44

Heavy quark effective theory

44.1 Introduction

Over a decade, a lot of experimental informations on heavy-quark decays and masses have been obtained from e^+e^- and hadron collider experiments. These have led to a detailed knowledge of the flavour sector of the standard model and to the discoveries of the $B^0 - \bar{B}^0$ mixing, rare decays induced by penguin operators, ... The experimental progress in the heavy flavour physics has been accompanied by some theoretical progress. Among other approaches, the discovery of the heavy-quark symmetry has led to the development of the heavy quark effective theory (HQET), which provides a systematic analysis of the properties of a hadron containing a heavy quark in terms of an expansion of the inverse of the heavy quark mass. Detailed discussions and references to the original works can be found in different reviews and lectures (see e.g. [545]).

44.2 Heavy-quark symmetry

When the mass of the heavy quark is much larger than the QCD scale $\Lambda_{\rm QCD}$, the QCD running coupling $\alpha_s(m_Q)$ is small, implying that at this scale of the order of the Compton wavelength $\lambda_Q \sim 1/m_Q$, one can safely use perturbative QCD for describing the hadrons. In this case the $\bar{Q}Q$-bound states with the size $\lambda_Q/\alpha_s(m_Q) \ll R_{\rm had} \sim 1$ fermi are like the hydrogen atom. However, systems composed of a heavy quark plus a light quark are more complicated because the size of the system is of the order of $R_{\rm had}$ while the typical momenta exchanged between the heavy and light quarks is of the order of $\Lambda_{\rm QCD}$. Therefore, the heavy quark is surrounded by strongly interacting clouds of light quarks, antiquarks and gluons. In this case, the simplification is provided by the fact that the Compton wavelength λ_Q is much smaller than the hadron size $R_{\rm had}$. To resolve the quantum numbers of the heavy quark would require a hard probe; the soft gluons exchanged between the heavy quark and the light constituents can only resolve distances much larger than λ_Q. Therefore, the light degrees of freedom are blind to the flavour (mass) and spin orientation of the heavy quark. They experience only its colour field, which extends over large distances because of confinement. In the rest frame of the heavy quark, it is in fact only the electric colour field that is important. Since the heavy-quark spin participates in interactions only through such relativistic effects,

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it decouples for $m_Q \to \infty$. It then follows that, in the limit $m_Q \to \infty$, hadronic systems which differ only in the flavour or spin quantum numbers of the heavy quark have the same configuration of their light degrees of freedom. Although this observation still does not allow us to calculate what this configuration is, it provides relations between the properties of, for example, the heavy mesons B, D, B^* and D^* in the ideal case where the b and c quark masses are infinitely heavy and the corrections to this limit are negligible. These relations result from some approximate symmetries of the effective strong interactions of heavy quarks at low energies. The configuration of light degrees of freedom in a hadron containing a single heavy quark with velocity v does not change if this quark is replaced by another heavy quark with different flavour or spin, but with the same velocity. Both heavy quarks lead to the same static colour field. For n_h heavy-quark flavours, there is thus an SU($2n_h$) spin-flavour symmetry group, under which the effective strong interactions are invariant. These symmetries are in close correspondence to familiar properties of atoms. The flavour symmetry is analogous to the fact that different isotopes have the same chemistry, since to good approximation the wave function of the electrons is independent of the mass of the nucleus. The electrons only see the total nuclear charge. The spin symmetry is analogous to the fact that the hyperfine levels in atoms are nearly degenerate. The nuclear spin decouples in the limit $m_e/m_N \to 0$. This heavy-quark symmetry looks quite similar to the chiral symmetry ($m \to 0$) but in the opposite way ($m_Q \to \infty$), although there is a striking difference.

Whereas chiral symmetry is a symmetry of the QCD Lagrangian in the limit of vanishing quark masses, heavy-quark symmetry is not a symmetry of the Lagrangian (not even an approximate one), but rather a symmetry of an effective theory that is a good approximation to QCD in a certain kinematic region. It is realized only in systems in which a heavy quark interacts predominantly by the exchange of soft gluons. In such systems the heavy quark is almost on-shell; its momentum fluctuates around the mass shell by an amount of order $\Lambda_{\rm QCD}$. The corresponding fluctuations in the velocity of the heavy quark vanish as $\Lambda_{\rm QCD}/m_Q \to 0$. The velocity becomes a conserved quantity and is no longer a dynamical degree of freedom [546]. Nevertheless, results derived on the basis of heavy-quark symmetry are model-independent consequences of QCD in a well-defined limit. To this end, it is however necessary to cast the QCD Lagrangian for a heavy quark:

$$\mathcal{L}_{\rm Q} = \bar{Q}\,(i\not{D} - m_Q)\,Q\,, \tag{44.1}$$

into a form suitable for taking the limit $m_Q \to \infty$.

44.3 Heavy quark effective theory

44.3.1 Introduction

As the effects of infinitely heavy quark are irrelevant at low energies, it becomes useful to built a low-energy effective theory, where the heavy quark no longer appears. This is very similar to the Fermi's theory where weak interactions in weak processes can be approximated

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by a four-fermion interaction governed by the weak coupling G_F. The removal of the heavy particle degrees of freedom can be done in the following ways [547–549]:

- One integrates out the heavy fields in the generating functional of the Green's functions of the theory, which is possible as the heavy particles do not appear as an external source. The resulting action is nonlocal, as in full QCD the heavy particle with mass $M \simeq m_Q$ can appear in virtual processes and propagate over a short but finite distance $\Delta x \sim 1/M$.
- Thus, one needs to get a local effective Lagrangian, which can be done by rewriting the non-local effective action as an infinite series of local terms in an operator product expansion (OPE) [222], which approximately corresponds to an expansion in powers of $1/M$. In this step, the short- and long-distance physics is disentangled, and their domain is separated by a scale ν such that $\Lambda_{\rm QCD} \ll \nu \ll m_Q$. The long-distance physics corresponds to interactions at low energies and is the same in the full and the effective theory below ν.
- In a third step, one needs to add, in a perturbative way using renormalization-group techniques, short-distance effects arising from quantum corrections involving large virtual momenta (of order M), which have not been described correctly in the effective theory once the heavy particle has been integrated out. These short-distance effects lead to a renormalization of the coefficients of the local operators in the effective Lagrangian. An example is the effective Lagrangian for non-leptonic weak decays, in which radiative corrections from hard gluons with virtual momenta in the range between m_W and some low renormalization scale μ give rise to Wilson coefficients, which renormalize the local four-fermion interactions [550–552]. The fact that the physics must be independent of the arbitrary scale ν allows us to derive renormalization-group equations, which can be employed to deal with the short-distance effects in an efficient way.

However, one should notice that the HQET approach is peculiar as it is motivated to describe the properties and decays of hadrons which do contain a heavy quark. Hence, it is not possible to remove the heavy quark completely from the effective theory, but only to integrate out the 'small components' in the full heavy-quark field, which describe the fluctuations around the mass shell.

44.3.2 The HQET Lagrangian

The starting point in the construction of the HQET is the observation that a heavy quark bound inside a hadron moves with the hadron's velocity v and is almost on-shell. Its momentum can be written as:

$$p_Q^\mu = m_Q v^\mu + k^\mu \,, \tag{44.2}$$

where the components of the so-called *residual momentum* k are much smaller than m_Q. Note that v is a four-velocity, so that $v^2 = 1$. Interactions of the heavy quark with light degrees of freedom change the residual momentum by an amount of order $\Delta k \sim \Lambda_{\rm QCD}$, but the corresponding changes in the heavy-quark velocity vanish as $\Lambda_{\rm QCD}/m_Q \to 0$. In this situation, it is appropriate to introduce large- and small-component fields, h_v and H_v, by:

$$h_v(x) = e^{im_Q v\cdot x}\, P_+\, Q(x)\,, \qquad H_v(x) = e^{im_Q v\cdot x}\, P_-\, Q(x)\,, \tag{44.3}$$

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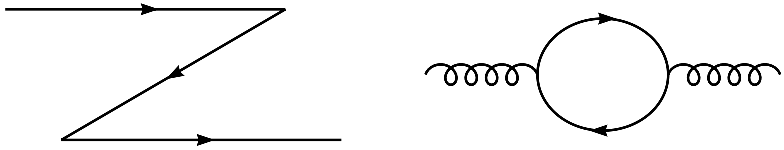

Fig. 44.1. Virtual fluctuations involving pair creation of heavy quarks. Time flows to the right.

where P_+ and P_- are projection operators defined as:

$$P_\pm = \frac{1 \pm \not{v}}{2}\,. \tag{44.4}$$

It follows that

$$Q(x) = e^{-im_Q v\cdot x}\,[h_v(x) + H_v(x)]\,. \tag{44.5}$$

Because of the projection operators, the new fields satisfy $\not{v}h_v = h_v$ and $\not{v}H_v = -H_v$. In the rest frame, i.e. for $v^\mu = (1, 0, 0, 0)$, h_v corresponds to the upper two components of Q, while H_v corresponds to the lower ones. Whereas h_v annihilates a heavy quark with velocity v, H_v creates a heavy antiquark with velocity v.

