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THE MIKI-GESSEL BERNOULLI NUMBER IDENTITY

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Abstract. A generalization due to Gessel [**3**] of Miki's identity between Bernoulli numbers is shown to be a direct consequence of a functional equation for the generating function.

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1. Introduction. Recall that the Bernoulli numbers B_n are defined by the formal power series:

$$\mathcal{T}(X) = \frac{X}{e^X - 1} = \sum_{n\geq 0} \frac{B_n}{n!} X^n \in \mathbb{Q}[[X]].$$

More generally, the Bernoulli polynomials $B_n(\lambda) \in \mathbb{Q}[\lambda]$ are given by:

$$\mathcal{T}_\lambda(X) = \frac{X e^{\lambda X}}{e^X - 1} = \sum_{n\geq 0} \frac{B_n(\lambda)}{n!} X^n \in (\mathbb{Q}[\lambda])[[X]].$$

In [**3**] Gessel established the following generalization of an identity due to Miki [**4**].

PROPOSITION 1.1. (Miki, Gessel *et al.*). *For* $k \geq 1$,

$$\sum_{i+j=k;\, i,j\geq 1} \frac{B_i(\lambda)}{i} \cdot \frac{B_j(\lambda)}{j} = \frac{2}{k}\left(B_k(\lambda)\cdot H_{k-1} + \sum_{r+2s=k;\, s\geq 1} \binom{k}{r} B_r(\lambda)\cdot \frac{B_{2s}}{2s}\right),$$

where H_{k-1} *is the harmonic number* $\sum_{l=1}^{k-1} 1/l$.

The original result of Miki is obtained by setting $\lambda = 0$; the case $\lambda = 1/2$ reduces to an identity of Faber and Pandharipande which appeared in [**2**] (with a proof supplied by D. Zagier). See [**3**] for the details.

We shall deduce Proposition 1.1 from a functional equation for the generating function $\mathcal{T}_\lambda(X)$. The proof is essentially a distillation of an argument given by Dunne and Schubert in the recent paper [**1**].

2. Proof. Let us write

$$\mathcal{L}(X) = \frac{X/2}{\tanh(X/2)} = \frac{X}{2}\left(\frac{e^X + 1}{e^X - 1}\right) = \sum_{m\geq 0} \frac{B_{2m}}{(2m)!} X^{2m} = \mathcal{T}(X) + \frac{X}{2} \in \mathbb{Q}[[X]].$$

(The notation '$\mathcal{T}_\lambda$' and '$\mathcal{L}$' is taken from Topology: for the Todd and Hirzebruch-L genera.)

It is then an elementary exercise to establish the functional equation:

$$\mathcal{T}_\lambda(tX)\cdot\mathcal{T}_\lambda((1-t)X)=\mathcal{T}_\lambda(X)((1-t)\mathcal{L}(tX)+t\mathcal{L}((1-t)X))\quad(t\in\mathbb{R}),\tag{2.1}$$

which may be read as an identity in the ring $(A[\lambda])[[X]]$, where A is the ring of polynomial functions $\mathbb{R}\to\mathbb{R}$.

Now recall that

$$\int_0^1 t^i(1-t)^j\frac{dt}{t(1-t)}=\frac{(i-1)!\,(j-1)!}{(i+j-1)!}\quad\text{for positive integers } i,\ j\geq 1,$$

as follows, for example, from the identity $(X-Y)\int_0^1 \mathrm{e}^{tX}\mathrm{e}^{(1-t)Y}\,dt=\mathrm{e}^X-\mathrm{e}^Y$ in $\mathbb{Q}[[X,Y]]$. It is also easy to verify that, for any integer $k\geq 1$,

$$\int_0^1(-t^k+1-(1-t)^k)\frac{dt}{t(1-t)}=\int_0^1\frac{(t-t^k)+(1-t-(1-t)^k)}{t(1-t)}\,dt=2\sum_{1\leq l<k}\frac{1}{l}=2H_{k-1}.$$

By rearranging the functional equation (2.1) as

$$\begin{aligned}(\mathcal{T}_\lambda(tX)-1)\cdot(\mathcal{T}_\lambda((1-t)X)-1)&=(1-\mathcal{T}_\lambda(tX)+\mathcal{T}_\lambda(X)-\mathcal{T}_\lambda((1-t)X))\\&\quad+\mathcal{T}_\lambda(X)((1-t)(\mathcal{L}(tX)-1)+t(\mathcal{L}((1-t)X)-1)),\end{aligned}$$

we obtain from the coefficient of X^k, for $k\geq 1$, the identity:

$$\begin{aligned}\sum_{i+j=k;\,i,j\geq 1}B_i(\lambda)\cdot B_j(\lambda)\frac{t^i(1-t)^j}{i!\,j!}&=(-t^k+1-(1-t)^k)\frac{B_k(\lambda)}{k!}\\&\quad+\sum_{r+2s=k;\,s\geq 1}\frac{B_r(\lambda)}{r!}\cdot((1-t)t^{2s}+t(1-t)^{2s})\frac{B_{2s}}{(2s)!}\end{aligned}$$

in $A[\lambda]$. Multiplying this identity by $dt/(t(1-t))$ and integrating between 0 and 1 we deduce that

$$\sum_{i+j=k;\,i,j\geq 1}\frac{(i-1)!(j-1)!}{(k-1)!i!\,j!}B_i(\lambda)\cdot B_j(\lambda)=2H_{k-1}\frac{B_k(\lambda)}{k!}+2\sum_{r+2s=k;\,s\geq 1}\frac{(2s-1)!}{(2s)!(2s)!}\cdot\frac{B_r(\lambda)}{r!}\cdot B_{2s}.$$

This equality is easily recast in the form (1.1). □

REMARK 2.2. Further identities, of the type considered in [**1**] involving the gamma function, may be obtained by multiplying (2.1) by $t^x(1-t)^y\,dt/(t(1-t))$ for any choice of real numbers $x,\ y\geq 0$.

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