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NON-ABELIAN TORSION THEORIES

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Torsion theories have proved a very useful tool in the theory of abelian categories; for example, in one proof of Mitchell's embedding theorem (Bucur and Deleanu [**3**]) and in ring theory (Lambek [**5**]). It is the purpose of this paper to initiate an analogous theory for non-abelian categories. Originally we had hoped to prove the non-abelian analogue of Mitchell's theorem this way (Barr, [**2**, Theorem III (1.3)]), but so far this had not been possible. Nonetheless an interesting variety of examples fit into this theory.

In an abelian category $\mathscr{A}$, a torsion theory is given by an idempotent subfunctor N of the identity functor. This means that there is a natural embedding $\epsilon : N \to \mathscr{A}$ (here $\mathscr{A}$ is also used to denote its identity functor; a similar convention applies to objects and functors) such that $N\epsilon : N^2 \to N$ is an isomorphism; this follows from $\epsilon \cdot N\epsilon = \epsilon \cdot \epsilon N$ (by naturality) and the fact that $N\epsilon = \epsilon N$. Thus taking $\delta = (N\epsilon)^{-1}$, it is clear that (N, ϵ, δ) is an idempotent cotriple on $\mathscr{A}$. Intuitively, NA may be thought of as the set of "almost-null elements" of A. In the familiar torsion theory on the category $\mathscr{A}b$ of abelian groups, NA consists of the torsion elements of A. There is a torsion theory on any category of $\mathscr{A}b$ valued presheaves for which NA consists of those sections which vanish identically on some cover. In that case the torsion free objects are the separated presheaves, while the torsion free divisible objects are the sheaves.

When one comes to generalize from abelian to non-abelian contexts, there is always the problem of what to do with exact sequences. In abelian torsion theories a central role is played by the exact sequence

$$0 \to NA \to A \to A/NA \to 0.$$

From the examples above it seems that the quotient A/NA is the more important construction: once we have it, we may then define the notions of torsion free (separated) and torsion free divisible (sheaf) objects. From this point of view NA, the set of quasi-null things, is replaced by the kernel pair of $A \to A/NA$ which may be thought of as the set of pairs of elements of A which are quasi-equal (e.g. differ by a torsion element or which become equal on a covering).

The functor $SA = A/NA$ comes equipped with a natural epimorphism $\eta : \mathscr{A} \to S$. One always supposes that $NS = 0$, which implies that ηS is an isomorphism. As above, $\eta S = S\eta$, so that taking $\mu = (\eta S)^{-1}$, (S, η, μ) is an idempotent triple on $\mathscr{A}$.

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An important property of abelian torsion theories is that when $A \subset A'$, $A \cap NA' = NA$. This is readily seen to be equivalent to the fact that $A \subset A'$ implies $SA \subset SA'$.

A word on notation. We will use the arrows $\twoheadrightarrow$ and $\rightarrowtail$ to denote epimorphisms and regular monomorphisms respectively. These are often used predicatively so that the "$X \twoheadrightarrow SX$", appearing in (1.1) below, is read "the morphism $X \rightarrow SX$ is an epimorphism". The assumptions we will be usually making suffice to guarantee that these two classes give a factorization system.

It will be a standing hypothesis that the category will have whatever finite limits and colimits are required in the proofs. For example, in the main theorem these are equalizers and pushouts.

1. Definitions.

(1.1) Let $\mathscr{X}$ be a category. A *torsion theory* on $\mathscr{X}$ is an idempotent triple $\mathbf{S} = (S, \eta, \mu)$ on $\mathscr{X}$ such that

(i) for all $X \in \mathscr{X}$, $\eta X : X \twoheadrightarrow SX$, and

(ii) for all $X \rightarrowtail X'$, $SX \rightarrowtail SX'$.

(1.2) Let $\mathscr{X}$ be a category. A *semi-topology* is an idempotent triple $\mathbf{Q} = (Q, \alpha, \nu)$ on $\mathscr{X}$ such that whenever $X' \rightarrow X \rightrightarrows X''$ is a coexact sequence in $\mathscr{X}$ (that is both an equalizer and a cokernel pair), then

$$QX' \rightarrow QX \rightrightarrows QX''$$

is an equalizer in $\mathscr{X}$. $\mathbf{Q}$ is called a *topology* if it preserves all finite inverse limits.

(1.3) Let $\mathscr{X}$ be a category and $X \in \mathscr{X}$. An *injective effacement* of X is a morphism $X \rightarrowtail I$ such that every diagram

$$\begin{array}{ccc} X & \rightarrowtail & I \\ \uparrow & & \\ Y & \rightarrowtail & Y' \end{array}$$

can be completed to a commutative square by a map $Y' \rightarrow I$. *$\mathscr{X}$ has injective effacements* if each object of $\mathscr{X}$ has one.