In terms of the new fields, the QCD Lagrangian (44.1) for a heavy quark takes the form:

$$\mathcal{L}_\mathrm{Q} = \bar{h}_v\, iv\cdot D\, h_v - \bar{H}_v\,(iv\cdot D + 2m_Q)\, H_v + \bar{h}_v\, i\not{D}_\perp H_v + \bar{H}_v\, i\not{D}_\perp h_v\,, \tag{44.6}$$

where $D_\perp^\mu = D^\mu - v^\mu\, v\cdot D$ is orthogonal to the heavy-quark velocity: $v\cdot D_\perp = 0$. In the rest frame, $D_\perp^\mu = (0, \vec{D}\,)$ contains the spatial components of the covariant derivative. From Eq. (44.6), it is apparent that h_v describes massless degrees of freedom, whereas H_v corresponds to fluctuations with twice the heavy-quark mass. These are the heavy degrees of freedom that will be eliminated in the construction of the effective theory. The fields are mixed by the presence of the third and fourth terms, which describe pair creation or annihilation of heavy quarks and antiquarks. As shown in the first diagram in Fig. 44.1, in a virtual process, a heavy quark propagating forward in time can turn into an antiquark propagating backward in time, and then turn back into a quark. The energy of the intermediate quantum state $hh\bar{H}$ is larger than the energy of the incoming heavy quark by at least $2m_Q$. Because of this large energy gap, the virtual quantum fluctuation can only propagate over a short distance $\Delta x \sim 1/m_Q$. On hadronic scales set by $R_\mathrm{had} = 1/\Lambda_\mathrm{QCD}$, the process essentially looks like a local interaction of the form:

$$\bar{h}_v\, i\not{D}_\perp \frac{1}{2m_Q}\, i\not{D}_\perp h_v\,, \tag{44.7}$$

where we have simply replaced the propagator for H_v by $1/2m_Q$. A more correct treatment is to integrate out the small-component field H_v, thereby deriving a non-local effective action for the large-component field h_v, which can then be expanded in terms of local operators. Before doing this, let us mention a second type of virtual corrections involving pair creation, namely heavy-quark loops. An example is shown in the second diagram in

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Fig. 44.1. Heavy-quark loops cannot be described in terms of the effective fields h_v and H_v, since the quark velocities inside a loop are not conserved and are in no way related to hadron velocities. However, such short-distance processes are proportional to the small coupling constant $\alpha_s(m_Q)$ and can be calculated in perturbation theory. They lead to corrections that are added onto the low-energy effective theory in the renormalization procedure.

On a classical level, the heavy degrees of freedom represented by H_v can be eliminated using the equation of motion. Taking the variation of the Lagrangian with respect to the field $\bar{H}_v$, we obtain:

$$(iv \cdot D + 2m_Q)\, H_v = i\not{D}_\perp h_v \,. \tag{44.8}$$

This equation can formally be solved to give:

$$H_v = \frac{1}{2m_Q + iv \cdot D}\, i\not{D}_\perp h_v \,, \tag{44.9}$$

showing that the small-component field H_v is indeed of order $1/m_Q$. We can now insert this solution into Eq. (44.6) to obtain the *non-local effective Lagrangian*:

$$\mathcal{L}_{\text{eff}} = \bar{h}_v\, iv \cdot D\, h_v + \bar{h}_v\, i\not{D}_\perp \frac{1}{2m_Q + iv \cdot D}\, i\not{D}_\perp h_v \,. \tag{44.10}$$

Clearly, the second term corresponds to the first class of virtual processes shown in Fig. 44.1.

One can derive this Lagrangian in a more elegant way using the generating functional for QCD Green functions containing heavy-quark fields [553]. To this end, one starts from the field redefinition in Eq. (44.5) and couples the large-component fields h_v to external sources ρ_v. Green functions with an arbitrary number of h_v fields can be constructed by taking derivatives with respect to ρ_v. No sources are needed for the heavy degrees of freedom represented by H_v. The functional integral over these fields is Gaussian and can be performed explicitly, leading to the effective action:

$$S_{\text{eff}} = \int \mathrm{d}^4x\, \mathcal{L}_{\text{eff}} - i \ln \Delta \,, \tag{44.11}$$

with $\mathcal{L}_{\text{eff}}$ as given in Eq. (44.10). The appearance of the logarithm of the determinant:

$$\Delta = \exp\left(\frac{1}{2}\,\mathrm{Tr}\,\ln[2m_Q + iv \cdot D - i\eta]\right) \tag{44.12}$$

is a quantum effect not present in the classical derivation presented above. However, in this case the determinant can be regulated in a gauge-invariant way, and by choosing the gauge $v \cdot A = 0$ one can show that $\ln \Delta$ is just an irrelevant constant [553,554].

Because of the phase factor in Eq. (44.5), the x dependence of the effective heavy-quark field h_v is weak. In momentum space, derivatives acting on h_v produce powers of the residual momentum k, which is much smaller than m_Q. Hence, the non-local effective Lagrangian

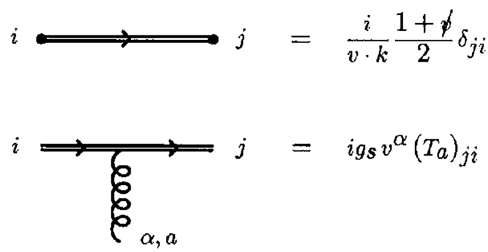

Fig. 44.2. Feynman rules of the HQET (i, j and a are colour indices). A heavy quark with velocity v is represented by a double line. The residual momentum k is defined in Eq. (44.2).

in Eq. (44.10) allows for a derivative expansion:

$$\mathcal{L}_{\text{eff}} = \bar{h}_v \, i v \cdot D \, h_v + \frac{1}{2m_Q} \sum_{n=0}^{\infty} \bar{h}_v \, i \not{D}_\perp \left(-\frac{i v \cdot D}{2m_Q} \right)^n i \not{D}_\perp h_v \, . \tag{44.13}$$

Taking into account that h_v contains a P_+ projection operator, and using the identity

$$P_+ \, i \not{D}_\perp \, i \not{D}_\perp P_+ = P_+ \left[(i D_\perp)^2 + \frac{g}{2} \sigma_{\mu\nu} \, G^{\mu\nu} \right] P_+ \, , \tag{44.14}$$

where $i[D^\mu, D^\nu] = g \, G^{\mu\nu}$ is the gluon field-strength tensor, one finds that [555,556]

$$\mathcal{L}_{\text{eff}} = \bar{h}_v \, i v \cdot D \, h_v + \frac{1}{2m_Q} \bar{h}_v \, (i D_\perp)^2 \, h_v + \frac{g_s}{4m_Q} \bar{h}_v \, \sigma_{\mu\nu} \, G^{\mu\nu} \, h_v + O\big(1/m_Q^2\big) \, . \tag{44.15}$$

In the limit $m_Q \to \infty$, only the first term remains:

$$\mathcal{L}_\infty = \bar{h}_v \, i v \cdot D \, h_v \, . \tag{44.16}$$

This is the effective Lagrangian of the HQET. It gives rise to the Feynman rules shown in Fig. 44.2.

