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AN EXTENSION OF THE ESSENTIAL SUPREMUM CONCEPT WITH APPLICATIONS TO NORMAL INTEGRANDS AND MULTIFUNCTIONS

E.J. BALDER

Let $(T, \mathcal{T}, \mu)$ be a σ-finite measure space and X a Suslin space. Let A be a class of normal integrands on $T \times X$. We discuss the existence of an essential supremum of A, namely, a normal integrand l with

$$l = \sup\{a : a \in A_0\} ,$$

where A_0 is a countable subclass of A, and, for each $a \in A$,

$$a(t, \cdot) \le l(t, \cdot) \quad \text{for almost every } t .$$

In this way we obtain an extension of the classical essential supremum concept. The applications include a result on measurable selectors of nonmeasurable multifunctions.

1. Countable reflection and generalized conjugation

Let X be a topological space and X' a given abstract space. Let $c : X \times X' \to (-\infty, +\infty)$ be a given functional; it will be referred to as the *coupling functional* [2], [7], [13].

Given a functional $f : X \to [-\infty, +\infty]$, we define its *c-conjugate* $f^c : X' \to [-\infty, +\infty]$ and *c-biconjugate* $f^{cc} : X \to [-\infty, +\infty]$ as follows [2], [7], [13]:

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(1.1) $$f^c(x') \equiv \sup_x \,[c(x, x')-f(x)] \ , \quad x' \in X' \ ;$$

(1.2) $$f^{cc}(x) \equiv \sup_{x'} \,[c(x, x')-f^c(x')] \ , \quad x \in X \ .$$

It follows trivially from (1.1)-(1.2) that, for each functional $f : X \to [-\infty, +\infty]$,

(1.3) $$f \geq f^{cc} \ .$$

A functional $f : X \to [-\infty, +\infty]$ is said to be *c-reflexive* if

$$f = f^{cc} \ .$$

We denote the class of c-reflexive functionals on X by $\Gamma^c(X)$; it plays an important role in optimization theory.

Let us say that a functional $f : X \to (-\infty, +\infty)$ is a (*finite*) *elementary c-functional* if there exist $x' \in X'$, $\eta \in \mathbb{R}$, such that

$$f = c(\cdot, x') + \eta \ .$$

Let us also denote the functional identically equal to $-\infty$ $(+\infty)$ by ω_1 (ω_2) . The following characterization of $\Gamma^c(X)$ is elementary [7, Satz 3.6].

LEMMA 1.1. *A functional* $f : X \to [-\infty, \infty]$ *belongs to* $\Gamma^c(X)$ *if and only if it is the supremum of a collection of elementary c-functionals.*

Proof. Sufficiency follows from (1.2). (Note that if $f^c(\bar{x}') = +\infty$ for some $\bar{x}'$, then $\bar{x}'$ can be omitted from (1.2); note also that if $f(\bar{x}) = -\infty$ for some $\bar{x}$, then $f = \omega_1$, which is the supremum over the empty collection.)

As for necessity, suppose $f = \sup_{\alpha\in A} \left[c\left(\cdot, x'_\alpha\right)+\eta_\alpha\right]$ for some index set A . It follows for each $\alpha \in A$ that $-\eta_\alpha \geq f^c\left(x'_\alpha\right)$, by applying (1.1). Hence $f \leq \sup_{\alpha\in A} \left[c\left(\cdot, x'_\alpha\right)-f^c\left(x'_\alpha\right)\right] \leq f^{cc}$ by applying (1.2). It must now be true that $f \in \Gamma^c(X)$, in view of (1.3).

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It follows from the above lemma that a necessary condition for $f : X \to [-\infty, +\infty]$, $f \neq \omega_1$, to belong to $\Gamma^c(X)$ is that f be *c-tempered* [2], that is, there exist $x' \in X'$, $\eta \in \mathbb{R}$, such that

$$f \geq c(\cdot, x') + \eta .$$

Let Γ_0 be a given class of functionals from X into $[-\infty, +\infty]$. We define Γ_0 to be *countably reflected* if X' contains a countable (said more precisely, at most countable) subset $\{x_i'\}$ such that, for each $f \in \Gamma_0$,

$$f = \sup_i \left[c\left(\cdot, x_i'\right)-f^c\left(x_i'\right)\right] . \tag{1.4}$$

Note that this implies automatically that Γ_0 is contained in $\Gamma^c(X)$ (Lemma 1.1).