(1.4) Let $\mathbf{S} = (S, \eta, \mu)$ be a torsion theory on $\mathscr{X}$. An object $X \in \mathscr{X}$ is called *torsion free* (TF) if $\eta X : X \rightarrow SX$ is an isomorphism. It is called *torsion free and divisible* (TFD) if it is TF and if every regular mono to another TF object is a regular mono in the full subcategory of TF objects. This means that for every $X \rightarrowtail X'$ where X' is TF, there is a TF object X'' and maps such that

$$X \rightarrow X' \rightrightarrows X''$$

is an equalizer.

2. The main theorem.

(2.1) THEOREM. *Let $\mathscr{X}$ be a category with finite limits and colimits and injective effacements. Then there is a one-one correspondence between torsion theories on $\mathscr{X}$ and semi-topologies on $\mathscr{X}$.*

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The essential idea is to associate to each torsion theory the full subcategory of TFD objects. We show that subcategory is coreflective (the inclusion has a left adjoint) and the resultant triple is a semi-topology. To go the other way, given a semi-topology, i.e. a coreflector with a certain property, we may form the corresponding epi-coreflector (Kennison [**4**]), where coreflectors are called reflectors) and show that this is a torsion theory.

At this point we prove some preliminary propositions.

(2.2) PROPOSITION. *If the square*

$$\begin{array}{ccc} X & \overset{h}{\rightarrowtail} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \underset{f}{\rightarrowtail} & Y \end{array}$$

is a pushout, then f is a regular monomorphism.

Proof. Let

$$Y \overset{k}{\rightarrowtail} I$$

be an injective effacement and choose $l : X' \to I$ with $l \cdot h = k \,.\, g$. The existence of such an l is guaranteed by the definition of injective effacement. There then results a map $\langle k, l \rangle : Y' \to I$ with $\langle k, l \rangle \cdot f = k$, $\langle k, l \rangle \cdot g = l$. In particular, f being an initial factor of the regular monomorphism k, is again one.

(2.3) PROPOSITION. *Every morphism in $\mathscr{X}$ has a factorization, unique up to isomorphism, of the form $\cdot \twoheadrightarrow \cdot \rightarrowtail \cdot$.*

Proof. For a proof, see [**2**, I, (2.3)].

(2.4) PROPOSITION. *Let $\mathscr{X}$ be any category. There is a one-one correspondence between natural equivalence classes of idempotent triples on $\mathscr{X}$ and full replete coreflective subcategories of $\mathscr{X}$ which associates to a triple the category of algebras and to a coreflective subcategory the corresponding triple.*

Proof. Let $\mathbf{T} = (T, \eta, \mu)$ be an idempotent triple. If (X, x) is an algebra, then $x \cdot \eta X = X$, so that x is a split epimorphism. If it were not a monomorphism, neither would $T\eta X \cdot \eta X \cdot x = T(\eta X \cdot x) \cdot T\eta X = T\eta X \cdot Tx \cdot T\eta X = T\eta X$ be, which contradicts the fact that $T\eta X = \mu X^{-1}$ is an isomorphism. Hence x is both a split epimorphism and a monomorphism and thus an isomorphism. Together with $x \cdot \eta X = X$, this implies that ηX is an isomorphism and $x = \eta X^{-1}$. Thus every algebra is of the form $(X, \eta X^{-1})$, where ηX is an isomorphism. Conversely, if ηX is an isomorphism, the equations $\eta X^{-1} \cdot \eta X = X$ and $\eta X^{-1} \cdot T\eta X^{-1} = \eta X^{-1} \cdot \mu X$ imply that $(X, \eta X^{-1})$ is an algebra. Finally, the naturality of η implies that for all $f : X \to Y$, $\eta Y \cdot f = Tf \cdot \eta X$, and that when ηX, ηY are isomorphisms, $f \cdot \eta X^{-1} = \eta Y^{-1} \cdot Tf$, and so f is an algebra morphism. Thus $\mathscr{X}^{\mathbf{T}}$ is the full subcategory of $\mathscr{X}$ consisting of those objects X for which ηX is an isomorphism. Such a subcategory is obviously replete (closed under isomorphism). Conversely, given a full replete coreflective subcategory

$$\mathscr{Y} \underset{J}{\overset{I}{\rightleftarrows}} \mathscr{X},$$

the triple associated to $J \dashv I$ is easily seen to be idempotent, since $Y \in \mathscr{Y}$ implies $JY \cong Y$.