44.3.3 Symmetries of the Lagrangian

Study of these symmetries can be, for example, found in [546]. Since there appear no Dirac matrices, interactions of the heavy quark with gluons leave its spin unchanged. Associated with this is an SU(2) symmetry group, under which $\mathcal{L}_\infty$ is invariant. The action of this symmetry on the heavy-quark fields becomes most transparent in the rest frame, where the generators S^i of SU(2) can be chosen as:

$$S^i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} ; \qquad [S^i, S^j] = i \epsilon^{ijk} S^k \, . \tag{44.17}$$

Here σ^i are the Pauli matrices. An infinitesimal SU(2) transformation $h_v \to (1 + i\vec{\epsilon} \cdot \vec{S}\,) \, h_v$ leaves the Lagrangian invariant:

$$\delta \mathcal{L}_\infty = \bar{h}_v \, [i v \cdot D, i\vec{\epsilon} \cdot \vec{S}\,] \, h_v = 0 \, . \tag{44.18}$$

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Another symmetry of the HQET arises since the mass of the heavy quark does not appear in the effective Lagrangian. For n_h heavy quarks moving at the same velocity, Eq. (44.16) can be extended by writing:

$$\mathcal{L}_\infty = \sum_{i=1}^{n_h} \bar{h}_v^i \, i v \cdot D \, h_v^i \, . \tag{44.19}$$

This is invariant under rotations in flavour space. When combined with the spin symmetry, the symmetry group is promoted to SU($2n_h$). This is the heavy-quark spin-flavour symmetry [557,546]. Its physical content is that, in the limit $m_Q \to \infty$, the strong interactions of a heavy quark become independent of its mass and spin.

Now, let us consider the operators appearing at order $1/m_Q$ in the effective Lagrangian in Eq. (44.15). They can be easily identified in the rest frame. The first operator:

$$\mathcal{O}_{\text{kin}} = \frac{1}{2m_Q} \, \bar{h}_v \, (i D_\perp)^2 \, h_v \to -\frac{1}{2m_Q} \, \bar{h}_v \, (i \vec{D}\,)^2 \, h_v \, , \tag{44.20}$$

is the gauge-covariant extension of the kinetic energy arising from the residual motion of the heavy quark. The second operator is the non-Abelian analogue of the Pauli interaction, which describes the colour-magnetic coupling of the heavy-quark spin to the gluon field:

$$\mathcal{O}_{\text{mag}} = \frac{g_s}{4m_Q} \, \bar{h}_v \, \sigma_{\mu\nu} \, G^{\mu\nu} \, h_v \to -\frac{g_s}{m_Q} \, \bar{h}_v \, \vec{S} \cdot \vec{B}_c \, h_v \, . \tag{44.21}$$

Here $\vec{S}$ is the spin operator defined in (44.17), and $B_c^i = -\frac{1}{2}\epsilon^{ijk} G^{jk}$ are the components of the colour-magnetic field. The chromo-magnetic interaction is a relativistic effect, which scales like $1/m_Q$. This is the origin of the heavy-quark spin symmetry.

44.3.4 Heavy quark wave-function renormalization in HQET

As an illustration of the previous discussion, we consider the heavy quark wave-function renormalization using dimensional regularization in $n = 4 - \epsilon$-space–time, which we have discussed in length in previous sections. For QCD, one introduces renormalized quantities by $Q^{\text{bare}} = Z_Q^{1/2} \, Q^{\text{ren}}$, $A^{\text{bare}} = Z_A^{1/2} A^{\text{ren}}$, $\alpha_s^{\text{bare}} = \mu^{2\epsilon} Z_\alpha \, \alpha_s^{\text{ren}}$, etc., where μ is an arbitrary mass scale introduced to render the renormalized coupling constant dimensionless. Similarly, in the HQET one defines the renormalized heavy-quark field by $h_v^{\text{bare}} = Z_h^{1/2} \, h_v^{\text{ren}}$. From now on, the superscript "ren" will be omitted. In the minimal subtraction MS scheme, Z_h can be computed from the $1/\epsilon$ pole in the heavy-quark self energy using:

$$1 - Z_h^{-1} = \frac{1}{\epsilon} \text{pole of } \frac{\partial \Sigma(v \cdot k)}{\partial v \cdot k} \, . \tag{44.22}$$

As long as $v \cdot k < 0$, the self-energy is IR finite and real. The result is gauge-dependent, however. Evaluating the diagram shown in Fig. 44.3 in the Feynman gauge, we obtain at

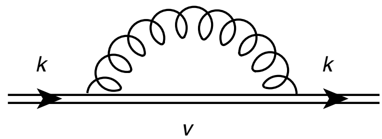

Fig. 44.3. One-loop self-energy $-i\,\Sigma(v \cdot k)$ of a heavy quark in the HQET.

one-loop order:

$$\begin{aligned}\Sigma(v \cdot k) &= -ig_s^2\, t_a t_a \int \frac{\mathrm{d}^n t}{(2\pi)^n}\, \frac{1}{(t^2 + i\eta)[v \cdot (t+k) + i\eta]} \\ &= -2iC_F g_s^2 \int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}^n t}{(2\pi)^n}\, \frac{1}{[t^2 + 2\lambda\, v \cdot (t+k) + i\eta]^2} \\ &= \frac{C_F \alpha_s}{2\pi}\, \Gamma(\epsilon) \int_0^\infty \mathrm{d}\lambda \left(\frac{\lambda^2 + \lambda\omega}{4\pi\mu^2}\right)^{-\epsilon}, \qquad (44.23)\end{aligned}$$

where $C_F = 4/3$ is a colour factor, λ is a dimensionful Feynman parameter, and $\omega = -2v \cdot k > 0$ acts as an IR cutoff. A straightforward calculation leads to:

$$\begin{aligned}\frac{\partial\Sigma(v \cdot k)}{\partial v \cdot k} &= \frac{C_F \alpha_s}{\pi}\, \Gamma(1+\epsilon) \left(\frac{\omega^2}{4\pi\mu^2}\right)^{-\epsilon} \int_0^1 \mathrm{d}z\, z^{-1+2\epsilon}\, (1-z)^{-\epsilon} \\ &= \frac{C_F \alpha_s}{\pi}\, \Gamma(2\epsilon)\, \Gamma(1-\epsilon) \left(\frac{\omega^2}{4\pi\mu^2}\right)^{-\epsilon}, \qquad (44.24)\end{aligned}$$

where we have substituted $\lambda = \omega\,(1-z)/z$. From an expansion around $\epsilon = 0$, we obtain:

$$Z_h = 1 + \frac{C_F \alpha_s}{2\pi\epsilon}\,. \qquad (44.25)$$

This result was first derived by Politzer and Wise [572].

44.3.5 Residual mass term and definition of the heavy quark mass

In the derivation presented earlier in this section, m_Q has been chosen to be the *mass in the Lagrangian*. Using this parameter in the phase redefinition in Eq. (44.5) we obtained the effective Lagrangian in Eq. (44.16), in which the heavy-quark mass no longer appears. However, this treatment has its subtleties. The symmetries of the HQET allow a *residual mass* δm for the heavy quark, provided that δm is of order Λ_{QCD} and is the same for all heavy-quark flavours. Even if we arrange that such a mass term is not present at the tree level, it will in general be induced by quantum corrections. (This is unavoidable if the theory is regulated with a dimensionful cutoff.) Therefore, instead of Eq. (44.16) we should write

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the effective Lagrangian in the more general form [558]:

$$\mathcal{L}_\infty = \bar{h}_v \, i v \cdot D \, h_v - \delta m \, \bar{h}_v h_v \, . \tag{44.26}$$

If we redefine the expansion parameter according to $m_Q \to m_Q + \Delta m$, the residual mass changes in the opposite way: $\delta m \to \delta m - \Delta m$. This implies that there is a unique choice of the expansion parameter m_Q such that $\delta m = 0$. Requiring $\delta m = 0$, as it is usually done implicitly in the HQET, defines a heavy-quark mass, which in perturbation theory coincides with the pole mass [133,147,148]. This, in turn, defines for each heavy hadron H_Q a parameter $\bar{\Lambda}$ (sometimes called the *binding energy*) through

$$\bar{\Lambda} = (m_{H_Q} - m_Q)|_{m_Q \to \infty} \, . \tag{44.27}$$

If one prefers to work with another choice of the expansion parameter, the values of non-perturbative parameters such as $\bar{\Lambda}$ change, but at the same time one has to include the residual mass term in the HQET Lagrangian. However, like the pole mass, the previous definition might be affected by renormalons as we have discussed in previous chapters.

