Solutions of Laplace's equation in an n-dimensional space of constant curvature

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This paper is a sequel to an earlier one[1] containing a tensor formulation and generalisation of well-known solutions of Laplace's equation and of the classical wave-equation. The partial differential equation considered was

$$\Delta_2 V \equiv g^{ij} \left\{ \frac{\partial^2 V}{\partial x^i \, \partial x^j} - \Gamma^k_{ij} \frac{\partial V}{\partial x^k} \right\} = 0,$$

where Γ^k_{ij} is the Christoffel symbol of the second kind, and the work was restricted to the case in which the associated line-element

$$ds^2 = g_{ij} \, dx^i \, dx^j$$

was that of an n-dimensional flat space. It is shown below that similar solutions exist for any n-dimensional space of constant positive or negative curvature K.

Tensor methods are employed except occasionally when the results are applied to particular spaces. The geometrical nature of the solutions is discussed in § 6, where it is shown that they are closely connected with well-known solutions of the ordinary cylindrical-polar form of Laplace's equation $\nabla^2 V = 0$.

Although the paper is intended principally as a contribution to the theory of partial differential equations, it is offered as much for its incidental results as for its main theorems. These results include an invariant definition of Beltrami's coordinates for a two-dimensional space of constant curvature (§ 3), a definition of spatial distance in a de Sitter space-time (§ 4), and relations between certain fundamental scalars belonging to the analytical theory of spaces of constant curvature (§ 6).

A more precise statement of the main problem solved is given at the beginning of § 2.

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[1] *Proc. Edinburgh Math. Soc.* (2), 2 (1930-31), 181.

§ 1. *Preliminaries.*

Let then

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{1.1}$$

define the metric of an n-dimensional space of constant curvature K. Covariant derivatives with respect to the x^i will be denoted by the simple addition of suffixes, so that the partial differential equation to be solved may be written

$$\Delta_2 V \equiv V_i^i = 0, \tag{1.2}$$

where

$$V_i^i \equiv g^{ij} V_{ij}.$$

In (1.1) the coordinates x^i and the coefficients g_{ij} need not be real, the theory being valid for any space in which the curvature tensor is given by

$$R_{ijkl} = K\,(g_{ik}\, g_{jl} - g_{il}\, g_{jk}).$$

Let (x^i) and $(\bar{x}^i)$ be any two points of the space and let s be the function of the x's and $\bar{x}$'s representing the length of the arc of the geodesic joining them. When $K = 0$, s is completely determinate; so, for example, when $n = 3$ and the coordinates are rectangular cartesian,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1.3}$$

and

$$s = \{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})\}^{\frac{1}{2}}.$$

When $K \neq 0$, s is determinate but for additive multiples of $2\pi/K^{\frac{1}{2}}$; thus if $n = 2$ and $K = 1/a^2$, the metric may be taken in the form

$$ds^2 = a^2\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1.4}$$

and s is given by

$$\cos\,(s/a) = \cos\theta\,\cos\bar{\theta} + \sin\theta\,\sin\bar{\theta}\,\cos\,(\phi - \bar{\phi}). \tag{1.5}$$

We suppose that any one particular value of s has been chosen, and write

$$\Omega \equiv \tfrac{1}{2}s^2, \tag{1.6}$$

so that, when the metric has the special form (1.3),

$$\Omega = \tfrac{1}{2}\{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2\}. \tag{1.7}$$

Although it has hitherto gained relatively little attention, the function Ω undoubtedly occupies a fundamental place in Riemannian geometry and tensor analysis. It is, so to speak, an integrated form of the scalar differential $\frac{1}{2}ds^2$ which defines the metric. In terms of Ω it is possible to obtain for any Riemannian space a simple expression for the Riemann normal coordinates of origin $(\bar{x}^i)$ and also a tensor form of Taylor's theorem[1]. It also arises in the solution of tensor partial differential equations[2], it can be used in the development of Riemannian trigonometry[3], and in General Relativity it appears in the formulae for the 4-potential of a charged particle[4] and for spatial distance as determined by measurements of stellar magnitudes and luminosities[5].

Covariant derivatives of Ω with respect to the x's, the $\bar{x}$'s being kept constant, will be denoted by the addition of simple Latin subscripts. Bracketed Latin subscripts will similarly denote its covariant derivatives with respect to the $\bar{x}$'s, the x's being kept constant. Suffixes will be raised and lowered by means of the fundamental tensor, which will be evaluated at (x^i) for ordinary suffixes and at $(\bar{x}^i)$ for bracketed suffixes. All functions evaluated at the latter point will be denoted by a superposed dash ("bar"). Thus

$$\Omega_i \equiv \frac{\partial \Omega}{\partial x^i}, \quad \Omega_{(i)} \equiv \frac{\partial \Omega}{\partial \bar{x}^i},$$

$$\Omega_{ij} \equiv \frac{\partial^2 \Omega}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \Omega}{\partial x^k},$$

$$\Omega_{(ij)} \equiv \frac{\partial^2 \Omega}{\partial \bar{x}^i \partial \bar{x}^j} - \bar{\Gamma}^k_{ij} \frac{\partial \Omega}{\partial \bar{x}^k},$$

$$\Omega^i \equiv g^{ij} \Omega_j, \quad \Omega^{(i)} \equiv \bar{g}^{ij} \Omega_{(j)},$$

and so on. The same notation will be used for any function which, like Ω, depends upon the $\bar{x}$'s as well as upon the x's.

In any Riemannian space Ω satisfies the identities

$$\Omega_i \Omega^i \equiv 2\Omega, \qquad \Omega_{(i)} \Omega^{(i)} \equiv 2\Omega, \tag{1.8}$$

which follow immediately from the well-known[6] relation $s_i s^i \equiv 1$ satisfied by the function s, while in a flat space the identities

$$\Omega_{ij} \equiv g_{ij}, \qquad \Omega_{(ij)} \equiv \bar{g}_{ij} \tag{1.9}$$

[1] Ruse, *Proc. London Math. Soc.*, **32** (1931), 90.

[2] See the paper quoted above and also Ruse, *Quarterly J. of Math.* (Oxford), **3** (1932), 15.

[3] Synge, *Proc. London Math. Soc.*, **32** (1931), 241.

[4] Ruse, *Quarterly J. of Math.* (Oxford), **1** (1930), 146.

[5] Etherington, *Phil. Mag.*, **15** (1933), 761.

[6] Darboux, *Leçons sur la théorie générale des surfaces*, Vol. II, Book V, Ch. V (p. 450, § 536 in 2nd edition, Paris 1915).

also hold. These may be verified immediately by choosing a system of rectangular cartesian coordinates.