3. Semi-topologies to torsion theories. Let $\mathbf{Q} = (Q, \alpha, \nu)$ be a semi-topology on $\mathscr{X}$ and let $\mathscr{Y}$ be the category of $\mathbf{Q}$-algebras. Let $\mathscr{Z}$ be the full subcategory whose objects are regular subobjects of objects of $\mathscr{Y}$.

(3.1) PROPOSITION. *Each of the inclusions $\mathscr{Y} \subset \mathscr{Z} \subset \mathscr{X}$ is coreflective.*

Proof. The coreflector to $\mathscr{Y} \subset \mathscr{Z}$ is provided by the same Q (restricted to $\mathscr{Y}$). As for $\mathscr{Z} \subset \mathscr{X}$, let SX be defined by factoring αX as

$$X \xrightarrow{\eta X} SX \xrightarrow{\tau X} QX.$$

If $Z \in \mathscr{Z}$ and we choose $Z \rightarrowtail Y$ with $Y \in \mathscr{Y}$, we apply Q to get the commutative square

$$\begin{array}{ccc} Z & \rightarrowtail & Y \\ {\scriptstyle \alpha Z}\downarrow & & \downarrow{\scriptstyle \cong} \\ QZ & \longrightarrow & QY \end{array}$$

from which we see that $\alpha Z : Z \rightarrowtail QZ$. Thus $\mathscr{Z}$ is the full subcategory of objects Z for which αZ is a regular monomorphism. Now if $X \to Y$ is any map with $Y \in \mathscr{Y}$, we have a commutative square

$$\begin{array}{ccccc} X & \twoheadrightarrow & SX & \rightarrowtail & QX \\ \downarrow & & & & \downarrow \\ Y & & \longrightarrow & & QY \end{array}$$

and the diagonal fill-in (possessed by any factorization system, see [**2**, I, (2.6)]) provides a map $SX \to Y$ which makes the diagram commute. The uniqueness comes from the fact that $X \twoheadrightarrow SX$. This shows adjointness.

The functor S and the natural transformation η extend to an idempotent triple $\mathbf{S} = (S, \eta, \mu)$ on $\mathscr{X}$.

(3.2) PROPOSITION. *The triple* $\mathbf{S}$ *is a torsion theory.*

Proof. It is an idempotent triple and obviously $\eta X : X \twoheadrightarrow SX$ for all X. We need only show that S preserves regular monomorphisms. But Q does (since it preserves equalizers in coexact sequences) and S is a regular subfunctor of Q.

This process then gives the correspondence in one direction.

4. Torsion theories to semi-topologies. In the previous section we saw how to construct a torsion theory from a semi-topology. This construction used the existence of injective effacements at only one place and in a very minor way (i.e., it could have been replaced by a far weaker assumption). Here we will make much more serious use of the hypothesis to go in the other direction.

Let $\mathbf{S} = (S, \eta, \mu)$ be a torsion theory on $\mathscr{X}$ and let $\mathscr{Z}$ be the category of $\mathbf{S}$ algebras. Evidently $\mathscr{Z}$ is the full subcategory of TF objects. Let $\mathscr{Y}$ be the full subcategory of TFD objects. We will show that $\mathscr{Y}$ is coreflective and the triple associated to the coreflector will be the desired one.

Throughout this section the arrow $\rightarrowtail$ will be used for a map which is a regular monomorphism in $\mathscr{X}$, regardless of whether it is one in $\mathscr{Z}$.

(4.1) *Definition.* Let $f : Z \rightarrowtail Z'$, where $Z, Z' \in \mathscr{Z}$. We say that f is dense if f is an epimorphism in $\mathscr{Z}$. We say that f is closed if it is a regular monomorphism in $\mathscr{Z}$ (not merely in $\mathscr{X}$).

(4.2) PROPOSITION. *Let $f : Z \rightarrowtail Z'$ in $\mathscr{Z}$. Then f has a factorization as $Z \rightarrowtail \bar{Z} \rightarrowtail Z'$, where the first map is dense and the second closed.*

Proof. A closed map is a regular monomorphism in $\mathscr{X}$, so that any pushout of it is a regular monomorphism in $\mathscr{X}$, hence at least a monomorphism in $\mathscr{Z}$. It follows from [**2**, I, 2.3] that every morphism in $\mathscr{Z}$ has a factorization as an epimorphism followed by a map which is a regular monomorphism in $\mathscr{Z}$. Applying this to $f : Z \rightarrowtail Z'$, we get $f = f_2 \cdot f_1$, where f_1 is an epimorphism in $\mathscr{Z}$ and f_2 is a regular monomorphism in $\mathscr{Z}$. But f_1 is a first factor of a regular monomorphism in $\mathscr{X}$ and thus is one itself, and so f_1 is dense. Evidently f_2 is closed.