44.4 Hadron spectroscopy from HQET

The spin-flavour symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states [559,560]. In the limit $m_Q \to \infty$, the spin of the heavy quark and the total angular momentum j of the light degrees of freedom are separately conserved by the strong interactions. Because of heavy-quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavour, spin, parity, etc.) of their light degrees of freedom [561]. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm \frac{1}{2}$.

The flavour symmetry relates the properties of states with different heavy-quark flavour.

In general, the mass of a hadron H_Q containing a heavy quark Q obeys an expansion of the form:

$$m_{H_Q} = m_Q + \bar{\Lambda} + \frac{\Delta m^2}{2m_Q} + O(1/m_Q^2) \, . \tag{44.28}$$

The parameter $\bar{\Lambda}$ represents contributions arising from terms in the Lagrangian that are independent of the heavy-quark mass [558], whereas the quantity Δm^2 originates from the terms of order $1/m_Q$ in the effective Lagrangian of the HQET. For the ground-state pseudoscalar and vector mesons, one can parametrize the contributions from the kinetic energy and the chromomagnetic interaction in terms of two quantities λ_1 and λ_2, in such a way that [562]:

$$\Delta m^2 = -\lambda_1 + 2\big[J(J+1) - \tfrac{3}{2}\big] \lambda_2 \, , \tag{44.29}$$

where $J = j \pm 1/2$ is the total spin of the states. The hadronic parameters $\bar{\Lambda}$, λ_1 and λ_2 are independent of m_Q. They characterize the properties of the light constituents.

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Consider, as a first example, the SU(3) mass splittings for heavy mesons. The heavy quark expansion predicts that:

$$\begin{aligned} m_{B_S} - m_{B_d} &= \bar{\Lambda}_s - \bar{\Lambda}_d + O(1/m_b)\,, \\ m_{D_S} - m_{D_d} &= \bar{\Lambda}_s - \bar{\Lambda}_d + O(1/m_c)\,, \end{aligned} \tag{44.30}$$

where we have indicated that the value of the parameter $\bar{\Lambda}$ depends on the flavour of the light quark. Thus, to the extent that the charm and bottom quarks can both be considered sufficiently heavy, the mass splittings should be similar in the two systems. This prediction is confirmed experimentally, since:

$$\begin{aligned} m_{B_S} - m_{B_d} &= (90 \pm 3)\ \text{MeV}\,, \\ m_{D_S} - m_{D_d} &= (99 \pm 1)\ \text{MeV}\,. \end{aligned} \tag{44.31}$$

As a second example, consider the spin splittings between the ground-state pseudoscalar ($J = 0$) and vector ($J = 1$) mesons, which are the members of the spin-doublet with $j = \frac{1}{2}$. From Eqs. (44.28) and (44.29), it follows that

$$\begin{aligned} m_{B^*}^2 - m_B^2 &= 4\lambda_2 + O(1/m_b)\,, \\ m_{D^*}^2 - m_D^2 &= 4\lambda_2 + O(1/m_c)\,. \end{aligned} \tag{44.32}$$

The data are compatible with this prediction:

$$\begin{aligned} m_{B^*}^2 - m_B^2 &\approx 0.49\ \text{GeV}^2\,, \\ m_{D^*}^2 - m_D^2 &\approx 0.55\ \text{GeV}^2\,. \end{aligned} \tag{44.33}$$

Assuming that the B system is close to the heavy-quark limit, one can obtain the value:

$$\lambda_2 \approx 0.12\ \text{GeV}^2 \tag{44.34}$$

for one of the hadronic parameters in Eq. (44.29). This quantity plays an important role in the phenomenology of inclusive decays of heavy hadrons. Similar relations can also be obtained in the case of heavy baryons:

$$\begin{aligned} m_{\Lambda_b} - m_B - \frac{3\lambda_2}{2m_B} &\simeq 311\ \text{MeV}\,, \\ m_{\Lambda_c} - m_D - \frac{3\lambda_2}{2m_D} &\simeq 320\ \text{MeV}\,, \end{aligned} \tag{44.35}$$

which are close to each other to be compared with the data. The dominant correction in Eq. (44.35) comes from the contribution of the chromo-magnetic interaction to the masses of the heavy mesons,[1] which adds a term $3\lambda_2/2m_Q$ on the right-hand side.

The mass formula in Eq. (44.28) can also be used to derive information on the heavy-quark masses from the observed hadron masses. Introducing the 'spin-averaged' meson masses

[1] Because of spin symmetry, there is no such contribution to the masses of Λ_Q baryons.

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$\bar{m}_B = \frac{1}{4}(m_B + 3m_{B^*}) \approx 5.31$ GeV and $\bar{m}_D = \frac{1}{4}(m_D + 3m_{D^*}) \approx 1.97$ GeV, we find that:

$$m_b - m_c = (\bar{m}_B - \bar{m}_D)\left\{1 - \frac{\lambda_1}{2\bar{m}_B\bar{m}_D} + O\left(1/m_Q^3\right)\right\}. \tag{44.36}$$

Using theoretical estimates for the parameter λ_1, which lie in the range (for a complete reference, see e.g. [545]):

$$\lambda_1 = -(0.3 \pm 0.2)\ \mathrm{GeV}^2\,, \tag{44.37}$$

this relation leads to:

$$m_b - m_c = (3.39 \pm 0.03 \pm 0.03)\ \mathrm{GeV}\,, \tag{44.38}$$

where the first error reflects the uncertainty in the value of λ_1, and the second one takes into account unknown higher-order corrections. The fact that the difference of the pole masses, $m_b - m_c$, is known rather precisely is important for the analysis of inclusive decays of heavy hadrons.

44.5 The $\bar{B} \to D^* l \bar{\nu}$ exclusive process

We shall be concerned here with the semi-leptonic decay process $\bar{B} \to D^* l \bar{\nu}$ shown schematically in Fig. 44.4, and which has the largest branching fraction of all B-meson decay modes.

The strength of the $b \to c$ transition vertex is governed by the element V_{cb} of the CKM matrix, which is a fundamental parameter of the Standard Model. A primary goal of the study of semi-leptonic decays of B mesons is to extract with high precision the values of $|V_{cb}|$ (as well as $|V_{ub}|$ for $b \to u$ transitions).

44.5.1 Semi-leptonic form factors: the Isgur–Wise function

Heavy-quark symmetry implies relations between the weak decay form factors of heavy mesons, which are of particular interest. These relations have been derived by Isgur and Wise [557], generalizing ideas developed by Nussinov and Wetzel [563], and by Voloshin and Shifman [564,565].

Consider the elastic scattering of a B meson, $\bar{B}(v) \to \bar{B}(v')$, induced by a vector current coupled to the b quark. Before the action of the current, the light degrees of freedom inside the B meson orbit around the heavy quark, which acts as a static source of colour. On

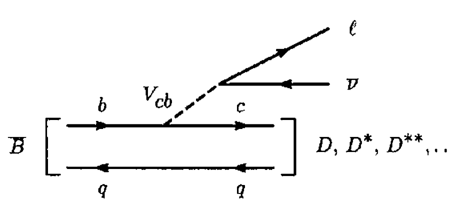

Fig. 44.4. Semi-leptonic decays of B mesons.

