eta)} W_{\xi+\eta}$. The linear combinations of the $W_\xi$ are dense in $B(K)$, and hence the prescription

$$\alpha_t(W_\xi) = W_{S_t\xi}, \quad \xi \in H,\ t \geqslant 0,$$

defines a semigroup of endomorphisms of $B(K)$ well. This is the CAR/CCR flow of index $d$.

The $E_0$-semigroups divide into three main subclasses: types I, II and III. The type I $E_0$-semigroups may be roughly described as the ones that possess sufficiently many 'units', where the latter term refers to a strongly continuous semigroup $\{T_t : t \geqslant 0\}$ of bounded operators that intertwines $\alpha$ and the identity (that is, $\alpha_t(A)T_t = T_tA$ for all $A \in B(H)$ and $t \geqslant 0$). Type I semigroups have a beautiful theory, and are completely classified by their 'numerical index', up to cocycle conjugacy. Remarkably, the type I $E_0$-semigroups turn out to be precisely those that are cocycle conjugate to the 'CAR/CCR flow' of index $d$. The other types are not currently very well understood. An $E_0$-semigroup $\alpha$ is called 'type III' if it possesses no units at all. Several deep and interesting examples of these semigroups, some of them rather recent, are discussed in detail in the final part of the book. The remaining $E_0$-semigroups are called type II. Powers and others have developed a 'standard form' for type II semigroups; however, this is not treated in the text under review. Indeed, the theory of $E_0$-semigroups is a rapidly evolving research area, and we can expect many exciting developments in the future.

We summarize briefly the contents of the book. It begins with a preface and an introductory chapter that describes some of the basic ideas, motivations and applications of the subject to certain of the noncommutative dynamical systems that arise naturally in mathematical physics and noncommutative probability. Part 1 of the text contains some basic theory of $E_0$-semigroups. $E_0$-semigroups are in a natural correspondence with more abstract objects called *product systems*, which may be thought of, loosely, as 'continuous tensor products' of Hilbert spaces, or as giving a 'continuous analogue of Fock space' (the title of a massive series of papers that the author wrote around 1990 (see, for example, [**2**, **3**])). Two $E_0$-semigroups are cocycle conjugate if and only if the canonically associated product systems are isomorphic. Thus the analysis of $E_0$-semigroups may be replaced by an analysis of product systems. The integrable sections of product system $E$ form a Banach algebra, which naturally gives rise to the *spectral $C^*$-algebra* $C^*(E)$, whose representation theory is in correspondence with that of $E$. In fact, the $C^*$-algebra $C^*(E)$ has many nice properties that make it quite amenable to analysis, and which are developed in Chapter 4. The simplest spectral $C^*$-algebras are the $C^*$-algebras $\mathcal{W}_n$ coming from sequences $\{U_n\}$ of semigroups of isometries satisfying $U_k(t)^*U_j(t) = e^{-t}1$ for all $t \geqslant 0$ and $k \neq j$. They are thus clearly 'continuous-time' analogues of the Cuntz $C^*$-algebras generated by $n$ isometries. Remarkably, it has recently been shown via the Kirchberg–Phillips classification results that the $\mathcal{W}_n$, unlike the Cuntz algebras, are all isomorphic! (See, for example, the survey article by Hirshberg and Zacharias in [**5**], and references therein.)

The main theme of Part 2 is the classification of type I $E_0$-semigroups. This is accomplished via the product systems mentioned above, and is a technical tour de force. Part 3 widens the scope of the semigroups considered by replacing the endomorphisms with completely positive (CP) maps. Of course Arveson was one of the originators of the theory of completely positive maps [**1**]. Some physical motivation of the importance of 'CP-semigroups' is nicely presented at the beginning of Part 3, via a discussion of the heat semigroup associated with a Riemannian manifold; the generators of CP-semigroups in many ways act like the Laplacian operator. However, CP-semigroups are clearly interesting in their own right, and it is nice that the author has devoted so much space here to their theory. A main point is that every CP-semigroup may be 'dilated' to an $E_0$-semigroup that has nice properties. Part 4 is titled 'Causality and dynamics'. It discusses 'eigenvalue lists', namely sequences of eigenvalues of positive trace class operators associated with the causal structure we mentioned earlier, and uses these to produce nontrivial 'past' and 'future' $E_0$-semigroups. Part 5, the final part, discusses several fascinating examples, including Powers's examples based on quasi-free states, and the probabilistic examples of Tsirelson, related to 'off-white noise'.

The book has clearly been written with the goal of teaching the subject; the exposition is crisp, clear, and will be accessible to graduate students. Nonetheless, the author's sophistication, profundity, and broad culture sparkles throughout the 434 pages. The book is handsomely bound in the bright yellow Springer Monographs in Mathematics series. It has a fine index, and the reader will greatly appreciate the trouble the author takes to describe what is going on, relate how a result or formula ought to be viewed, and so on. There are many new results, and significant reformulations and simplifications of the theory. It is superbly written for the most part; there are a few minor imperfections, such as the misspellings that occur from time to time throughout the book. In addition to being the essential reference for those working on $E_0$-semigroups, this magnificent book will be useful and inspirational to a wide range of mathematicians and mathematical physicists.

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Duren, P. *Harmonic mappings in the plane* (Cambridge University Press, 2004), 0 521 64121 7 (hardback), £40.

Harmonic mappings are plane bijective maps of the form

$$w = u + \mathrm{i}v = f(x, y), \tag{1}$$

where $u$ and $v$ are harmonic (but in general not conjugate) functions of $(x, y)$. Let us quote from the introduction to the book (p. xi).