Now suppose that the point $(\bar{x}^i)$ lies upon a given curve C of equations

$$\bar{x}^i = \bar{x}^i(\tau), \tag{1.10}$$

τ being a scalar parameter. That the $\bar{x}^i$, as here, occasionally represent current coordinates as well as those of a particular point should cause no confusion. Then if we substitute from (1.10) in Ω it becomes a function of the x's and of τ. Symbolically

$$\Omega \equiv \Omega(x, \bar{x}) \equiv \Omega(x^i; \tau). \tag{1.11}$$

Let Ω_τ, $\Omega_{\tau\tau}$, ... denote the scalars obtained by successive partial differentiation of Ω with respect to τ, the x's being kept constant; that is, let

$$\Omega_\tau \equiv \partial\,\Omega/\partial\tau, \quad \Omega_{\tau\tau} \equiv \partial^2\,\Omega/\partial\tau^2, \ldots .$$

Then in accordance with the notation described above, we write

$$\left.\begin{aligned} \Omega_{\tau i} &\equiv \frac{\partial^2\,\Omega}{\partial\tau\,\partial x^i}, \quad \Omega_{\tau\tau i} \equiv \frac{\partial^3\,\Omega}{\partial\tau^2\,\partial x^i}, \\ \Omega_{\tau ij} &\equiv \frac{\partial^3\,\Omega}{\partial\tau\,\partial x^i\,\partial x^j} - \Gamma^k_{ij}\,\frac{\partial^2\,\Omega}{\partial\tau\,\partial x^k}, \end{aligned}\right\} \tag{1.13}$$

and so on, τ *being treated as a constant* when the scalars Ω_τ, $\Omega_{\tau\tau}$, ... are differentiated covariantly with respect to the x^i.

Next suppose that we define τ as a function of the x's by solving the equation

$$\Omega(x^i; \tau) = 0 \tag{1.14}$$

for τ and taking any one root. This amounts to choosing upon C, which will be called the *base-curve*, one of the points $\bar{x}^i(\tau)$ at which the curve is cut by the null-cone of vertex (x^i), the equation of which, with the $\bar{x}$'s as current coordinates, is $\Omega(x, \bar{x}) = 0$. Thus $(\bar{x}^i)$ is now one of the points upon C at zero distance from (x^i).

If we substitute for τ as a function of the x^i in Ω_τ, this scalar becomes a function of the x's only which is in general non-zero even though Ω itself vanishes identically for that value of τ.

Now it was shown in the first of the papers quoted above that, for an n-dimensional *flat* space, solutions of the partial differential equation $\Delta_2 V = 0$ are given by

$$V = f(\Omega_\tau), \tag{1.15}$$

$$V = \phi(\tau)/\Omega_\tau^{\frac{1}{2}(n-2)}, \tag{1.16}$$

where f and ϕ are arbitrary functions of their arguments, provided that the functions $\bar{x}^i(\tau)$ satisfy the differential equations

$$\frac{\delta^2 \bar{x}^i}{\delta \tau^2} \equiv \frac{d^2 \bar{x}^i}{d\tau^2} + \bar{\Gamma}^i_{jk} \frac{d\bar{x}^j}{d\tau} \frac{d\bar{x}^k}{d\tau} = 0, \tag{1.17}$$

$$\bar{g}_{ij} \frac{d\bar{x}^i}{d\tau} \frac{d\bar{x}^j}{d\tau} = 0, \tag{1.18}$$

that is, provided that the base-curve C is a null geodesic. If τ is chosen so that it is zero at some fixed point (a^i) of C, then it is determined by (1.14) as a function of the x's except for an arbitrary multiplicative constant which arises because of the homogeneity of (1.17) and (1.18) in $d\tau$. It was found also that the second solution (1.16) holds in the cases $n = 2$ and $n = 4$ whatever the functions $\bar{x}^i(\tau)$, *i.e.* whether C be a null geodesic or no.

The question arises whether these theorems are extensible to more general Riemannian spaces, and it is shown below that similar results are in fact obtainable for a space of constant curvature K. In such a space it is convenient to work with the function defined by

$$Q \equiv \cos(K^{\frac{1}{2}} s) \equiv \cos\sqrt{(2K\,\Omega)} \tag{1.19}$$

instead of with Ω itself. Here K may be positive or negative but not zero, though we note that approximately

$$Q = 1 - K\,\Omega \tag{1.20}$$

when K is small. Suffixes attached to Q will have a meaning similar to those defined above for Ω.

By (1.8), Q satisfies the identities

$$Q_i\, Q^i \equiv K(1 - Q^2), \tag{1.21}$$

$$Q_{(i)}\, Q^{(i)} \equiv K(1 - Q^2), \tag{1.22}$$

in which the superscript 2 denotes the square. Also[1]

$$Q_{ij} \equiv -K g_{ij} Q, \quad Q_{(ij)} \equiv -K \bar{g}_{ij} Q. \tag{1.23}$$

By (1.20) these reduce to (1.9) when we make $K \to 0$.

[1] Ruse, *Quarterly J. of Math.* (Oxford), **1** (1930), 148; *cf.* Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, Vol. II (Leipzig, 1930), p. 143.

§ 2. *Statement and analysis of the problem.*

We suppose once again that

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{2.1}$$

defines the metric of an n-dimensional space of constant curvature K, and that in it we have chosen two points (x^i), $(\bar{x}^i)$ at which the coefficients g_{ij} are analytic. Suppose further that we have selected a base-curve C of equations

$$\bar{x}^i = \bar{x}^i(\tau) \tag{2.2}$$

upon which the latter point lies. For the moment we make no assumption about the nature of this curve except to require that it shall not pass through (x^i). Let (a^i) be the fixed point upon C at which τ is zero, so that $a^i = \bar{x}^i(0)$. The scalar Q defined by (1.19) is then a function of the x's and of τ, and we seek solutions of the partial differential equation

$$\Delta_2 V \equiv V^i_{\ i} = 0 \tag{2.3}$$

of the form

$$V = V(u, \tau), \text{ where } u \equiv Q_\tau, \tag{2.4}$$

τ being a function of the x's, say

$$\tau = \tau(x^i), \tag{2.5}$$

determined, except perhaps for an arbitrary constant factor, by the equation

$$Q = 1. \tag{2.6}$$

The last equation is equivalent to (1.14) and expresses the fact that upon C we have chosen one of the points $(\bar{x}^i)$ at which the null cone of vertex (x^i) cuts C.