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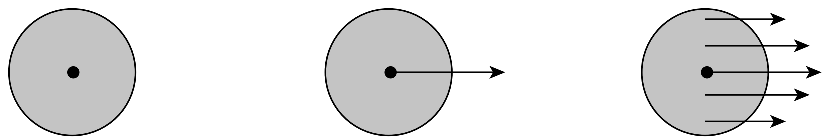

Fig. 44.5. Elastic transition induced by an external heavy-quark current.

average, the b quark and the B meson have the same velocity v. The action of the current is to replace instantaneously (at time $t = t_0$) the colour source by one moving at a velocity v', as indicated in Fig. 44.5. If $v = v'$, nothing happens; the light degrees of freedom do not realize that there was a current acting on the heavy quark. If the velocities are different, however, the light constituents suddenly find themselves interacting with a moving colour source. Soft gluons have to be exchanged to rearrange them so as to form a B meson moving at velocity v'. This rearrangement leads to a form-factor suppression, reflecting the fact that, as the velocities become more and more different, the probability for an elastic transition decreases. The important observation is that, in the limit $m_b \to \infty$, the form factor can only depend on the Lorentz boost $\gamma = v \cdot v'$ connecting the rest frames of the initial- and final-state mesons. Thus, in this limit a dimensionless probability function $\xi(v \cdot v')$ describes the transition. It is called the Isgur–Wise function [557]. In the HQET, which provides the appropriate framework for taking the limit $m_b \to \infty$, the hadronic matrix element describing the scattering process can thus be written as:

$$\frac{1}{m_B} \langle \bar{B}(v')| \, \bar{b}_{v'} \gamma^\mu b_v \, |\bar{B}(v)\rangle = \xi(v \cdot v')\,(v + v')^\mu \,. \tag{44.39}$$

Here b_v and $b_{v'}$ are the velocity-dependent heavy-quark fields of the HQET. It is important that the function $\xi(v \cdot v')$ does not depend on m_b. The factor $1/m_B$ on the left-hand side compensates for a trivial dependence on the heavy-meson mass caused by the relativistic normalization of meson states, which is conventionally taken to be:

$$\langle \bar{B}(p')|\bar{B}(p)\rangle = 2m_B v^0\,(2\pi)^3\,\delta^3(\vec{p} - \vec{p}\,') \,. \tag{44.40}$$

Note that there is no term proportional to $(v - v')^\mu$ in Eq. (44.39). This can be seen by contracting the matrix element with $(v - v')_\mu$, which must give zero since $\not{v} b_v = b_v$ and $\bar{b}_{v'} \not{v}' = \bar{b}_{v'}$.

It is more conventional to write the above matrix element in terms of an elastic form factor $F_{\rm el}(q^2)$ depending on the momentum transfer $q^2 = (p - p')^2$:

$$\langle \bar{B}(v')| \, \bar{b}\, \gamma^\mu b \, |\bar{B}(v)\rangle = F_{\rm el}(q^2)\,(p + p')^\mu \,, \tag{44.41}$$

where $p^{(\prime)} = m_B v^{(\prime)}$. Comparing this with Eq. (44.39), we find that

$$F_{\rm el}(q^2) = \xi(v \cdot v') \,, \qquad q^2 = -2m_B^2(v \cdot v' - 1) \,. \tag{44.42}$$

Because of current conservation, the elastic form factor is normalized to unity at $q^2 = 0$. This condition implies the normalization of the Isgur–Wise function at the kinematic point

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$v \cdot v' = 1$, i.e. for $v = v'$:

$$\xi(1) = 1\,. \tag{44.43}$$

It is in accordance with the intuitive argument that the probability for an elastic transition is unity if there is no velocity change. Since for $v = v'$ the final-state meson is at rest in the rest frame of the initial meson, the point $v \cdot v' = 1$ is referred to as the zero-recoil limit.

The heavy-quark flavour symmetry can be used to replace the b quark in the final-state meson by a c quark, thereby turning the B meson into a D meson. Then the scattering process turns into a weak decay process. In the infinite-mass limit, the replacement $b_{v'} \to c_{v'}$ is a symmetry transformation, under which the effective Lagrangian is invariant. Hence, the matrix element:

$$\frac{1}{\sqrt{m_B m_D}}\,\langle D(v')|\,\bar{c}_{v'}\gamma^\mu b_v\,|\bar{B}(v)\rangle = \xi(v \cdot v')\,(v + v')^\mu \tag{44.44}$$

is still determined by the same function $\xi(v \cdot v')$. This is interesting, since in general the matrix element of a flavour-changing current between two pseudoscalar mesons is described by two form factors:

$$\langle D(v')|\,\bar{c}\,\gamma^\mu b\,|\bar{B}(v)\rangle = f_+(q^2)\,(p + p')^\mu - f_-(q^2)\,(p - p')^\mu\,. \tag{44.45}$$

Comparing the above two equations, we find that:

$$\begin{aligned} f_\pm(q^2) &= \frac{m_B \pm m_D}{2\sqrt{m_B m_D}}\,\xi(v \cdot v')\,, \\ q^2 &= m_B^2 + m_D^2 - 2m_B m_D\,v \cdot v'\,. \end{aligned} \tag{44.46}$$

Thus, the heavy-quark flavour symmetry relates two a priori independent form factors to one and the same function. Moreover, the normalization of the Isgur–Wise function at $v \cdot v' = 1$ now implies a non-trivial normalization of the form factors $f_\pm(q^2)$ at the point of maximum momentum transfer, $q^2_{\text{max}} = (m_B - m_D)^2$:

$$f_\pm\big(q^2_{\text{max}}\big) = \frac{m_B \pm m_D}{2\sqrt{m_B m_D}}\,. \tag{44.47}$$

The heavy-quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons. A vector meson with longitudinal polarization is related to a pseudoscalar meson by a rotation of the heavy-quark spin. Hence, the spin-symmetry transformation $c_{v'}^{\Uparrow} \to c_{v'}^{\Downarrow}$ relates $\bar{B} \to D$ with $\bar{B} \to D^*$ transitions. The result of this transformation is [557]:

$$\begin{aligned} \frac{1}{\sqrt{m_B m_{D^*}}}\,\langle D^*(v', \varepsilon)|\,\bar{c}_{v'}\gamma^\mu b_v\,|\bar{B}(v)\rangle &= i\epsilon^{\mu\nu\alpha\beta}\,\varepsilon^*_\nu\,v'_\alpha v_\beta\;\xi(v \cdot v')\,, \\ \frac{1}{\sqrt{m_B m_{D^*}}}\,\langle D^*(v', \varepsilon)|\,\bar{c}_{v'}\gamma^\mu\gamma_5\,b_v\,|\bar{B}(v)\rangle &= [\varepsilon^{*\mu}\,(v \cdot v' + 1) - v'^\mu\,\varepsilon^* \cdot v]\xi(v \cdot v')\,, \end{aligned} \tag{44.48}$$

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where ε denotes the polarization vector of the D^* meson. Once again, the matrix elements are completely described in terms of the Isgur–Wise function. Now this is even more remarkable, since in general four form factors, $V(q^2)$ for the vector current, and $A_i(q^2)$, $i = 0, 1, 2$, for the axial current, are required to parametrize these matrix elements. In the heavy-quark limit, they obey the relations [566]

$$\begin{aligned}\frac{m_B + m_{D^*}}{2\sqrt{m_B m_{D^*}}}\,\xi(v\cdot v') &= V(q^2) = A_0(q^2) = A_1(q^2)\\ &= \left[1 - \frac{q^2}{(m_B + m_D)^2}\right]^{-1} A_1(q^2)\,,\\ q^2 &= m_B^2 + m_{D^*}^2 - 2m_B m_{D^*}\, v\cdot v'\,. \end{aligned} \tag{44.49}$$

Equations (44.46) and (44.49) summarize the relations imposed by heavy-quark symmetry on the weak decay form factors describing the semi-leptonic decay processes $\bar{B} \to D\,\ell\,\bar{\nu}$ and $\bar{B} \to D^*\ell\,\bar{\nu}$. These relations are model-independent consequences of QCD in the limit where $m_b, m_c \gg \Lambda_{\mathrm{QCD}}$. They play a crucial role in the determination of the CKM matrix element $|V_{cb}|$. In terms of the recoil variable $w = v\cdot v'$, the differential semi-leptonic decay rates in the heavy-quark limit become [567]:

$$\begin{aligned}\frac{\mathrm{d}\Gamma(\bar{B}\to D\,\ell\,\bar{\nu})}{\mathrm{d}w} &= \frac{G_F^2}{48\pi^3}\,|V_{cb}|^2\,(m_B + m_D)^2\, m_D^3\,(w^2-1)^{3/2}\,\xi^2(w)\,,\\ \frac{\mathrm{d}\Gamma(\bar{B}\to D^*\ell\,\bar{\nu})}{\mathrm{d}w} &= \frac{G_F^2}{48\pi^3}\,|V_{cb}|^2\,(m_B - m_{D^*})^2\, m_{D^*}^3\,\sqrt{w^2-1}\,(w+1)^2\\ &\quad\times\left[1 + \frac{4w}{w+1}\,\frac{m_B^2 - 2w\,m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2}\right]\xi^2(w)\,. \end{aligned} \tag{44.50}$$