LEMMA 1.2. *The class* $\Gamma_0 \subset [-\infty, +\infty]^X$ *is countably reflected if and only if there exists a countable collection of elementary c-functionals on* X *such that each* f *in* Γ_0 *is the supremum of a subset of this collection.*

Proof. First we prove sufficiency. Let $\{x_i'\} \subset X'$ be as in (1.4). Let $\{r_j\}$ be an enumeration of the rational numbers. Take an arbitrary $f \in \Gamma_0$. Define $M \subset \mathbb{N} \times \mathbb{N}$ to be the set of (i, j) such that $c\left(\cdot, x_j'\right) + r_j \leq f$. Then $(i, j) \in M$ if and only if $r_j \leq -f^c\left(x_j'\right)$. For each $i \in \mathbb{N}$ there exists a subsequence of $\{r_j\}$ that converges to $-f^c\left(x_i'\right)$ from below. We conclude from this and (1.4) that $\sup_{(i,j)\in M} \left[c\left(\cdot, x_i'\right)+r_j\right] = f$.

Necessity is demonstrated as follows. Denote the countable collection of elementary c-functionals figuring in the statement by $\left\{c\left(\cdot, x_i'\right)+\eta_i\right\}$. Take an arbitrary $f \in \Gamma_0$. By supposition there exists $I \subset \mathbb{N}$ such that

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$f = \sup_{i \in I} [c(\cdot, x_i') + \eta_i]$. Clearly $-\eta_i \geq f^c(x_i')$ for each $i \in I$. Hence

$$f = \sup_{i \in I} [c(\cdot, x_i') + \eta_i] \leq \sup_{i \in I} \left[c(\cdot, x_i') - f^c(x_i')\right]$$

$$\leq \sup_i \left[c(\cdot, x_i') - f^c(x_i')\right] \leq f^{cc} ,$$

so by (1.3) we have $f = \sup_i \left[c(\cdot, x_i') - f^c(x_i')\right]$. The potential usefulness of the above concepts is illustrated by the following examples.

EXAMPLE A.1. X is a metric space, its metric being denoted by d , X' is the product set $\mathbb{R} \times X$ and c is defined by setting, for $x \in X$, $x' = (r, x'') \in X'$,

$$c(x, x') \equiv -rd(x, x'') . \tag{1.5}$$

It is well known that in this case the class $\Gamma^c(X)$ consists of the c-tempered lower semicontinuous functionals on X plus the functional ω_1 (*cf.* [8, IX.42]). Although an *ad hoc* proof can easily be given, using Lemma 1.1, we observe instead that the result follows immediately from [2, Theorem 1, Lemma 1] (*cf.* [6, Theorem 4.2] for a somewhat weaker version of this result . The results of [2] apply, since c is a coupling functional of needle type at every $x \in X$ [2], [12], that is, for each $\bar{x}' \in X$, $\bar{\eta} \in \mathbb{R}$, $\bar{\delta}, \varepsilon > 0$, there exist $x' \in X'$, $\delta \leq \bar{\delta}$, such that, for every $y \in X$,

$$c(y, x') \leq c(y, \bar{x}') + \bar{\eta} \quad \text{if} \quad d(x, y) \geq \delta ,$$

$$c(y, x') \leq \varepsilon \quad \text{if} \quad d(x, y) < \delta .$$

EXAMPLE A.2. As Example A.1, but with X a separable metric space (metric d). In this case the class $\Gamma_1 \equiv \Gamma^c(X)$ is countably reflected. Using Lemma 1.2, an *ad hoc* proof could easily be given. Rather than to do so, we observe the following [3]. Let $\{x_i\}$ be a countable dense subset of X , define $X_0' \equiv \{(n, x_i) : n, i \in \mathbb{N}\}$ and verify that the restriction c_0 of c to $X \times X_0'$ is still of needle type (c is as in (1.5)). Note also that by the fact that $\{x_i\}$ is dense in X and the triangle

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inequality, c-temperedness and c_0-temperedness are equivalent for functionals in $[-\infty, +\infty]^X$. It now follows by [2, Theorem 1] that, for each $f \in \Gamma_1$,

$$f = \sup_{x' \in X'_0} \left[c_0(\cdot, x') - f^{c_0}(x')\right] = \sup_{x' \in X'_0} \left[c(\cdot, x') - f^{c}(x')\right] .$$

EXAMPLE B.1. X is a Hausdorff locally convex space, X' is the set of all linear continuous functionals on X, c is the usual duality between X and X'. In this classical case $\Gamma^c(X)$ is the collection of all proper convex lower semicontinuous functionals on X plus the functionals ω_1, ω_2 [11, 6.3.4].