To find whether solutions of the required type exist, we begin by substituting from (2.4) in (2.3). In doing so we must remember that u is a function of the x^i both in its own right, so to speak, and because it depends upon τ which is itself a function of the x's. Its partial derivative is therefore not simply u_i, where the subscript as usual denotes a covariant derivative with τ constant, but $u_i + u_\tau \tau_i$, that is, $Q_{\tau i} + Q_{\tau\tau}\tau_i$. A similar remark applies to any function of the x's and of τ. We thus have

$$V_i = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x^i} + \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial x^i} = V_u\,(Q_{\tau i} + Q_{\tau\tau}\,\tau_i) + V_\tau\,\tau_i,$$

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where the suffixes u, τ denote (as also below) ordinary partial derivatives. Hence

$$
\begin{aligned}
V^i_i = {} & V_{uu}(Q_{\tau i} + Q_{\tau\tau}\tau_i)(Q^i_\tau + Q_{\tau\tau}\,\tau^i) \\
& + 2V_{u\tau}(Q_{\tau i} + Q_{\tau\tau}\tau_i)\,\tau^i + V_{\tau\tau}\,\tau_i\,\tau^i \\
& + V_u(Q^i_{\tau i} + 2Q_{\tau\tau i}\,\tau^i + Q_{\tau\tau\tau}\,\tau_i\,\tau^i + Q_{\tau\tau}\,\tau^i_i) \\
& + V_\tau\,\tau^i_i.
\end{aligned} \tag{2.7}
$$

We have to show that the coefficients in this expression can in certain circumstances be expressed as functions of u and τ only.

Since (2.6) is an identity when we substitute for τ from (2.5), we have on differentiating partially with respect to x^i,

$$Q_i + Q_\tau\,\tau_i = 0,$$

whence

$$\tau_i = -\,Q_i/Q_\tau. \tag{2.8}$$

This relation is an identity in the x's when τ is expressed as a function of those variables, but we restrict the use of the identity sign to relations like (1.21), (1.22) and (1.23) which are identities in τ as well as in the x's. Only the latter kind may be differentiated partially with respect to τ.

From (2.8),

$$\tau_i\,\tau^i = Q_i\,Q^i/Q^2_\tau = K(1 - Q^2)/Q^2_\tau$$

by (1.21), whence

$$\tau_i\,\tau^i = 0 \tag{2.9}$$

because $Q = 1$. It is to be noticed that the latter relation is used only after all the differentiations required by (2.7) have been performed.

Again, differentiating (1.21) with respect to τ, we get

$$Q_{\tau i}\,Q^i \equiv -\,KQQ_\tau, \tag{2.10}$$

whence, by (2.8),

$$Q_{\tau i}\,\tau^i = KQ = K \tag{2.11}$$

by (2.6). If now (2.10) be differentiated with respect to τ, we obtain

$$Q_{\tau i}\,Q^i_\tau + Q_{\tau\tau i}\,Q^i \equiv -\,KQ^2_\tau - KQQ_{\tau\tau},$$

and this, with (2.6), gives

$$Q_{\tau i}\,Q^i_\tau = -\,Q_{\tau\tau i}\,Q^i - KQ^2_\tau - KQ_{\tau\tau}. \tag{2.12}$$

Substituting from (2.9), (2.11) and (2.12) in (2.7), we get

$$V^i_i = V_{uu}(-Q_{\tau\tau i}Q^i - KQ^2_\tau + KQ_{\tau\tau}) + 2KV_{u\tau} + V_u(Q^i_{\tau i} + 2Q_{\tau\tau i}\tau^i + Q_{\tau\tau}\tau^i_i) + V_\tau\tau^i_i. \tag{2.13}$$

Now by (1.23),

$$Q^i_i \equiv -nKQ, \tag{2.14}$$

whence

$$Q^i_{\tau i} \equiv -nKQ_\tau. \tag{2.15}$$

Also by (2.8),

$$\begin{aligned}\tau^i_i &= -\{Q_\tau(Q^i_i + Q_{\tau i}\tau^i) - Q_i(Q^i_\tau + Q_{\tau\tau}\tau^i)\}/Q^2_\tau,\\ &= (n-2)K/Q_\tau\end{aligned} \tag{2.16}$$

by (2.14), (2.6), (2.11), (2.10), (2.8) and (2.9). Substituting from (2.15) and (2.16) in (2.13), replacing Q_τ by u, we obtain

$$V^i_i = V_{uu}(-Q_{\tau\tau i}Q^i - Ku^2 + KQ_{\tau\tau}) + 2KV_{u\tau} + V_u\{-nKu + 2Q_{\tau\tau i}\tau^i + (n-2)KQ_{\tau\tau}/u\} + (n-2)KV_\tau/u. \tag{2.17}$$

But

$$Q_\tau \equiv \frac{\partial Q}{\partial \bar{x}^i}\frac{d\bar{x}^i}{d\tau},$$

whence

$$\begin{aligned}Q_{\tau\tau} &\equiv \frac{\partial Q}{\partial \bar{x}^i}\frac{d^2\bar{x}^i}{d\tau^2} + \frac{\partial^2 Q}{\partial \bar{x}^i\partial \bar{x}^j}\frac{d\bar{x}^i}{d\tau}\frac{d\bar{x}^j}{d\tau}\\ &\equiv \frac{\partial Q}{\partial \bar{x}^i}\left\{\frac{d^2\bar{x}^i}{d\tau^2} + \bar{\Gamma}^i_{jk}\frac{d\bar{x}^j}{d\tau}\frac{d\bar{x}^k}{d\tau}\right\} + \left\{\frac{\partial^2 Q}{\partial \bar{x}^i\partial \bar{x}^j} - \bar{\Gamma}^k_{ij}\frac{\partial Q}{\partial \bar{x}^k}\right\}\frac{d\bar{x}^i}{d\tau}\frac{d\bar{x}^j}{d\tau}\\ &\equiv Q_{(i)}\frac{\delta^2\bar{x}^i}{\delta\tau^2} + Q_{(ij)}\frac{d\bar{x}^i}{d\tau}\frac{d\bar{x}^j}{d\tau},\\ &\equiv Q_{(i)}\frac{\delta^2\bar{x}^i}{\delta\tau^2} - KQ\bar{g}_{ij}\frac{d\bar{x}^i}{d\tau}\frac{d\bar{x}^j}{d\tau}\end{aligned} \tag{2.18}$$

by (1.23).

The functions $\bar{x}^i(\tau)$ have hitherto been left arbitrary. We now choose them so that

$$\frac{\delta^2\bar{x}^i}{\delta\tau^2} = 0 \tag{2.19}$$

and

$$\bar{g}_{ij}\frac{d\bar{x}^i}{d\tau}\frac{d\bar{x}^j}{d\tau} = e \text{ (constant)}. \tag{2.20}$$

When $e \neq 0$, the base-curve C is an ordinary geodesic and τ is proportional to its arc-length measured from the point (a^i) to the point $(\bar{x}^i)$. It is actually equal to that arc-length when $e = \pm 1$. But if $e = 0$, C is a null geodesic with τ as a privileged parameter, *i.e.* a parameter such that the differential equations of the null geodesics have the canonical form (2.19), (2.20). In that case τ is indeterminate as a function of the x's to the extent of a constant factor.