44.5.2 The Luke's theorem for the $1/m_Q$ corrections

These expressions receive symmetry-breaking corrections, since the masses of the heavy quarks are not infinitely large. Perturbative corrections of order $\alpha_s^n(m_Q)$ can be calculated order-by-order in perturbation theory. A more difficult task is to control the non-perturbative power corrections of order $(\Lambda_{\mathrm{QCD}}/m_Q)^n$. The HQET provides a systematic framework for analysing these corrections. For the case of weak-decay form factors the analysis of the $1/m_Q$ corrections was performed by Luke [568], where, in the zero-recoil limit, an analogue of the Ademollo–Gatto theorem [569] can be proved. This is *Luke's theorem* [568], which states that the matrix elements describing the leading $1/m_Q$ corrections to weak decay amplitudes vanish at zero recoil. This theorem is valid to all orders in perturbation theory [562,570,571], and then protects the $\bar{B} \to D^*\ell\,\bar{\nu}$ decay rate from receiving first-order $1/m_Q$ corrections at zero recoil [567]. A similar statement is not true for the decay $\bar{B} \to D\,\ell\,\bar{\nu}$. The reason is simple but somewhat subtle. Luke's theorem protects only those form factors not multiplied by kinematic factors that vanish for $v = v'$. By angular momentum

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conservation, the two pseudoscalar mesons in the decay $\bar{B} \to D\,\ell\,\bar{\nu}$ must be in a relative p wave, and hence the amplitude is proportional to the velocity $|\vec{v}_D|$ of the D meson in the B-meson rest frame. This leads to a factor $(w^2 - 1)$ in the decay rate. In such a situation, kinematically suppressed form factors can contribute [566]. Later, the authors in [562] have analysed the structure of $1/m_Q^2$ corrections for both meson and baryon weak decay form factors [562].

44.5.3 Short-distance corrections and matching conditions

We have shown previously that HQET reproduces correctly the non-perturbative part of the full theory but does not contain correctly its short-distance part. This can be understood by denoting that the heavy quark only participates to strong interactions through its interaction with gluons, where hard gluons can resolve the spin and flavour quantum numbers of a heavy quark. Their effects lead to a renormalization of the coefficients of the operators in the HQET. A new feature of such short-distance corrections is that through the running coupling constant they induce a logarithmic dependence on the heavy-quark mass [564], which can be calculated in perturbation theory, since $\alpha_s(m_Q)$ is small.

Let us for example, consider the matrix elements of the vector current $V = \bar{q}\,\gamma^\mu Q$. In QCD this current is partially conserved and needs no renormalization. Therefore, its matrix elements are free of UV divergences. Still, these matrix elements have a logarithmic dependence on m_Q from the exchange of hard gluons with virtual momenta of the order of the heavy-quark mass. If one goes over to the effective theory by taking the limit $m_Q \to \infty$, these logarithms diverge. Consequently, the vector current in the effective theory does require a renormalization [572]. Its matrix elements depend on an arbitrary renormalization scale ν, which separates the regions of short- and long-distance physics. If ν is chosen such that $\Lambda_{\rm QCD} \ll \nu \ll m_Q$, the effective coupling constant in the region between ν and m_Q is small, and perturbation theory can be used to compute the short-distance corrections. These corrections have to be added to the matrix elements of the effective theory, which contain the long-distance physics below the scale ν. The relation between matrix elements in the full and in the effective theory is:

$$\langle\, V(m_Q)\,\rangle_{\rm QCD} = C_0(m_Q,\nu)\,\langle V_0(\nu)\rangle_{\rm HQET} + \frac{C_1(m_Q,\nu)}{m_Q}\,\langle V_1(\nu)\rangle_{\rm HQET} + \cdots\,, \quad (44.51)$$

where we have indicated that matrix elements in the full theory depend on m_Q, whereas matrix elements in the effective theory are mass-independent, but do depend on the renormalization scale. The Wilson coefficients $C_i(m_Q,\nu)$ are defined by this relation. Order by order in perturbation theory, they can be computed from a comparison of the matrix elements in the two theories. Since the effective theory is constructed to reproduce correctly the low-energy behaviour of the full theory, this *matching* procedure is independent of any long-distance physics, such as IR singularities, non-perturbative effects, and the nature of the external states used in the matrix elements.

The coefficient functions can be evaluated in perturbation theory using the renormalization group equation. Most of the existing calculations of short-distance corrections in the HQET can be found, for example, in [545].

44.5.4 Determination of $|V_{cb}|$ from HQET

For this purpose, one considers the decay rate given Eq. (44.50), where the Isgur–Wise function $\xi^2(w)$ is replaced by the function $\mathcal{F}(w)$, which takes into account corrections of the order $\alpha_s(m_Q)$ and Λ_{QCD}/m_Q to the Isgur–Wise function. The aim is to measure the quantity $|V_{cb}|\mathcal{F}(w)$ as a function of w, and and to extract $|V_{cb}|$ from an extrapolation of the data to the zero-recoil point $w = 1$, where the B and the D^* mesons have a common rest frame. At this kinematic point, heavy-quark symmetry helps us to calculate the normalization $\mathcal{F}(1)$ with small and controlled theoretical errors. Since the range of w values accessible in this decay is rather small ($1 < w < 1.5$), the extrapolation can be done using an expansion around $w = 1$:

$$\mathcal{F}(w) = \mathcal{F}(1)\,[1 - \hat{\varrho}^2\,(w-1) + \hat{c}\,(w-1)^2 \ldots]\,. \tag{44.52}$$

The slope $\hat{\varrho}^2$ and the curvature $\hat{c}$, and indeed more generally the complete shape of the form factor, are tightly constrained by analyticity and unitarity requirements [573,574]. In the long run, the statistics of the experimental results close to zero recoil will be such that these theoretical constraints will not be crucial to get a precision measurement of $|V_{cb}|$. They will, however, enable strong consistency checks. Measurements of the recoil spectrum have been performed by several experimental groups. Figure 44.6 shows, as an example, the data reported some time ago by the CLEO Collaboration. The weighted average of the

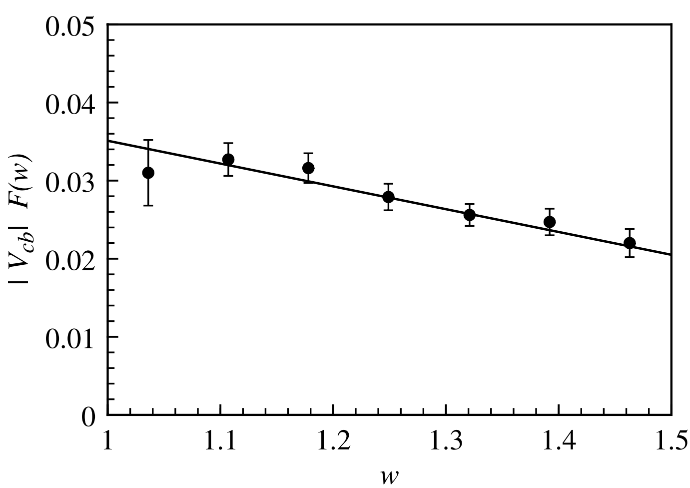

Fig. 44.6. CLEO data for the product $|V_{cb}|\,\mathcal{F}(w)$, as extracted from the recoil spectrum in $\bar{B} \to D^*\ell\,\bar{\nu}$ decays [575]. The line shows a linear fit to the data.