EXAMPLE B.2. As Example B.1, but with X a Suslin locally convex space. Consider the subclass Γ_2 of $\Gamma^c(X)$ consisting of the proper convex lower semicontinuous functionals on X whose epigraph is locally compact and contains no straight line, plus the functionals ω_1, ω_2. In this case the class Γ_2 is countably reflected, as is well known (*cf.* [10]). To see this, note that there exists a countable subset of X' which is dense in the Mackey topology $\tau(X', X)$ [5, III.32]. By [5, I.14] the assumptions on the epigraph of each $f \in \Gamma_2$ imply that f^c is finite and continuous at some point in X'. Hence (1.4) follows from (1.2) by applying [5, III.33].

2. Essential suprema of sets of integrands

Let $(T, \mathcal{T}, \mu)$ be a σ-finite measure space and X a Suslin space. A functional $l : T \times X \to [-\infty, +\infty]$ is called *integrand* (on $T \times X$); it is said to be *measurable* in case it is $\big(\mathcal{T} \otimes B(X)\big)$-measurable where $B(X)$ denotes the Borel subsets of X. Suppose Γ denotes a class in $[-\infty, +\infty]^X$ (for instance, the set of lower semicontinuous functionals on X). An integrand l on $T \times X$ is said to be a Γ-*integrand* if the functional $l(t, \cdot)$ on X belongs to Γ for each $t \in T$. In case the class Γ can be verbalized, our terminology for Γ-integrands will be verbalized similarly. For instance, we speak of lower semicontinuous integrands, and so on.

Let X', c have the same meaning as before, but suppose in addition that for each $x' \in X'$, $c(\cdot, x')$ is $B(X)$-measurable. Let Γ_0 denote a fixed subclass of $\Gamma^c(X)$ and let A be a collection of measurable Γ_0-integrands on $T \times X$. An integrand l on $T \times X$ is said to be the *essential supremum* of A if there exists a countable subcollection A_0 of A such that

$$l = \sup\{a : a \in A_0\} \tag{2.1}$$

and, for each $a \in A$,

$$a(t, \cdot) \leq l(t, \cdot) \text{ for almost every } t . \tag{2.2}$$

We shall investigate the existence of such essential suprema.

REMARK 2.1. Suppose l_1, l_2 are essential suprema of A. It follows then from (2.1)-(2.2) that l_1, l_2 are essentially equal, that is,

$$l_1(t, \cdot) = l_2(t, \cdot) \text{ for almost every } t .$$

This allows us to speak about "the" essential supremum of A, although it is more accurate to call l_1, l_2 *versions* of the essential supremum of A.

THEOREM 2.2. *Suppose Γ_0 is a countably reflected subclass of $\Gamma^c(X)$ and suppose A is a collection of measurable Γ_0-integrands on $T \times X$. Then the essential supremum of A exists.*

Proof. Let $\overline{T}$ denote the μ-completion of T. By our suppositions the functional $(t, x) \to c(x, x') - a(t, x)$ is $(T \otimes B(X))$-measurable for each $a \in A$, $x' \in X'$. Hence it follows from [5, III.39] and the fact that X is Suslin, that for each $a \in A$, $x' \in X'$, the functional $a^c(\cdot, x')$ is $\overline{T}$-measurable (conjugation takes place with respect to the second variable only). Let $\{x_i'\}$ be as in the definition of countable reflection. For each $i \in \mathbb{N}$ there exists a countable subset A_i of A such that $\inf\left\{a^c\left(\cdot, x_i'\right) : a \in A_i\right\}$ is a version of the (classical)

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essential infimum [14, II.4] of the set $\left\{a^c(\cdot, x_i') : a \in A\right\}$ of $\bar{T}$-measurable functionals on T (note that μ is equivalent to a finite measure on $(T, \bar{T})$). We set $A_0 \equiv \bigcup_i A_i$ and define l as in (2.1). By the fact that A is composed of Γ_0-integrands we have, for each $a \in A$, $t \in T$,

$$a(t, \cdot) = \sup_i \left[c(\cdot, x_i')-a^c(t, x_i')\right] . \tag{2.3}$$

It follows from (2.1) that, for each $i \in \mathbb{N}$, $t \in T$,

$$l^c(t, x_i') = \sup_x \inf_{a\in A_0} \left[c(x, x_i')-a(t, x)\right] \leq \inf_{a\in A_0} a^c(t, x_i') . \tag{2.4}$$