(4.3) PROPOSITION. *Let Z be a TF object. Then Z has, in $\mathscr{X}$, an injective effacement $Z \rightarrowtail I$, with I also TF.*

Proof. Choose any injective effacement $Z \rightarrowtail I$. Since S preserves regular monomorphisms,

$$Z \cong SZ \rightarrowtail SI.$$

Now for any $X \to Z$ and $X \rightarrowtail X'$, consider the diagram

$$\begin{array}{ccccc} Z & \rightarrowtail & I & \longrightarrow & SI \\ \uparrow & & \uparrow & & \\ X & \rightarrowtail & X' & & \end{array}$$

where the map $X' \to I$ exists because $Z \to I$ is an injective effacement and the composite $X' \to I \to SI$ gives the required commutative square.

(4.4) PROPOSITION. *Suppose $Z \rightarrowtail I$ is an injective effacement with Z and I TF. If the map is closed, then Z is TFD.*

Proof. Suppose we are given $Z \rightarrowtail Z'$. Consider the diagram

$$\begin{array}{ccc} Z & \rightarrowtail & I \\ \| & & \uparrow \\ Z & \rightarrowtail & Z' \end{array}$$

with the indicated fill-in of $Z' \to I$. Thus $Z \rightarrowtail Z'$ is a first factor of a regular monomorphism (in $\mathscr{Z}$) and hence is one itself.

(4.5) PROPOSITION. *Let $Z \rightarrowtail I$ be an injective effacement where Z and I are TF. Then if $Z \rightarrowtail \bar{Z} \rightarrowtail I$ is its dense/closed factorization, $\bar{Z} \rightarrowtail I$ is also an injective effacement.*

Proof. First consider a diagram

$$\begin{array}{ccccc} Z & \overset{f}{\rightarrowtail} & \bar{Z} & \overset{g}{\rightarrowtail} & I \\ & & \| & & \\ & & \bar{Z} & \overset{h}{\rightarrowtail} & I' \end{array}$$

where $\bar{Z} \rightarrowtail I'$ is an injective effacement. Observe that since $Z \rightarrowtail I$ is an injective effacement, there is a map $k : I' \to I$ with $k \cdot h \cdot f = g \cdot f$. Now $k \cdot h$ and $g : \bar{Z} \rightarrowtail I$ are maps between objects of $\mathscr{Z}$ and f is an epimorphism in $\mathscr{Z}$, so $k \cdot h = g$. The result now follows from consideration of the diagram

$$\begin{array}{ccccc} \bar{Z} & \rightarrowtail & I' & \longrightarrow & I \\ \uparrow & & & & \\ X & \rightarrowtail & X' & & \end{array} .$$

(4.6) PROPOSITION. *Let $Z \rightarrowtail \bar{Z}$ be a dense embedding into a TFD object. Then any map $Z \to Y$ where Y is TFD extends to a unique map $\bar{Z} \to Y$.*

Proof. Consider a diagram

$$\begin{array}{ccc} Z & \rightarrowtail & \bar{Z} \\ \downarrow & & \downarrow \\ Y & \rightarrowtail & I \end{array}$$

where $Y \rightarrowtail I$ is an injective effacement and $\bar{Z} \to I$ is any map making the square commute. Since $Z \rightarrowtail \bar{Z}$ is dense and $Y \rightarrowtail I$ is closed, the usual diagonal fill-in provides a map $\bar{Z} \to Y$ which makes both triangles commute. The uniqueness follows from the fact that $Z \rightarrowtail \bar{Z}$ is an epimorphism in $\mathscr{Z}$.

(4.7) PROPOSITION. *The category $\mathscr{Y}$ of TFD objects is a coreflexive subcategory of $\mathscr{Z}$ and hence of $\mathscr{X}$.*

Proof. The proof is trivial.

Let $\mathbf{Q} = (Q, \eta, \mu)$ be the triple corresponding to the coreflector. Evidently QX is the closure of SX in any injective effacement $SX \rightarrowtail I$ such that $I \in \mathscr{Z}$.

(4.8) PROPOSITION. *Let $Z \rightarrowtail Z' \rightarrowtail I$, where the second map is an injective effacement. Then so is the composite $Z \rightarrowtail I$.*

Proof. We have just to consider a diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Z' & \longrightarrow & I \\ \uparrow & & & & \\ X & & \longrightarrow & & X' . \end{array}$$

(4.9) PROPOSITION. *If $X \rightarrowtail X'$, then $QX \rightarrowtail QX'$.*

Proof. We know that $Y = SX \rightarrowtail Y' = SX'$. If $Y' \rightarrowtail I$ is an injective effacement, then so is $Y \rightarrowtail Y' \rightarrowtail I$. Then we have

$$\begin{array}{ccc} Y & \rightarrowtail & Y' \\ \downarrow & & \downarrow \\ \bar{Y} & \longrightarrow & \bar{Y}' \\ \downarrow & \cong & \downarrow \\ I & \longrightarrow & I \end{array}$$

from which we see that $QX = \bar{Y} \rightarrowtail QX' = \bar{Y}'$.