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experimental results is [576]:

$$|V_{cb}|\,\mathcal{F}(1) = (35.2 \pm 2.6) \times 10^{-3}\,. \tag{44.53}$$

Heavy-quark symmetry implies that the general structure of the symmetry-breaking corrections to the form factor at zero recoil is [567]:

$$\mathcal{F}(1) = \eta_A \left(1 + 0 \times \frac{\Lambda_{\rm QCD}}{m_Q} + {\rm const} \times \frac{\Lambda^2_{\rm QCD}}{m_Q^2} + \cdots\right) \equiv \eta_A\,(1 + \delta_{1/m^2})\,, \tag{44.54}$$

where η_A is a short-distance correction arising from the finite renormalization of the flavour-changing axial current at zero recoil, and δ_{1/m^2} parametrizes second-order (and higher) power corrections. The absence of first-order power corrections at zero recoil is a consequence of Luke's theorem [568]. The one-loop expression for η_A has been known for a long time [577,565,578]:

$$\eta_A = 1 + \frac{\alpha_s(M)}{\pi}\left(\frac{m_b + m_c}{m_b - m_c}\ln\frac{m_b}{m_c} - \frac{8}{3}\right) \approx 0.96\,. \tag{44.55}$$

The scale $M \approx 0.51\sqrt{m_b m_c}$ in the running coupling constant can be fixed [579] by adopting the BLM prescription [173]. This lowest order value has been confirmed by the two-loop result [580]:

$$\eta_A|_{2-{\rm loop}} \simeq 0.960 \pm 0.007\,. \tag{44.56}$$

The different analysis of power corrections are more uncertain. The results are in the range:

$$\delta_{1/m^2} \simeq -(0.055 \pm 0.025)\,. \tag{44.57}$$

These different results lead to:

$$\mathcal{F}(1) = 0.91 \pm 0.03 \tag{44.58}$$

for the normalization of the hadronic form factor at zero recoil. Thus, the corrections to the heavy-quark limit amount to a moderate decrease of the form factor of about 10%. This can be used to extract from the experimental result Eq. (44.53) the model-independent value

$$|V_{cb}| = (38.7 \pm 2.8_{\rm exp} \pm 1.3_{\rm th}) \times 10^{-3}\,. \tag{44.59}$$

There are some other predictions on the different form factors which one can obtain in the same way from HQET, and which agree with the still present inaccurate data.

44.6 The inclusive $\bar{B} \to X l \bar{\nu}$ weak process

We have already discussed different inclusive processes ($e^+e^- \to$ hadrons, τ semi-leptonic decays, ...) in the second part of this book. Here, we shall be concerned with the inclusive $\bar{B} \to X l \bar{\nu}$ weak process involving a heavy quark. From a theoretical point of view such decays have two advantages: first, bound-state effects related to the initial state, such as

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the 'Fermi motion' of the heavy quark inside the hadron [581,582], can be accounted for in a systematic way using the heavy-quark expansion; secondly, the fact that the final state consists of a sum over many hadronic channels eliminates bound-state effects related to the properties of individual hadrons. This second feature is based on the hypothesis of quark-hadron duality, which is an important concept in QCD phenomenology. The assumption of duality is that cross-sections and decay rates, which are defined in the physical region (i.e. the region of time-like momenta), are calculable in QCD after a 'smearing' or 'averaging' procedure has been applied [583]. In semi-leptonic decays, it is the integration over the lepton and neutrino phase space that provides a smearing over the invariant hadronic mass of the final state (so-called global duality). For non-leptonic decays, on the other hand, the total hadronic mass is fixed, and it is only the fact that one sums over many hadronic states that provides an averaging (so-called local duality[2]). Clearly, local duality is a stronger assumption than global duality. It is important to stress that quark-hadron duality cannot yet be derived from first principles; still, it is a necessary assumption for many applications of QCD. The success of the QCD predictions for the hadronic τ widths is a strong test of the validity of global duality [325,328,346,345].

Using the optical theorem, the inclusive decay width of a hadron H_b containing a b quark can be written in the form:

$$\Gamma(H_b \to X) = \frac{1}{m_{H_b}} \,\mathrm{Im}\, \langle H_b|\, \mathbf{T}\, |H_b\rangle\,, \tag{44.60}$$

where the transition operator $\mathbf{T}$ is given by:

$$\mathbf{T} = i \int \mathrm{d}^4x\, T\{\mathcal{L}_{\mathrm{eff}}(x), \mathcal{L}_{\mathrm{eff}}(0)\}\,. \tag{44.61}$$

Inserting a complete set of states inside the time-ordered product, we recover the standard expression

$$\Gamma(H_b \to X) = \frac{1}{2m_{H_b}} \sum_X (2\pi)^4\, \delta^4(p_H - p_X)\, |\langle X|\, \mathcal{L}_{\mathrm{eff}}\, |H_b\rangle|^2 \tag{44.62}$$

for the decay rate. For the case of semi-leptonic and non-leptonic decays, $\mathcal{L}_{\mathrm{eff}}$ is the effective Fermi weak Lagrangian, which, in practice is corrected for short-distance effects [550,551,584–586] arising from the exchange of gluons with virtualities between m_W and m_b. In the case of the inclusive semi-leptonic decay rate, for instance, the sum would include only those states X containing a lepton-neutrino pair. In perturbation theory, some contributions to the transition operator are given by the two-loop diagrams shown on the left-hand side in Fig. 44.7. Because of the large mass of the b quark, the momenta flowing through the internal propagator lines are large. It is thus possible to construct an OPE for the transition operator, in which $\mathbf{T}$ is represented as a series of local operators containing the heavy-quark fields. The operator with the lowest dimension, $d = 3$, is $\bar{b}b$. It arises by contracting the internal lines of the first diagram. In the usual OPE, the only gauge-invariant

[2] This terminology may differ with the local duality used in the QCD spectral sum rules analysis which will be discussed in a future part of this book.

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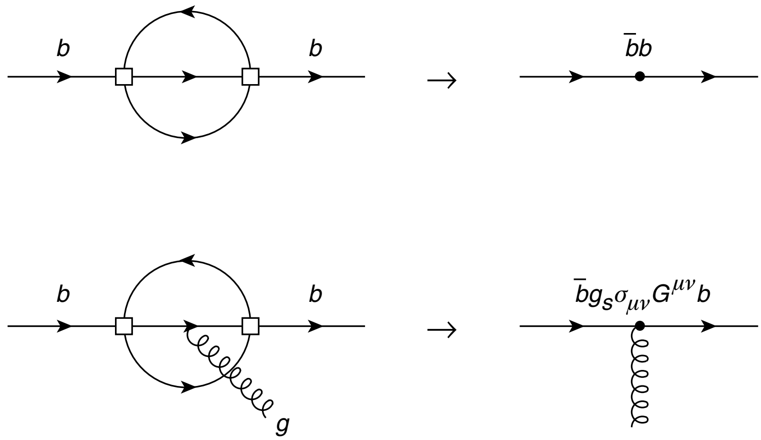

Fig. 44.7. Perturbative contributions to the transition operator **T** (left), and the corresponding operators in the OPE (right). The open squares represent a four-fermion interaction of the effective Lagrangian $\mathcal{L}_{\text{eff}}$, and the black circles represent local operators in the OPE.

operator with dimension four is $\bar{b}\, i\not{D}\, b$; however, the equations of motion imply that between physical states this operator can be replaced by $m_b\bar{b}b$. The first operator that is different from $\bar{b}b$ has dimension five and contains the gluon field. It is given by $\bar{b}\, g_s\sigma_{\mu\nu}G^{\mu\nu}b$. This operator arises from diagrams in which a gluon is emitted from one of the internal lines, such as the second diagram shown in Fig. 44.7. For dimensional reasons, the matrix elements of such higher-dimensional operators are suppressed by inverse powers of the heavy-quark mass. Thus, any inclusive decay rate of a hadron H_b can be written as [587–589]:

$$\Gamma(H_b \to X_f) = \frac{G_F^2 m_b^5}{192\pi^3}\left\{c_3^f\,\langle\bar{b}b\rangle_H + c_5^f\,\frac{\langle\bar{b}\,g_s\sigma_{\mu\nu}G^{\mu\nu}b\rangle_H}{m_b^2} + \cdots\right\}, \tag{44.63}$$

where the prefactor arises naturally from the loop integrations, c_n^f are calculable coefficient functions (which also contain the relevant CKM matrix elements) depending on the quantum numbers f of the final state, and $\langle O\rangle_H$ are the (normalized) forward matrix elements of local operators, for which we use the short-hand notation:

$$\langle O\rangle_H = \frac{1}{2m_{H_b}}\,\langle H_b|\,O\,|H_b\rangle\,. \tag{44.64}$$

In the next step, these matrix elements are systematically expanded in powers of $1/m_b$, using the technology of the HQET. The result is [562,587,589]:

$$\begin{aligned}\langle\bar{b}b\rangle_H &= 1 - \frac{\mu_\pi^2(H_b) - \mu_G^2(H_b)}{2m_b^2} + O\big(1/m_b^3\big)\,,\\ \langle\bar{b}\,g_s\sigma_{\mu\nu}G^{\mu\nu}b\rangle_H &= 2\mu_G^2(H_b) + O(1/m_b)\,,\end{aligned} \tag{44.65}$$