Substituting from (2.19) and (2.20) in (2.18), we obtain

$$Q_{\tau\tau} \equiv -KeQ, \tag{2.21}$$

whence $$Q_{\tau\tau i} \equiv -KeQ_i,$$

so that $$Q_{\tau\tau i}\, Q^i = 0 \tag{2.22}$$

by (1.21) and (2.6). Also

$$Q_{\tau\tau i}\, \tau^i = 0 \tag{2.23}$$

by (2.22) and (2.8). Using these results in (2.17), we get

$$\Delta_2 V \equiv V^i_{\ i} = -K\left[(u^2+Ke)\, V_{uu} - 2V_{u\tau} + \{nu + (n-2)\,Ke/u\}\, V_u - (n-2)\, V_\tau/u\right]. \tag{2.24}$$

We thus have the following theorem:

If in an n-dimensional space of constant curvature K the functions $\bar{x}^i(\tau)$ of the parameter τ are any that satisfy the differential equations

$$\frac{\delta^2 \bar{x}^i}{\delta \tau^2} = 0, \qquad \bar{g}_{ij}\frac{d\bar{x}^i}{d\tau}\frac{d\bar{x}^j}{d\tau} = e,$$

where e is any constant, and if $Q = \cos(K^{\frac{1}{2}}s)$, s being the geodesic distance between the point of coordinates $\bar{x}^i(\tau)$ and the point (x^i), then a solution of the partial differential equation $\Delta_2 V = 0$ is

$$V = V(u, \tau),$$

where $u \equiv \partial Q/\partial\tau$, $V(u, \tau)$ is any solution of the partial differential equation

$$(u^2+Ke)\frac{\partial^2 V}{\partial u^2} - 2\frac{\partial^2 V}{\partial u\, \partial\tau} + \frac{nu^2+(n-2)\,Ke}{u}\frac{\partial V}{\partial u} - \frac{n-2}{u}\frac{\partial V}{\partial \tau} = 0, \tag{2.25}$$

and τ is finally replaced by any function of the x^i obtained by solving the equation $Q = 1$ for τ.

We now consider the nature of the solutions in the two cases $e = 0$ and $e \neq 0$.

§ 3. *The case $e = 0$.*

When $e = 0$ the base-curve C is a null geodesic and (2.25) reduces to

$$u^2\frac{\partial^2 V}{\partial u^2} - 2\frac{\partial^2 V}{\partial u\, \partial\tau} + nu\frac{\partial V}{\partial u} - \frac{n-2}{u}\frac{\partial V}{\partial \tau} = 0. \tag{3.1}$$

From the identity (2.21) we also have

$$Q_{\tau\tau} \equiv 0,$$

which shows that Q is a linear function of τ and therefore of the form

$$Q \equiv f(x^i) + a\tau g(x^i),$$

where f and g are functions of the x's only and a is the constant factor of indeterminacy referred to above. Consequently in this case the equation $Q = 1$ yields only one essentially distinct value of τ, which corresponds to the geometrical fact that the null cone of vertex (x^i) cuts a null geodesic C not lying in it in one point only.

In (3.1) put

$$\tau = y + x, \quad u = 1/x, \tag{3.2}$$

so that
$$x = \frac{1}{u} \equiv \frac{1}{Q_\tau}, \quad y = \tau - \frac{1}{u} \equiv \tau - \frac{1}{Q_\tau}, \tag{3.3}$$

and the partial differential equation transforms into

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} = \frac{n-2}{x}\frac{\partial V}{\partial x}. \tag{3.4}$$

When the number of dimensions n of the space is even, this equation is solvable in finite terms. It is in fact the example given by Forsyth[1] of equations that have no intermediate integral but whose primitive involves two arbitrary functions. Thus if $n-2=2m$, where m is a positive integer or zero, its solution is

$$V = \sum_{r=0}^{m} (-1)^r \frac{2^r}{r!} \frac{\binom{m}{r}}{\binom{2m}{r}} x^r \{\phi^{(r)}(x-y) + \psi^{(r)}(x+y)\},$$

where ϕ, ψ are arbitrary functions of their arguments, or, by (3.3),

$$V = \sum_{r=0}^{m} (-1)^r \frac{2^r}{r!} \frac{\binom{m}{r}}{\binom{2m}{r}} \frac{1}{Q_\tau^r} \left\{\phi^{(r)}\left(\frac{2}{Q_\tau} - \tau\right) + \psi^{(r)}(\tau)\right\}.$$

It can be shown by direct substitution of this value of V in the partial differential equation $\Delta_2 V = 0$ that, *when* $\phi \equiv 0$ *and* $n = 2$ *or* $n = 4$, *the solution holds whatever the nature of the curve* C, *i.e.* whether or no the functions $\bar{x}^i(\tau)$ satisfy the differential equations of the null geodesics. In all other cases the restriction that C be a null geodesic and τ a privileged parameter is essential.

[1] *A treatise on differential equations* (5th ed., 1921), p. 534, § 276.

The case $n=2$ presents some special points of interest. Replacing y by $-iy$ in (3.2), (3.3) and (3.4), we find that for any two-dimensional space

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1, 2) \tag{3.5}$$

of constant curvature K, the transformation from the general coordinates (x^i) to the special set (x, y) defined by

$$\tau = x - iy, \quad Q_\tau = 1/x, \tag{3.6}$$

or

$$x = 1/Q_\tau, \qquad y = i\,(\tau - 1/Q_\tau), \tag{3.7}$$

where the base-curve C is a null geodesic and τ, Q_τ are the functions of x^i described above, reduces the partial differential equation $\Delta_2 V = 0$ to the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{3.8}$$

This suggests that ds^2 will have some specially simple form when expressed in terms of x and y. To find whether this is so, take (3.5) in the form (1.4), viz.,

$$ds^2 = a^2\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{3.9}$$

for which $K = 1/a^2$ and

$$Q = \cos\theta\cos\bar\theta + \sin\theta\sin\bar\theta\cos(\phi - \bar\phi). \tag{3.10}$$

The most general equations of a null geodesic are

$$\cos\bar\theta = i\,(A\tau + B),$$
$$\tan(\bar\phi - \kappa) = A\tau + B,$$

where $\bar\theta$, $\bar\phi$ are current coordinates, A, B, κ are constants and τ is a privileged parameter. Substituting for $\bar\theta$, $\bar\phi$ in (3.10), we get

$$Q = A\tau\,\{i\cos\theta + \sin\theta\sin(\phi-\kappa)\} \\ + [\sin\theta\cos(\phi-\kappa) + B\,\{i\cos\theta + \sin\theta\sin(\phi-\kappa)\}],$$

whence $Q_\tau = A\,\{i\cos\theta + \sin\theta\sin(\phi-\kappa)\}$.

Also the equation $Q = 1$ gives

$$\tau = \frac{1}{A}\left\{\frac{1 - \sin\theta\cos(\phi-\kappa)}{i\cos\theta + \sin\theta\sin(\phi-\kappa)} - B\right\}, \tag{3.11}$$

and the transformation (3.7) becomes in this case

$$x = \frac{1}{A\,\{i\cos\theta + \sin\theta\sin(\phi-\kappa)\}}, \tag{3.12}$$

$$y = -\frac{i}{A}\left\{\frac{\sin\theta\cos(\phi-\kappa)}{i\cos\theta + \sin\theta\sin(\phi-\kappa)} + B\right\}. \tag{3.13}$$

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It may be noted in (3.11) that the expression for τ involves the arbitrary constant factor $1/A$.