(4.10) PROPOSITION. *Let*

$$X \rightarrow X' \rightrightarrows X''$$

be a coexact sequence in $\mathscr{X}$. Then

$$QX \rightarrow QX' \rightrightarrows QX''$$

is coexact in $\mathscr{Y}$.

Proof. We know that $QX \rightarrowtail QX'$, and the nature of $\mathscr{Y}$ is such that this is a regular monomorphism in $\mathscr{Y}$ as well. Hence it is the equalizer in $\mathscr{Y}$ of its cokernel pair in $\mathscr{Y}$. The coreflector preserves colimits, so that the cokernel pair of $QX \rightarrow QX'$ is $QX' \rightrightarrows QX''$, and so the sequence is coexact in $\mathscr{Y}$. The inclusion preserves limits, so that the diagram is an equalizer in $\mathscr{X}$.

Combining these results, we have

(4.11) THEOREM. *Let $\mathbf{S} = (S, \eta, \mu)$ be a torsion theory on a category $\mathscr{X}$ which has injective effacements. Then the full subcategory of TFD objects is coreflective; the triple associated with the coreflector is a semi-topology.*

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5. Proof of the main theorem.

(5.1) PROPOSITION. *Let* $\mathbf{Q} = (Q, \alpha, \gamma)$ *be a semi-topology on* $\mathscr{X}$ *and* $\mathbf{S} = (S, \eta, \mu)$ *be the associated torsion theory. Then* $\mathbf{Q}$ *is the semi-topology associated to* $\mathbf{S}$.

Proof. We must show that an object is TFD if and only if it is a $\mathbf{Q}$ algebra. Suppose Y is TFD. Then $Y \cong SY \rightarrowtail QY$, so that $\alpha Y : Y \rightarrowtail QY$. Since Y is TFD, there is an equalizer diagram

$$Y \rightarrowtail QY \rightrightarrows Y'$$

in which Y' is TF. Thus $Y' \to QY'$, from which it follows readily that

$$Y \rightarrowtail QY \rightrightarrows QY'$$

is also an equalizer. Since a coreflective category is closed under inverse limits, if follows that Y must be a $\mathbf{Q}$ algebra as well.

Conversely, let Y be a $\mathbf{Q}$-algebra. Then it is certainly an $\mathbf{S}$-algebra, hence TF. If $Y \rightarrowtail Z$ with Z also TF, then $Y \rightarrowtail Z \rightarrowtail QZ$. This composite is a regular monomorphism in $\mathscr{X}$; hence it is one in $\mathscr{X}^{\mathbf{Q}}$, since $\mathbf{Q}$ is a semi-topology; hence it is one in $\mathscr{X}^{\mathbf{S}}$, since $\mathscr{X}^{\mathbf{Q}}$ is a limit closed sub-category of $\mathscr{X}^{\mathbf{S}}$; and hence, finally, the first factor is a regular monomorphism in $\mathscr{X}^{\mathbf{S}}$. This shows that Y is TFD.

(5.2) PROPOSITION. *Let* $\mathbf{S} = (S, \eta, \mu)$ *be a torsion theory on* $\mathscr{X}$ *and* $\mathbf{Q} = (Q, \alpha, \gamma)$ *be the associated semi-topology. Then* $\mathbf{S}$ *is the torsion theory associated to* $\mathbf{Q}$.

Proof. We must show that an object is TF if and only if $\alpha Z : Z \rightarrowtail QZ$. If Z is TF, QZ is the closure of Z in any injective effacement $Z \rightarrowtail I$ for which I is TF. Thus $\alpha Z : Z \rightarrowtail QZ$. Conversely, if that condition is satisfied, apply S to get

$$\begin{array}{ccc} S & \rightarrowtail & QZ \\ {\scriptstyle \eta Z}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\ SZ & \rightarrowtail & SQZ \end{array}$$

from which we see that ηZ, being an initial factor of a regular monomorphism, is one, and thus is an isomorphism.

This completes the proof of the main theorem. We conclude this section with a theorem on the hereditary property of injective effacements and injectives.

(5.3) THEOREM. *Let* $\mathbf{Q}$ *be a semi-topology on* $\mathscr{X}$. *Then the category* $\mathscr{Y}$ *of* $\mathbf{Q}$ *algebras has injective effacements.*

Proof. Let $Y \in \mathscr{Y}$ and let $Y \rightarrowtail I$ be an injective effacement with I TF. Then it is clear that $Y \rightarrowtail I \rightarrowtail QI$ is an injective effacement in $\mathscr{Y}$.