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where we have defined the HQET matrix elements:

$$\mu_\pi^2(H_b) = \frac{1}{2m_{H_b}} \langle H_b(v)|\, \bar{b}_v\, (i\vec{D})^2\, b_v\, |H_b(v)\rangle\,,$$
$$\mu_G^2(H_b) = \frac{1}{2m_{H_b}} \langle H_b(v)|\, \bar{b}_v \frac{g_s}{2}\sigma_{\mu\nu}G^{\mu\nu} b_v\, |H_b(v)\rangle\,. \tag{44.66}$$

Here $(i\vec{D})^2 = (iv\cdot D)^2 - (iD)^2$; in the rest frame, this is the square of the operator for the spatial momentum of the heavy quark. Inserting these results into Eq. (44.63) yields:

$$\Gamma(H_b \to X_f) = \frac{G_F^2 m_b^5}{192\pi^3}\left\{c_3^f\left(1 - \frac{\mu_\pi^2(H_b) - \mu_G^2(H_b)}{2m_b^2}\right) + 2c_5^f\,\frac{\mu_G^2(H_b)}{m_b^2} + \cdots\right\}. \tag{44.67}$$

It is instructive to understand the appearance of the 'kinetic energy' contribution μ_π^2, which is the gauge-covariant extension of the square of the b-quark momentum inside the heavy hadron. This contribution is the field-theory analogue of the Lorentz factor $(1-\vec{v}_b^{\,2})^{1/2} \simeq 1 - \vec{k}^{\,2}/2m_b^2$, in accordance with the fact that the lifetime, $\tau = 1/\Gamma$, for a moving particle increases due to time dilatation.

The main result of the heavy-quark expansion for inclusive decay rates is the observation that the free quark decay (i.e. the parton model) provides the first term in a systematic $1/m_b$ expansion [590]. For dimensional reasons, the corresponding rate is proportional to the fifth power of the b-quark mass. The non-perturbative corrections, which arise from bound-state effects inside the B meson, are suppressed by at least two powers of the heavy-quark mass; thus they are of relative order $(\Lambda_{\rm QCD}/m_b)^2$. Note that the absence of first-order power corrections is a consequence of the equations of motion, as there is no independent gauge-invariant operator of dimension four that could appear in the OPE. The fact that bound-state effects in inclusive decays are strongly suppressed explains a posteriori the success of the parton model in describing such processes [591,592].

The hadronic matrix elements appearing in the heavy-quark expansion in Eq. (44.67) can be determined to some extent from the known masses of heavy hadron states. For the B meson, one finds that:

$$\mu_\pi^2(B) = -\lambda_1 = (0.3 \pm 0.2)\ \text{GeV}^2\,,$$
$$\mu_G^2(B) = 3\lambda_2 \approx 0.36\ \text{GeV}^2\,, \tag{44.68}$$

where λ_1 and λ_2 are the parameters appearing in the mass formula of Eq. (44.29). For the ground-state baryon Λ_b, in which the light constituents have total spin zero, it follows that:

$$\mu_G^2(\Lambda_b) = 0\,, \tag{44.69}$$

while the matrix element $\mu_\pi^2(\Lambda_b)$ obeys the relation:

$$(m_{\Lambda_b} - m_{\Lambda_c}) - (\bar{m}_B - \bar{m}_D) = \left[\mu_\pi^2(B) - \mu_\pi^2(\Lambda_b)\right]\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right) + O\big(1/m_Q^2\big)\,, \tag{44.70}$$

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where $\bar{m}_B$ and $\bar{m}_D$ denote the spin-averaged masses introduced in connection with Eq. (44.36). The above relation implies:

$$\mu_\pi^2(B) - \mu_\pi^2(\Lambda_b) = (0.01 \pm 0.03)\ \mathrm{GeV}^2\,. \tag{44.71}$$

What remains to be calculated, then, is the coefficient functions c_n^f for a given inclusive decay channel.

To illustrate this general formalism, we discuss as an example the determination of $|V_{cb}|$ from inclusive semi-leptonic B decays. In this case the short-distance coefficients in the general expression (44.67) are given by [587–589]

$$\begin{aligned} c_3^{\mathrm{SL}} &= |V_{cb}|^2[1 - 8x^2 + 8x^6 - x^8 - 12x^4 \ln x^2 + O(\alpha_s)]\,, \\ c_5^{\mathrm{SL}} &= -6|V_{cb}|^2(1-x^2)^4\,. \end{aligned} \tag{44.72}$$

Here $x = m_c/m_b$, and m_b and m_c are the masses of the b and c quarks, defined to a given order in perturbation theory [133,147,148]. The $O(\alpha_s)$ terms in c_3^{SL} are known exactly [593], and reliable estimates exist for the $O(\alpha_s^2)$ corrections [594]. The theoretical uncertainties in this determination of $|V_{cb}|$ are quite different from those entering the analysis of exclusive decays. The main sources are the dependence on the heavy-quark masses, higher-order perturbative corrections, and above all the assumption of global quark-hadron duality. A conservative estimate of the total theoretical error on the extracted value of $|V_{cb}|$ yields [595]:

$$|V_{cb}| = (0.040 \pm 0.003)\left[\frac{\mathrm{B}_{\mathrm{SL}}}{10.5\%}\right]^{1/2}\left[\frac{1.6\,\mathrm{ps}}{\tau_B}\right]^{1/2} = (40 \pm 1_{\mathrm{exp}} \pm 3_{\mathrm{th}}) \times 10^{-3}\,. \tag{44.73}$$

The value of $|V_{cb}|$ extracted from the inclusive semi-leptonic width is in excellent agreement with the value in Eq. (44.59) obtained from the analysis of the exclusive decay $\bar{B} \to D^*\ell\,\bar{\nu}$. This agreement is gratifying given the differences of the methods used, and it provides an indirect test of global quark-hadron duality.

Combining the two measurements gives the final result:

$$|V_{cb}| = 0.039 \pm 0.002\,. \tag{44.74}$$

After V_{ud} and V_{us}, this is the third-best known entry in the CKM matrix.

44.7 Rare B decays and CP-violation

One of the main objectives of B-factories is to test the CKM mechanism, which predicts that all CP violation results from a single complex phase in the quark mixing matrix.

Indeed, the determination of the sides and angles of the 'unitarity triangle' $V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0$ depicted in Fig. 44.8 plays a central role in the B factory program. Adopting the standard phase conventions for the CKM matrix, only the two smallest elements in this relation, V_{ub}^* and V_{td}, have non-vanishing imaginary parts (to an excellent approximation). In the standard model the angle $\beta = -\arg(V_{td})$ can be determined in a theoretically clean way by measuring the mixing-induced CP asymmetry in the decays

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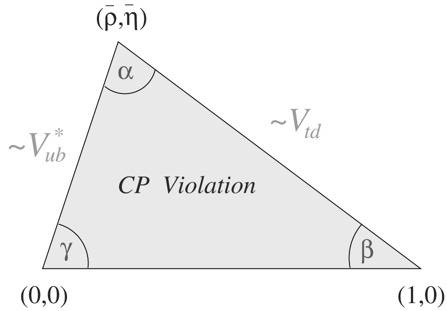

Fig. 44.8. The rescaled unitarity triangle representing the relation $1 + \frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}} + \frac{V_{tb}^* V_{td}}{V_{cb}^* V_{cd}} = 0$. The apex is determined by the Wolfenstein parameters $(\bar{\rho}, \bar{\eta})$. The area of the triangle is proportional to the strength of CP violation in the standard model.

$B \to J/\psi\ K_S$. Recents results from CDF [596] and especially from B-factories: Babar [597] and Belle [598] indicate a large value of β. The angle $\gamma = \arg(V_{ub}^*)$, or equivalently the combination $\alpha = 180^\circ - \beta - \gamma$, is much harder to determine [595]. After the different announcements of evidence for a CP asymmetry in the decays $B \to J/\psi\ K_S$ and by direct CP violation in $K \to \pi\pi$ decays by the KTeV and NA48 groups [599], there are a lot of efforts for investigating theoretically these rare B decay processes. Among others, two competing groups [600,601] work actively on these processes, but they have not yet reached any mutual agreements for their results.

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