It may now be shown by direct calculation that under the transformation (3.12), (3.13) the line-element (3.9) assumes Beltrami's form[1]

$$ds^2 = -\frac{1}{Kx^2}(dx^2 + dy^2). \tag{3.14}$$

It is easy to see that (3.8) is in fact the form taken by the equation $\Delta_2 V = 0$ when the metric is given by (3.14).

We thus have the result that *for any two-dimensional space of constant curvature K referred to a general coordinate-system x^i, equations* (3.7) *give a transformation which reduces ds^2 to the canonical form* (3.14). In particular, if ds^2 is initially in the form (3.14), say

$$ds^2 = -\frac{1}{Kx_1^2}(dx_1^2 + dx_2^2),$$

equations (3.7), calculated with an arbitrary null geodesic as base-curve, give a transformation of the type[2]

$$x = Ax_1/\{x_1^2 + (x_2 - \beta)^2\},$$
$$y - \gamma = \pm A(x_2 - \beta)/\{x_1^2 + (x_2 - \beta)^2\},$$

(A, β, γ const.), which preserves the form of ds^2.

We have been considering the case when $e = 0$ and n is even, and in particular when $n = 2$. To find the nature of the solution for any positive integral value of n, return to the original partial differential equation (3.1) and make the transformation of independent variables defined by

$$\xi = \tau, \quad \eta = \tau - 2/u \equiv \tau - 2/Q_\tau. \tag{3.15}$$

The equation becomes

$$\frac{\partial^2 V}{\partial \xi\, \partial \eta} + \frac{n-2}{2(\xi - \eta)}\left(\frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \eta}\right) = 0. \tag{3.16}$$

This is a particular case of the Euler-Poisson partial differential equation[3], the properties of which are too well known to need repeating

[1] Darboux, *op. cit.*, Vol. III, Book VII, Ch. XI (p. 394, § 782 in 1st edition, 1894).

[2] *Cf.* Darboux, *op. cit.*, Vol. III, p. 396, § 783, formula (6).

[3] Euler, *Opera omnia*, 1st series, Vol. 13 (Leipzig 1913), p. 212 *et seq.*; also *Oeuvres de Laplace*, Vol. IX (Paris 1893), p. 41, § IX; and Darboux, *op. cit.*, Vol. II, Book IV, Ch. III (p. 54 in 2nd ed.).

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here. Any solution of (3.16), where ξ, η are given by (3.15), is a solution of the original partial differential equation $\Delta_2 V = 0$. It may however be noted that if we put

$$V = (\xi - \eta)^{n-1} U,$$

then U satisfies the Euler-Poisson equation

$$\frac{\partial^2 U}{\partial \xi \, \partial \eta} - \frac{n}{2(\xi - \eta)} \left(\frac{\partial U}{\partial \xi} - \frac{\partial U}{\partial \eta} \right) = 0,$$

and that, if n is even, say $n = 2m + 2$, the general solution of (3.16) is[1]

$$V = (\xi - \eta)^{2m+1} \frac{\partial^{2m}}{\partial \xi^m \, \partial \eta^m} \left(\frac{X - Y}{\xi - \eta} \right),$$

where X is a function of ξ only and Y of η only. This is equivalent to the solution given above for the case when n is even.

§ 4. *The case* $e \neq 0$.

When $e \neq 0$ the base-curve C is an ordinary geodesic and τ is proportional to its arc-length measured from the fixed point (a^i) upon it. If we put

$$Ke = k^2, \tag{4.1}$$

so that k is real or imaginary according as Ke is positive or negative, the equation (2.25) to be solved becomes

$$(u^2 + k^2) \frac{\partial^2 V}{\partial u^2} - 2 \frac{\partial^2 V}{\partial u \, \partial \tau} + \frac{nu^2 + (n-2) k^2}{u} \frac{\partial V}{\partial u} - \frac{n-2}{u} \frac{\partial V}{\partial \tau} = 0. \tag{4.2}$$

If we make the transformation of independent variables defined by[2]

$$\xi = \tau, \quad \eta = \tau + \frac{2}{k} \arc \tan \frac{u}{k} \equiv \tau + \frac{2}{k} \arc \tan \frac{Q_\tau}{k}, \tag{4.3}$$

we get

$$\frac{\partial^2 V}{\partial \xi \, \partial \eta} - \frac{(n-2) k}{2 \sin k (\xi - \eta)} \left(\frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \eta} \right) = 0. \tag{4.4}$$

Equations (4.3) and (4.4) reduce respectively to (3.15) and (3.16) when, having replaced η by $\eta + \pi/k$, we make e, and hence k, tend to zero. Equation (4.4) may therefore be regarded as a natural

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[1] Darboux, *op. cit.*, Vol. II, p. 65, equation (26).

[2] The ξ, η of (4.3) are of course different from the ξ, η of (3.15).

generalisation of the Euler-Poisson equation (3.16). It is beyond the scope of the present paper to study it in detail, but it may be remarked that it possesses solutions of the type

$$V = \theta(\xi - \eta) . \chi(\xi + \eta), \tag{4.5}$$

where θ, χ are respectively functions of $\xi - \eta$ and $\xi + \eta$ only. One such solution, for example, is given by

$$\theta \equiv F\left\{\frac{p}{k}, \; -\frac{p}{k}; \; \frac{n-1}{2}; \; \sin^2 \tfrac{1}{2}k(\xi - \eta)\right\},$$
$$\chi \equiv A \cos\{p(\xi + \eta) + \epsilon\},$$

where F is the hypergeometric function and A, p, ϵ are arbitrary constants.

It was remarked at the beginning of § 3 that, when the base-curve C is a null geodesic, the equation $Q = 1$ gives only one value of τ as a function of the x's [*cf.* (3.11)]. When C is an ordinary geodesic there are two essentially distinct values of τ. For by (2.21) and (4.1),

$$\frac{\partial^2 Q}{\partial \tau^2} \equiv -k^2 Q, \tag{4.6}$$

whence

$$Q \equiv A(x^i) \cos k\tau + B(x^i) \sin k\tau, \tag{4.7}$$

(A and B being functions of the x^i only), so that $Q = 1$ gives a quadratic equation in $\tan \frac{1}{2}k\tau$.