The interesting question of existence of injective effacements will be the subject of forthcoming papers. It will be shown that in the presence of cogenerators a necessary and sufficient condition can be given in terms of exactness.

6. When is Q a topology?

(6.1) In the abelian case the answer to the above question is "always" — provided S is additive. Here we show that under certain conditions on $\mathscr{X}$ this will happen if and only if S preserves finite products. If we suppose that in $\mathscr{X}$ a product of epimorphisms is an epimorphism, then it easily follows that if Q preserves finite products, so does S. What is interesting is that we give a condition on $\mathscr{X}$ which guarantees that any functor which preserves finite products and regular monomorphisms preserves all finite limits.

(6.2) *Definition.* Let $\mathscr{X}$ be a category such that for any object X, the class of subobjects is a lattice. Then $\mathscr{X}$ is said to have *effective unions* if for any object X and any two regular subobjects $A, B \rightarrowtail X$, the diagram

$$\begin{array}{ccc} A \cap B & \rightarrowtail & A \\ \downarrow & & \downarrow \\ B & \rightarrowtail & A \cup B \end{array}$$

is a pushout, and moreover $A \cup B$ is a regular subobject of X.

(6.3) Note that in a coregular category the intersection $A \cap B$ will automatically be a regular subobject of X. (See this by using the composite map $A \cap B \rightarrowtail A \rightarrowtail X$.)

It is trivial to see that any abelian category has effective unions.

(6.4) Proposition. *Any topos has effective unions.*

Proof. The union of two subobjects A and B of X is the coequalizer of

$$(A + B) \times_X (A + B) \rightrightarrows A + B.$$

Since sums are universal, $(A + B) \times_X (A + B) \cong A \times_X A + A \times_X B + B \times_X A + B \times_X B \cong A + A \cap B + B \cap A + B$. Hence

$$(*) \qquad A + A \cap B + B \cap A + B \underset{d^1}{\overset{d^0}{\rightrightarrows}} A + B \to A \cup B$$

is a coequalizer. Here the map d^0 is on the four summands, respectively: the inclusion $A \to A + B$; the injection $A \cap B \to A$, followed by inclusion $A \to A + B$; the similar map $B \cap A \to B \to A + B$; and the inclusion $B \to A + B$. The map d^1 has the same action on the first and last summand

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and reverses the action on the middle ones. One easily sees that (*) is a coequalizer if and only if

$$A \cap B \rightrightarrows A + B \to A \cup B$$

is. But that being a coequalizer is equivalent to the union being effective.

(6.5) This argument shows that universal sums imply effective unions. Abelian categories, however, do not have universal sums, so the assumption of universal sums is a stronger condition.

(6.6) PROPOSITION. *Let $\mathscr{X}$ have finite limits and the functor $U : \mathscr{X} \to \mathscr{Y}$ preserve finite products and equalizers of split monomorphisms. Then U preserves finite limits.*

Proof. The diagram

$$X \xrightarrow{d} X' \underset{d_1}{\overset{d_0}{\rightrightarrows}} X''$$

is an equalizer if and only if

$$X \xrightarrow{\quad d \quad} X' \underset{(X', d_1)}{\overset{(X', d_0)}{\rightrightarrows}} X' \times X''$$

is. The maps (X', d_0) and (X', d_1) are monos, each split by the product projection. From this one easily sees that U preserves equalizers and then all finite limits.

(6.7) PROPOSITION. *Let $\mathscr{X}$ have finite limits and colimits and effective unions. If the functor $U : \mathscr{X} \to \mathscr{Y}$ preserves finite products, regular monomorphisms, and cokernel pairs, it preserves all finite limits.*

Proof. Consider an equalizer diagram

$$X \xrightarrow{d} X' \underset{d_1}{\overset{d_0}{\rightrightarrows}} X''$$

in which d_0 and d_1 are split—in particular regular—monomorphisms. Then the cokernel pair $X' +_X X'$ is a regular subobject of X'' and we have

$$\begin{array}{ccccc} X & \longrightarrow & X' & \rightrightarrows & X' +_X X' \\ \| & & \| & & \downarrow \\ X & \longrightarrow & X' & \rightrightarrows & X'' \end{array}$$

in which the top row is coexact. If we apply U, we get

$$\begin{array}{ccccc} UX & \longrightarrow & UX' & \rightrightarrows & UX' +_{UX} UX' \\ \| & & \| & & \downarrow \\ UX & \longrightarrow & UX' & \rightrightarrows & UX'' \end{array} .$$

Since $UX \to UX'$, the top row continues to be coexact and the bottom row is then easily seen to be an equalizer.