To show (2.2), let $\bar{a} \in A$ be arbitrary. From the definition of essential infimum and (2.4) it follows that, for each $i \in \mathbb{N}$,

$$l^c(t, x_i') \leq \inf_{a\in A_i} a^c(t, x_i') \leq \bar{a}^c(t, x_i') \quad \text{for almost every } t .$$

In conjunction with (2.3) *this gives*

$$\bar{a}(t, \cdot) \leq \sup_i \left[c(\cdot, x_i')-l^c(t, x_i')\right] \quad \text{for almost every } t . \tag{2.5}$$

Now by (2.1) and Lemma 1.1 we have that l is a $\Gamma^c(X)$-integrand. Hence, (2.5) implies

$$l(t, \cdot) \geq \bar{a}(t, \cdot) \quad \text{for almost every } t ,$$

which is what we had to demonstrate.

REMARK 2.3. By taking X to be a singleton one readily inspects that Theorem 2.2 is a generalization of the classical result concerning the existence of the essential supremum for an arbitrary collection of measurable functionals from T into $[-\infty, +\infty]$ (note that the case where X is a singleton ranks under Examples A.2, B.2).

3. Applications

In this section we shall apply the result in Theorem 2.2 to collections of normal integrands and measurable multifunctions. Let us

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remember that a measurable lower semicontinuous integrand is also called a *normal integrand* [4], [5].

PROPOSITION 3.1. *Suppose* X *is a metrizable Suslin space and* m *is a nonmeasurable integrand on* $T \times X$. *Then there exists a normal integrand* l *on* $T \times X$ *such that*

$$(3.1) \qquad l(t, \cdot) \le m(t, \cdot) \text{ for almost every } t ,$$

and, for every normal integrand a *on* $T \times X$,

$$(3.2) \quad a(t, \cdot) \le m(t, \cdot) \text{ for almost every } t \text{ implies } a(t, \cdot) \le l(t, \cdot) \text{ for almost every } t .$$

Proof. Suppose first that m is nonnegative. Define then $\bar{m}(t, \cdot)$ to be the lower semicontinuous hull of $m(t, \cdot)$, $t \in G$ [11, 6.2]. Then $\bar{m}$ is a nonnegative Γ_1-integrand (in the sense of Example A.2). Apply Theorem 2.2 to the nonempty class A of normal integrands a on $T \times X$ such that

$$a(t, \cdot) \le \bar{m}(t, \cdot) \text{ for almost every } t .$$

The essential supremum of A exists by Theorem 2.2 and Example A.2. It satisfies (3.1)-(3.2) by (2.1)-(2.2) and the definition of $\bar{m}$. If m is general, we apply the above result to the nonnegative normal integrand $\exp(m)$ and finish by an obvious argument.

REMARK 3.2. The apparently new concept introduced in Proposition 3.1 will be called the normal hull of the integrand m.

PROPOSITION 3.3. *Suppose* X *is a locally convex Suslin space and suppose* m *is a nonmeasurable* inf-*compact proper convex integrand. Then there exists a proper convex* inf-*compact normal integrand* l *on* $T \times X$ *such that, for every proper convex* inf-*compact normal integrand* a *on* $T \times X$,

$$a(t, \cdot) \le m(t, \cdot) \text{ for almost every } t \text{ implies } a(t, \cdot) \le l(t, \cdot) \text{ for almost every } t .$$

Proof. Note that m is a Γ_2-integrand by Example B.2. Consider the possibly empty class A of proper convex inf-compact normal integrands a on $T \times X$ with

$$a(t, \cdot) \le m(t, \cdot) \text{ for almost every } t .$$

If A is empty, the result holds by taking l to be an arbitrary proper convex inf-compact normal integrand. If A is nonempty, the result holds by an application of Theorem 2.2 and the observation that the supremum of a countable subcollection in A is a proper convex inf-compact normal integrand on $T \times X$.