We now show that, *if τ is one root of the equation*

$$Q(x^i; \; \tau) = 1, \tag{4.8}$$

then $\tau + \dfrac{2}{k}$ *arc tan* $\dfrac{Q_\tau}{k}$ *is another*; that is, the distinct roots are precisely the variables ξ, η introduced in (4.3). For if we write for brevity

$$h = \frac{2}{k} \text{ arc tan } \frac{Q_\tau}{k}, \tag{4.9}$$

we have by Taylor's theorem,

$$Q(x^i; \tau + h) = Q + hQ_\tau + \frac{h^2}{2!} Q_{\tau\tau} + \frac{h^3}{3!} Q_{\tau\tau\tau} + \ldots, \tag{4.10}$$

where Q, Q_τ, ... mean $Q(x^i; \tau)$, $Q_\tau(x^i; \tau)$, ... By (4.6),

$$Q_{\tau\tau} \equiv -k^2 Q,$$

whence

$$Q_{\tau\tau\tau} \equiv -k^2 Q_\tau,$$
$$Q_{\tau\tau\tau\tau} \equiv -k^2 Q_{\tau\tau} \equiv k^4 Q.$$

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and so on. Substituting in (4.10), we get

$$Q(x^i;\ \tau+h) = Q + hQ_\tau - \frac{k^2 h^2}{2!}Q - \frac{k^2 h^3}{3!}Q_\tau + \frac{k^4 h^4}{4!}Q + \frac{k^4 h^5}{5!}Q_\tau - \dots .$$
$$= Q\cos(kh) + Q_\tau k^{-1}\sin(kh).$$

Inserting the value (4.9) of h and using (4.8), we at once obtain

$$Q(x^i;\ \tau+h) = 1,$$

which proves that $\tau+h$ is also a root of (4.8). By (4.7), any other root differs from either τ or $\tau+h$ by an integral multiple of $2\pi/k$. The null cone of vertex (x^i) cuts the base-geodesic C in two points only, for the addition of a multiple of $2\pi/k$ to τ gives the same point $\bar{x}^i(\tau)$ upon C. When $e = \pm 1$, so that τ is the actual arc-length of C measured from the fixed point (a^i), *the difference h between the roots τ and $\tau+h$ is equal to the intercept made upon C by the null cone of vertex (x^i).* Since the inverse tangent in the expression (4.9) for h is indeterminate to the extent of an additive multiple of π, this intercept is indeterminate to the extent of an additive multiple of $2\pi/k$, a fact that corresponds to the possibility of "travelling round the universe."

It may be noted in passing that the last result may be interpreted physically. Suppose the geodesic C to be the world-line of an observer O in a four-dimensional space-time of constant curvature, and that the arc-length of C measures the proper-time of O. Suppose also that the x^i are the world-coordinates of a star S at the instant when it receives from the observer a light-pulse dispatched by him at his proper-time τ. If the light-pulse be immediately reflected back it reaches[1] the observer at his proper-time $\tau+h$. Thus h, the intercept made by the null-cone of vertex S on the world-line C, measures the interval of proper-time between the sending and reception of the light-pulse by the observer. Now for an observer at rest at the spatial origin of a galilean space-time

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2)/c^2,$$

the intercept made upon his world-line, the equations of which are

$$\bar{x} = 0, \quad \bar{y} = 0, \quad \bar{z} = 0, \quad \bar{t} = \tau,$$

by the null cone of a star at (x, y, z, t), is $h = 2\delta/c$, where $\delta \equiv \sqrt{(x^2+y^2+z^2)}$ is the spatial distance of the star from the observer. A not unnatural

[1] We are assuming $h > 0$ for the sake of the argument. If $h < 0$, $\tau + h$ is the time of dispatch and τ the time of reception.

definition of spatial distance for the space-time of constant curvature is therefore

$$\delta = \pm \tfrac{1}{2} ch$$

or

$$\delta = \pm \frac{c}{\sqrt{(Ke)}} \arctan \frac{Q_\tau}{\sqrt{(Ke)}}, \tag{4.11}$$

by (4.9) and (4.1), where the sign is chosen so as to make δ positive. Applying this to the de Sitter world

$$ds^2 = \left(1 - \frac{r^2}{R^2}\right) dt^2 - \frac{1}{c^2} \left\{ \frac{dr^2}{1 - r^2/R^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right\},$$

for which $K = -c^2/R^2$, $K^{\frac{1}{2}} = ci/R$ and[1]

$$Q = \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} \left(1 - \frac{\bar{r}^2}{R^2}\right)^{\frac{1}{2}} \cosh \frac{c\,(t - \bar{t})}{R}$$
$$+ \frac{r\bar{r}}{R^2} \left\{ \cos\theta \cos\bar{\theta} + \sin\theta \sin\bar{\theta} \cos(\phi - \bar{\phi}) \right\},$$

and taking the observer at the spatial origin, so that his world-line (a geodesic) is specified by $\bar{r} = 0$, $\bar{t} = \tau$, $(e = 1)$, we have

$$Q = \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}} \cosh \frac{c\,(t - \tau)}{R},$$

whence, differentiating with respect to τ and using the fact that $Q = 1$, we quickly get

$$\delta = R \operatorname{arg\,tanh}(r/R)$$

for the distance from the spatial origin of a star at the world-point (r, θ, ϕ, t).

We now return, once again, to the discussion of the partial differential equation $\Delta_2 V = 0$.

§ 5. *Solutions for a flat space.*

We have seen that for a space of constant curvature K the partial differential equation reduces to

$$(u^2 + Ke) \frac{\partial^2 V}{\partial u^2} - 2 \frac{\partial^2 V}{\partial u\, \partial \tau} + \frac{nu^2 + (n-2) Ke}{u} \frac{\partial V}{\partial u} - \frac{n-2}{u} \frac{\partial V}{\partial \tau} = 0. \tag{5.1}$$

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[1] Ruse, "On the definition of spatial distance in general relativity," *Proc. Roy. Soc., Edinburgh*, **52** (1931-2), 183-194, and in particular p. 193. For other work on the definition of spatial distance see E. T. Whittaker, *Proc. Roy. Soc.* (A) **133** (1931), 93; Kermack, M'Crea and Whittaker, *Proc. Roy. Soc., Edinburgh*, **53** (1932-3), 35; Ruse, *Proc. Roy. Soc., Edinburgh*, **53** (1932-3), 79; Etherington, *loc. cit. supra*; Walker, *Quarterly J. of Math.* (Oxford), **4** (1933), 71; Temple, *Proc. Roy. Soc.* (A) **168** (1938), 136.