This shows when a semi-topology $\mathbf{Q}$ preserves products, it is a topology. We wish to show that if $\mathbf{S}$ preserves products, so does $\mathbf{Q}$, for one often knows $\mathbf{S}$ quite explicitly and wishes to deduce properties of $\mathbf{Q}$.

(6.8) PROPOSITION. *Let $\mathscr{X}$ be a category with effective unions. Let $\mathbf{S} = (S, \eta, \mu)$ be a torsion theory on $\mathscr{X}$ and $\mathbf{Q} = (Q, \alpha, \gamma)$ be the associated semi-topology. Then if S preserves products, so does Q and $\mathbf{Q}$ is a topology.*

Proof. Let $Z_1, Z_2 \in \mathscr{Z}$, the category of TF objects. Then $Z_1 \rightarrowtail QZ_1$ and $Z_2 \rightarrowtail QZ_2$ are dense maps and $QZ_1 \times QZ_2$ is also TFD. If we know that $Z_1 \times Z_2 \rightarrowtail QZ_1 \times QZ_2$ is dense, then $Q(Z_1 \times Z_2) = QZ_1 \times QZ_2$. Since the composite of dense maps is dense, it suffices to show, e.g., that $Z_1 \times Z_2 \rightarrowtail Z_1 \times QZ_2$ is dense. Now

$$Z_2 \rightarrowtail QZ_2 \rightrightarrows QZ_2 +_{Z_2} QZ_2$$

is an equalizer of split monomorphisms. This means that Z_2 is the intersection of the two regular embeddings of QZ_2 into the cokernel pair. Then

$$Z_1 \times Z_2 \to Z_1 \times QZ_2 \rightrightarrows Z_1 \times (QZ_2 +_{Z_2} QZ_2)$$

is again an intersection of two regular embeddings, so that

$$(Z_1 \times QZ_2) +_{Z_1 \times Z_2} (Z_1 \times QZ_2) \rightarrowtail Z_1 \times (QZ_2 +_{Z_2} QZ_2),$$

and if we apply S,

$$S(Z_1 \times QZ_2) +_{Z_1 \times Z_2} S(Z_1 \times QZ_2) \rightarrowtail SZ_1 \times (SQZ_2 +_{Z_2} SQZ_2) = Z_1 \times QZ_2,$$

which means that the two maps

$$Z_1 \times QZ_2 \rightrightarrows (Z_1 \times QZ_2) +_{Z_1 \times Z_2} (Z_1 \times QZ_2)$$

become equal when S is applied and hence that $Z_1 \times Z_2 \rightarrowtail QZ_2$ is dense.

7. Examples.

(7.1) *Abelian groups.* The category of abelian groups has the usual torsion theory. The torsion free objects are the usual torsion free groups, while the torsion free divisible objects are again the usual ones. The category of TFD modules is the category of $\mathbf{Q}$-modules and the coreflector is $\mathbf{Q} \otimes -$. More generally, we could consider the category of R-modules and all the possible torsion theories on it. See [**5**] for many examples as well as references to many others.

(7.2) *Set-valued sheaves.* We begin with a small category $\mathscr{M}$ and a topology on $\mathscr{M}$. We use the "classical" definition of a topology as given in [**1**]. That is, we are given for each object $M \in \mathscr{M}$ the knowledge of which sieves $\{M_i \to M\}$ are covers. These are to be closed under pullback and composition and should

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include each identity sieve $M \to M$. Then a presheaf is a functor $F : \mathscr{M}^{op} \to \mathscr{S}$. We define a torsion theory on the functor category $(\mathscr{M}^{op}, \mathscr{S})$ by saying that F is TF if for each cover $\{M_i \to M\}$, the map

$$FM \rightarrowtail \prod FM_i.$$

The reflection SF can be described by SFM as FM modulo the equivalence relation $x \sim x'$ in FM if there is a cover $\{M_i \to M\}$ for which $x|M_i = x'|M_i$ for all i. (This notation means that if $f_i : M_i \to M$ is the given map, then $Ff_i(x) = Ff_i(x')$.) The TFD objects are easily seen to be exactly the sheaves, those F such that given any cover $\{M_i \to M\}$, the sequence

$$FM \to \prod FM_i \rightrightarrows \prod F(M_i \times_M M_j)$$

is an equalizer. One easily checks, using the definition of a topology, that S preserves products and hence that $\mathbf{Q}$ is a topology.