Next, we consider the essential supremum when we are dealing with measurable selectors of nonmeasurable multifunctions. Let F be a nonmeasurable multifunction from T into X. Remember that a measurable function $u : T \to X$ is said to be a *measurable selector* of F if

$$u(t) \in F(t) \quad \text{for almost every} \quad t .$$

We denote the set of measurable selectors of F (possibly empty) by S_F. The following result generalizes [1, Theorem 2.7]. See [15, III.14] for a related result.

PROPOSITION 3.4. *Suppose X is a metrizable Suslin space and F is a nonmeasurable multifunction from T into X. Then there exists a countable (possibly empty) collection $\{u_j\} \subset S_F$ such that, for every $u \in S_F$,*

$$u(t) \in \mathrm{cl}\{u_j(t)\} \quad \textit{for almost every} \quad t . \tag{3.3}$$

Proof. Denote by d a metric on X. For each $u \in S_F$ we define

$$a_u(t, \cdot) \equiv -d(u(t), \cdot) , \quad t \in T .$$

Note that this defines a (possibly empty) class A of measurable Γ_1-integrands on $T \times X$ [5, III.14]. By Theorem 2.2 there exists a countable subcollection $\{u_j\}$ of S_F such that, for each $u \in S_F$,

$$a_u(t, \cdot) \le -\mathrm{dist}(\{u_j(t)\}, \cdot) \quad \text{for almost every} \quad t , \tag{3.4}$$

where $\mathrm{dist}(\{u_j(t)\}, x) \equiv \inf_j d(u_j(t), x)$, $t \in T$, $x \in X$. It follows from (3.4) that, for each $u \in S_F$,

$$\mathrm{dist}(\{u_j(t)\}, u(t)) = 0 \quad \text{for almost every} \quad t ,$$

which proves (3.3).

REMARK 3.5. Proposition 3.4 is equivalent to saying that for the measurable multifunction $G : t \mapsto \mathrm{cl}\{u_j(t)\}$ with closed values [9] we have

$$G(t) \subset \mathrm{cl}\ F(t) \quad \text{for almost every} \quad t$$

and

$$S_G = S_F .$$

In [1, Theorem 2.7] an essential role is played by the assumption that T be a countably generated σ-algebra. In this respect Proposition 3.5 is the more general result. On the other hand, the argument in [1] seems to go through in the case where X is merely separable metric.

PROPOSITION 3.6. *Suppose that* X *is a metrizable Suslin space and that* F *is a nonmeasurable multifunction from* T *into* X. *Then there exists a countable collection* $\{G_j\}$ *of measurable multifunctions from* T *into* X *with closed values such that*

$$\mathrm{cl}\ F(t) \subset \bigcap_j G_j(t) \quad \textit{for almost every} \quad t ,$$

and for each measurable multifunction G *from* T *into* X *with closed values*

$$G(t) \supset F(t) \quad \textit{for almost every} \quad t \quad \textit{implies}$$
$$G(t) \supset \bigcap_j G_j(t) \quad \textit{for almost every} \quad t .$$

Proof. Consider the nonempty class A consisting of the indicator integrands δ_G of the measurable closed valued multifunctions G such that $G(t) \supset F(t)$ for almost every t $\left(\delta_G(t, x) \equiv 0 \text{ if } x \in G(t) ,\right.$ $\left.\delta_G(t, x) \equiv +\infty \text{ if } x \notin G(t) ,\ t \in T ,\ x \in X\right)$.

It is easy to see that A is a class of measurable Γ_1-integrands on $T \times X$. The result follows now directly from an application of Theorem 2.2.

PROPOSITION 3.7. *Suppose that* X *is a Suslin locally convex space and that* F *is a nonmeasurable multifunction from* T *into* X. *Then there exists a countable collection* $\{G_j\}$ *of measurable multifunctions* *from* T *into* X *with convex closed locally compact values not containing a straight line such that for each measurable multifunction* G *from* T *into* X *with convex closed locally compact values not containing a straight line*

$$G(t) \supset F(t) \text{ for almost every } t \text{ implies}$$
$$G(t) \supset \bigcup_j G_j(t) \text{ for almost every } t .$$

Proof. But for the fact that the collection A which we have to consider here may be empty, the proof is quite analogous to that of Proposition 3.6 and will not be written out.

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Mathematical Institute,
University of Utrecht,
Budapestlaan 6,
PO Box 80.010,
3508 TA Utrecht,
The Netherlands.