Now when K is small, we have by (1.20) the approximation

$$Q_\tau = - K\,\Omega_\tau,$$

that is, approximately,

$$u = - Kv, \tag{5.2}$$

where

$$v = \Omega_\tau.$$

Hence, approximately, $$\frac{\partial V}{\partial u} = - \frac{1}{K}\frac{\partial V}{\partial v}.$$

So for K small, (5.1) becomes

$$\frac{K^2 v^2 + Ke}{K^2}\frac{\partial^2 V}{\partial v^2} + \frac{2}{K}\frac{\partial^2 V}{\partial v\,\partial \tau} + \frac{nK^2 v^2 + (n-2)\,Ke}{K^2 v}\frac{\partial V}{\partial v} + \frac{n-2}{Kv}\frac{\partial V}{\partial \tau} = 0.$$

Multiply by K and make $K \to 0$. We get

$$e\frac{\partial^2 V}{\partial v^2} + 2\frac{\partial^2 V}{\partial v\,\partial \tau} + \frac{(n-2)\,e}{v}\frac{\partial V}{\partial v} + \frac{n-2}{v}\frac{\partial V}{\partial \tau} = 0. \tag{5.3}$$

This is the partial differential equation for an n-dimensional flat space corresponding to the equation (5.1) for a space of non-zero curvature K. It may be obtained directly from the equation $\Delta_2 V = 0$ by using methods similar to those in § 2, the identities (1.8) and (1.9) replacing (1.21), (1.22) and (1.23).

When $e = 0$ the base-curve is a null geodesic and V satisfies the equation

$$2\frac{\partial^2 V}{\partial v\,\partial \tau} + \frac{n-2}{v}\frac{\partial V}{\partial \tau} = 0, \tag{5.4}$$

of which the general solution is

$$\begin{aligned} V &= f(v) + \phi(\tau)/v^{\frac{1}{2}(n-2)} \\ &= f(\Omega_\tau) + \phi(\tau)/\Omega_\tau^{\frac{1}{2}(n-2)}, \end{aligned}$$

where f and ϕ are arbitrary functions. This re-establishes the result quoted in § 1 [equations (1.15) and (1.16)], though it does not put in evidence the exceptional nature of the ϕ-solution when $n = 2$ or 4.

When $e \neq 0$ the base-curve is an ordinary geodesic (straight line). Change the variables to ξ, η defined by the equations

$$\xi = \tau, \quad \eta = \tau - 2v/e, \tag{5.5}$$

which follow from (4.3) and (5.2) by making $K \to 0$. Equation (5.3) becomes

$$\frac{\partial^2 V}{\partial \xi\,\partial \eta} - \frac{n-2}{2(\xi - \eta)}\left(\frac{\partial V}{\partial \xi} - \frac{\partial V}{\partial \eta}\right) = 0, \tag{5.6}$$

an Euler-Poisson equation. It is interesting to notice that this differs from (3.16) by having a minus instead of a plus sign before the first-order terms. Any solution of (5.6) is a solution of $\Delta_2 V = 0$.

It can be shown either by methods similar to those used in § 4, or as in § 5 below, that, if τ is one root of the equation

$$\Omega\,(x^i;\ \tau) = 0, \tag{5.7}$$

which is quadratic in τ, then $\tau - 2\,\Omega_\tau/e$ is the other. Thus the variables ξ, η of (5.5) are the roots of (5.7) regarded as an equation for τ.

§ 6. *Geometrical discussion.*

We now consider the geometrical significance of the functions Ω_τ, Q_τ.

First take the case of a flat space referred to rectangular cartesian coordinates. For present purposes it will be sufficient to take $n = 3$, the final results being valid for any value of n. We suppose, therefore, that

$$ds^2 = dx^2 + dy^2 + dz^2,$$
$$2\,\Omega = (x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2, \tag{6.1}$$

and that the base-curve C is the straight line

$$\bar{x} = a + l\tau, \quad \bar{y} = b + m\tau, \quad \bar{z} = c + n\tau, \tag{6.2}$$

where (a, b, c) is the point A at which $\tau = 0$ and l, m, n are constants such that

$$l^2 + m^2 + n^2 = e. \tag{6.3}$$

Substituting from (6.2) in (6.1), we get

$$2\,\Omega = r^2 - 2\tau\,\Sigma\, l\,(x - a) + e\tau^2. \tag{6.4}$$

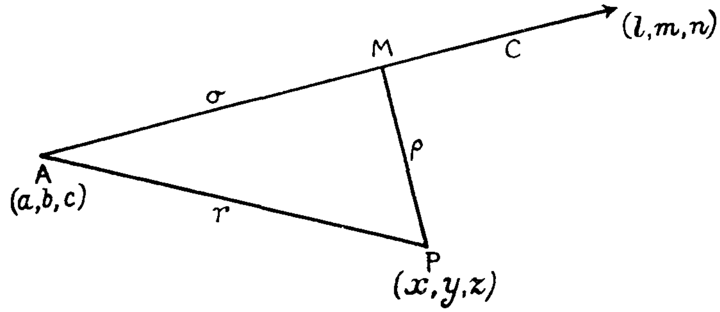
Here

$$r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2,$$

so that r is the distance of the point P, (x, y, z), from the fixed point A, (a, b, c), upon C.

From (6.4),

$$\Omega_\tau = -\,\Sigma\, l\,(x - a) + e\tau. \tag{6.5}$$



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Take $e = 1$. Then C is an ordinary (non-null) straight line, (l, m, n) are its direction cosines, and $\Sigma\, l\,(x - a)$ is the length of the orthogonal projection AM of AP upon C. Suppose that $AM = \sigma$ and $MP = \rho$. Then by (6.4) and (6.5),

$$2\,\Omega = r^2 - 2\sigma\tau + \tau^2, \tag{6.6}$$

$$\Omega_\tau = -\sigma + \tau, \tag{6.7}$$

and the equation $\Omega = 0$ therefore gives

$$\tau = \sigma \pm \sqrt{(\sigma^2 - r^2)},$$

that is

$$\tau = \sigma \pm i\rho. \tag{6.8}$$

Hence by (6.7),

$$\Omega_\tau = \pm\, i\rho, \tag{6.9}$$

and therefore

$$\tau - 2\,\Omega_\tau = \sigma \mp i\rho. \tag{6.10}$$

Equations (6.8) and (6.10) provide a verification that, if τ is one root of the equation $\Omega = 0$, then $\tau - 2\,\Omega_\tau$ is the other. These roots are the ξ, η of (5.5), and we may therefore write

$$\xi = \sigma + i\rho, \quad \eta = \sigma - i\rho. \tag{6.11}$$

If we transform the coordinates (x, y, z) to any new set having A as origin and the line C as z-axis, the variables ρ, σ become the ordinary non-angular cylindrical coordinates of P, ρ being the cylindrical radius-vector. Thus the solutions of $\Delta_2\, V = 0$ found above for the case $e \neq 0$ are the tensor generalisation of solutions, independent of the angular coordinate ϕ, of the cylindrical form of the ordinary Laplace's equation, *i.e.*, of

$$\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho}\,\frac{\partial^2 V}{\partial\phi^2} + \rho\,\frac{\partial^2 V}{\partial\sigma^2} = 0.$$

In an n-dimensional flat space ρ and σ have of course the same geometrical significance as in three dimensions: that is, ρ is the perpendicular distance from the base-line C of the point P, (x^i), and σ is the projection of AP upon C, A being the point (a^i) upon C at which $\tau = 0$.