(7.3) Let $\mathscr{X}$ be complete, cocomplete, coregular, and co-wellpowered. Let $\mathscr{J}$ be a class of injectives. The idea of this example is to let SX be the regular image of the natural map

$$X \to \prod_{J \in \mathscr{J}} J^{(X,J)}.$$

This makes sense only when $\mathscr{J}$ is a set (in which case we can drop the co-well-powered-ness hypothesis). When $\mathscr{J}$ is large, we must work harder. Consider each epimorphic image $X \twoheadrightarrow X_k$ for which X_k is a regular subobject of a product of things in $\mathscr{J}$. There is only a set of such X_k, so for each one choose a map $X_k \rightarrowtail \prod_{i \in I_k} J_i$. Then let I be the disjoint union of the I_k. There is a natural map $X \to \prod_{i \in I} J_i$ whose regular image we call SX. We see from the diagonal fill-in in the diagram

$$\begin{array}{ccc} X & \twoheadrightarrow & SX \\ \Big| & & \downarrow \\ \Big| & & \prod_{i \in I} J_i \\ \downarrow & & \downarrow \\ X_k & \rightarrowtail & \prod_{i \in I_k} J_i \end{array}$$

that SX is initial among those epimorphic images of X which are embeddable in products of J's. Now let $X \to X'$. In the definition of SX we have complete latitude in choosing I so long as it is sufficiently large. Let $\mathscr{J}_0$ be the sub*set* of $\mathscr{J}$ consisting of all the objects used to embed both SX and SX'. Then in fact SX is the image of X in $\prod_{J \in \mathscr{J}_0} J^{(X,J)}$ and similarly for X'. We have induced

$$(X', J) \to (X, J),\ J^{(X,J)} \to J^{(X',J)}$$

for each $J \in \mathscr{J}_0$, and then

$$\prod J^{(X,J)} \to \prod J^{(X',J)}.$$

Then the diagonal fill-in in the diagram

$$\begin{array}{ccc} X & \twoheadrightarrow & SX \\ \downarrow & & \downarrow \\ X' & & \prod J^{(X,J)} \\ \downarrow & & \downarrow \\ SX' & \rightarrowtail & \prod J^{(X',J)} \end{array}$$

makes S a functor. It is easily idempotent, for SX is evidently maximal among those of its quotients which are regularly embeddable in a product of J's. Finally $X \rightarrowtail X'$ gives, since the J's are injective, $(X', J) \twoheadrightarrow (X, J)$ and then $J^{(X,J)} \rightarrowtail J^{(X',J)}$, from which it readily follows that $SX \rightarrowtail SX'$. Thus S gives a torsion theory **S**. The TF objects are the regular subobjects of products of J's. The TFD objects are not so readily identifiable. The TFD reflector always exists, however, since any J and product of J's is TF, and an object which is TF and injective is always TFD.

In [**6**] this situation is studied in greater detail. We give examples to show that S may or may not preserve products.

(7.4) Let $\mathscr{X} = \mathscr{S}^G$, where G is a group. Then $\mathscr{X}$ is a topos whose subobject classifier is 2 with trivial G-action. If $\mathscr{J}$ is taken to be $\{2\}$ alone, then powers of 2 are all discrete and subobjects of those are discrete as well. Evidently the TF objects are discrete G-sets and they are all TFD. The coreflector $S = Q$ is the set of orbits functor, and a moment's reflection shows that S does not preserve products; e.g., $SG = 1$, while $S(G \times G)$ is a discrete set with $\#(G)$ elements.

(7.5) If, on the other hand, we let $\mathscr{X}$ be any topos and let $\mathscr{J}$ be any class of injectives closed under internal exponentiation, the resultant S will preserve finite products. To see this, note that exponentiation, being a right adjoint, commutes with products and monomorphisms. Thus if Z is TF, say $Z \rightarrowtail \prod J_i$, then for any X,

$$Z^X \rightarrowtail \prod J_i^X$$

shows that Z^X is TF. Now for any $X, X' \in \mathscr{X}$, both $SX \times SX'$ and $S(X \times X')$ are TF. Then for any TF object Z, we have

$$\begin{aligned} (S(X \times X'), Z) &\cong (X \times X', Z) \cong (X', Z^X) \cong (SX', Z^X) \\ &\cong (X \times SX', Z) \cong (X, Z^{SX'}) \cong (SX, Z^{SX'}) \cong (SX \times SX', Z). \end{aligned}$$

Thus $S(X \times X') \cong SX \times SX'$, and by following the above chain of isomorphisms, we see that it is induced by the natural map.

Note that the resultant coreflector being exact implies that the category of TFD objects is again a topos. This example can be modified to work for elementary toposes as well.

I would like to thank the referee for calling my attention to [**7**] in which related questions are considered, although with the hypothesis of existence of injective envelopes. Note that the example (7.4) provides a counterexample to the statement of Proposition 3, p. 305, of [**7**].

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