When $e = 0$ the line C is isotropic and l, m, n may no longer be properly described as direction cosines. We may however conventionally regard $\Sigma\, l\,(x - a)$ as measuring the projection σ of AP upon C, and we have by (6.4) and (6.5),

$$2\,\Omega = r^2 - 2\sigma\tau,$$

$$\Omega_\tau = -\,\sigma. \tag{6.12}$$

The equation $\Omega = 0$ gives

$$\tau = \tfrac{1}{2}r^2/\sigma. \tag{6.13}$$

(6.12) and (6.13) give a geometrical meaning to the variables v, τ appearing in (5.4).

When the space is of non-zero curvature K no such simple interpretations of τ and Ω_τ (or Q_τ) are immediately possible. If $e \neq 0$ we can however obtain a geometrical description of these functions in the manner outlined below.

Consider once again a flat space of n dimensions referred to any coordinates (x^i), and take $e = 1$. Then the variable ρ, which has by (6.9) the value

$$\rho = \mp i\, \Omega_\tau, \tag{6.14}$$

measures the perpendicular distance and therefore the minimum distance of P, (x^i), from the base-line C. But the square of the distance of P from any point $\bar{x}^i(\tau)$ on C is $2\,\Omega\,(x^i; \tau)$, and this is a minimum for variable τ if $\Omega_\tau = 0$. Thus the latter relation, regarded as an equation for τ, gives the value of τ at the foot of the perpendicular M from P upon the line C. But since τ is the arc-length of C measured from the fixed point A, this value is σ, and we thus have

$$\rho^2 = 2\,\Omega\,(x^i;\ \sigma), \tag{6.15}$$

where σ is the function of the x's obtained by solving the equation

$$\frac{\partial}{\partial \sigma}\Omega\,(x^i;\ \sigma) = 0. \tag{6.16}$$

To avoid confusion of notation we continue to represent $\Omega_\tau\,(x^i;\ \tau)$ by Ω_τ, τ being the function of the x's obtained from the equation $\Omega = 0$, and denote $\Omega\,(x^i;\, \sigma)$ by ω. Then by (6.15) we have

$$\rho^2 = 2\omega, \tag{6.17}$$

whence, by (6.14),

$$\Omega_\tau^2 = -\,2\omega. \tag{6.18}$$

Also by (6.7),

$$\tau - \Omega_\tau = \sigma. \tag{6.19}$$

To sum up: Instead of using as the basis of our calculations the functions

$$\tau \text{ and } \Omega_\tau, \text{ where } \Omega = 0, \tag{6.20}$$

we could use the functions

$$\tau \text{ and } \Omega, \text{ where } \Omega_\tau = 0, \tag{6.21}$$

the τ, Ω, Ω_τ of (6.21) being denoted respectively by σ, ω, ω_σ to distinguish them from the τ, Ω, Ω_τ of (6.20). The two sets of functions are connected by the relations (6.18), (6.19). Geometrically, 2ω is the square of the perpendicular distance of P, (x^i), from C, and σ is the projection of AP upon C.

We now establish the corresponding results for a space of non-zero curvature K. As before let τ and Q_τ be the functions of the x's obtained by solving the equation

$$Q(x^i;\tau) = 1 \tag{6.22}$$

for τ. Also let

$$q \equiv Q(x^i;\sigma), \tag{6.23}$$

where $\sigma \equiv \sigma(x^i)$ is the function of the x's obtained by solving the equation

$$q_\sigma \equiv \frac{\partial Q}{\partial \sigma} = 0 \tag{6.24}$$

for σ. Then we shall show that

$$\tau = \sigma \pm \frac{1}{k} \operatorname{arc\,cos} \frac{1}{q}, \tag{6.25}$$

$$Q_\tau = \mp k \sqrt{(q^2 - 1)}, \tag{6.26}$$

where, as usual, $k = \sqrt{(Ke)}$. To prove (6.25) we merely have to show that (6.22) is satisfied by the two values of τ given by (6.25). Now we have by Taylor's theorem, remembering that $Q(x^i;\sigma) \equiv q$,

$$Q\left(x^i;\ \sigma \pm \frac{1}{k} \operatorname{arc\,cos} \frac{1}{q}\right)$$

$$= q \pm \frac{1}{k}\left(\operatorname{arc\,cos} \frac{1}{q}\right) q_\sigma + \frac{1}{2!\,k^2}\left(\operatorname{arc\,cos} \frac{1}{q}\right)^2 q_{\sigma\sigma} \pm \ldots. \tag{6.27}$$

But

$$q_{\sigma\sigma} \equiv -k^2 q, \tag{6.28}$$

as follows by replacing τ by σ in (4.6). Hence

$$q_{\sigma\sigma\sigma} \equiv -k^2 q_\sigma, \quad q_{\sigma\sigma\sigma\sigma} \equiv k^4 q, \ \ldots.$$

and the right-hand side of (6.27) becomes

$$q \cos\left(\operatorname{arc\,cos} \frac{1}{q}\right) \pm \frac{q_\sigma}{k} \sin\left(\operatorname{arc\,cos} \frac{1}{q}\right),$$

which is equal to unity because of (6.24). That (6.25) gives the roots of (6.22) is thus established.

To prove (6.26) we use (6.25). We have

$$q \equiv Q(x^i;\ \sigma)$$
$$= Q\left(x^i;\ \tau \mp \frac{1}{k} \operatorname{arc\,cos} \frac{1}{q}\right).$$

Expanding the right-hand side by Taylor's theorem and remembering that Q means $Q(x^i;\ \tau)$, we obtain

$$q = Q \mp \frac{1}{k}\left(\operatorname{arc\,cos} \frac{1}{q}\right) Q_\tau + \frac{1}{2!\,k^2}\left(\operatorname{arc\,cos} \frac{1}{q}\right)^2 Q_{\tau\tau} \mp \ldots .$$

Using the relation (4.6), namely $Q_{\tau\tau} = -k^2 Q$, just as we used (6.28), we get

$$q = Q \cos\left(\operatorname{arc\,cos} \frac{1}{q}\right) \mp \frac{Q_\tau}{k} \sin\left(\operatorname{arc\,cos} \frac{1}{q}\right),$$

and therefore, since $Q = 1$,

$$q = \frac{1}{q} \mp \frac{Q_\tau}{k}\sqrt{\left(1 - \frac{1}{q^2}\right)}.$$

This at once gives (6.26).

Now by (6.23) and (1.19),

$$q = \cos(K^{\frac{1}{2}} s),$$

where s is the geodesic distance of the point P, (x^i), from the point M of coordinates $\bar{x}^i(\sigma)$ upon the base-curve C. By (6.24) this distance is a maximum or minimum, and is thus the perpendicular distance of P from the curve C. Also when $e = 1$, σ measures the arc-length of C from A to M. We thus obtain from (6.25) and (6.26) expressions for the functions τ and Q_τ (which may be complex) in terms of quantities σ, q that have a geometrical meaning.

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