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DOMAINS OF PARACOMPACTNESS AND LOCAL COMPACTNESS

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Introduction. Given a class $\mathscr{B}$ of topological spaces and a class $\mathscr{F}$ of mappings of topological spaces, the $\mathscr{F}$*-resolvant* of $\mathscr{B}$ is defined to be the class $\mathscr{R}_{\mathscr{F}}(\mathscr{B})$ of topological spaces all of whose $\mathscr{F}$-images lie in $\mathscr{B}$. Whenever $\mathscr{F}$ is closed under composition and includes identity maps, $\mathscr{R}_{\mathscr{F}}(\mathscr{B})$ is easily seen to be the largest class of spaces smaller than $\mathscr{B}$ which is closed under $\mathscr{F}$-images.

The class $\mathscr{R}_{\mathscr{F}}(\mathscr{B})$ was isolated as an object of study in a recent paper by MacDonald and Willard [**1**]. In the present paper, we continue the investigation begun by these authors. In particular, in Section 1 we provide a counterexample to their conjecture concerning the $\mathscr{F}$-resolvant of the class of paracompact spaces, where $\mathscr{F}$ is the class of quotient mappings. The second section is devoted to a characterization of the $\mathscr{F}$-resolvant of the class of locally compact spaces, where $\mathscr{F}$ represents either the class of closed mappings or the class of hereditarily quotient mappings.

All spaces considered here are assumed to be Hausdorff topological spaces and a mapping is always a continuous surjection. Following [**1**], acc X is used to denote the set of non-isolated (or accumulation) points of a space X.

1. A counterexample. The problem of characterizing the quotient resolvant of the class of paracompact spaces appears to be difficult. Some progress in this direction is related in the following two theorems.

THEOREM 1.1 [**1**]. (a) *If X is regular and* acc X *is Lindelöf, then every regular quotient of X is paracompact.*

(b) *If X is paracompact and each point of a dense subset of* acc X *has a countable base, then every regular quotient of X is paracompact if and only if* acc X *is Lindelöf.*

THEOREM 1.2 [**3**]. *If X is a first countable paracompact space, then the following are equivalent:*

(a) *every Hausdorff quotient of X is paracompact;*

(b) *every Hausdorff quotient of X is regular;*

(c) acc X *is compact, or else X is locally compact and* acc X *is Lindelöf.*

In view of Theorem 1.1, MacDonald and Willard [**1**] conjectured that every regular quotient of X is paracompact only if acc X is Lindelöf. The example

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below, however, shows that such is not the case even for quotient mappings with Hausdorff range.

A covering $\mathscr{A}$ of a space X will be called *compatible with the topology of* X (or simply, *compatible*) if a subset K of X is closed whenever $K \cap A$ is closed for every $A \in \mathscr{A}$. Note that if $\mathscr{A}$ is a closed, closure-preserving covering of X, then it is compatible if and only if X has the weak topology with respect to $\mathscr{A}$ in the sense of Morita [**2**].

Example 1.3. *A paracompact space* X *all of whose Hausdorff quotients are paracompact, although* acc X *is not Lindelöf.*

Let Ω denote the set of ordinals less than or equal to the first uncountable ordinal ω_1, equipped with the topology rendering each point isolated except ω_1. Neighbourhoods of ω_1 will be the usual neighbourhoods in the order topology. Let X be the topological sum of $\aleph_1$-copies of Ω; that is, $X = [0, \omega_1) \times \Omega$ where $[0, \omega_1)$ carries the discrete topology.

That acc X is not Lindelöf is immediate. Suppose f is any quotient mapping of X onto a space Y. Before showing that Y must be paracompact, we shall first establish a few properties of the quotient. Set $\Omega_\alpha = \{\alpha\} \times \Omega$ for each countable ordinal α. It is not difficult to show that every G_δ-subset of X, and hence of Y, is open. Therefore every countable subset of Y is closed. This in turn implies that $f(\Omega_\alpha)$ is closed in Y for each $\alpha < \omega_1$.

Although f need not be a closed mapping, its restriction to each Ω_α is closed. To see this, let F be a closed subset of Ω_α. If $(\alpha, \omega_1) \notin F$, then $f(F)$ is countable and hence closed in Y. So suppose $(\alpha, \omega_1) \in F$. If $f(F)$ is not closed, then every neighbourhood of some $y \in Y - f(F)$ meets $f(F)$ in an uncountable set. But this contradicts the fact that $f(\Omega_\alpha) - U$ is countable for every neighbourhood U of $f(\alpha, \omega_1)$. Hence $f(F)$ is closed.

Since paracompactness is preserved under closed mappings, $f(\Omega_\alpha)$ is paracompact for each $\alpha < \omega_1$. Clearly $\{\Omega_\alpha\}_{\alpha<\omega_1}$ is a closed compatible cover of X, whence $\{f(\Omega_\alpha)\}_{\alpha<\omega_1}$ is a closed compatible cover of Y by paracompact subspaces. Now define $K_\alpha = \bigcup_{\beta<\alpha} f(\Omega_\beta)$, for each $\alpha < \omega_1$. Note that each K_α is closed (since every F_σ-subset of Y is closed) and paracompact (see [**2**, Corollary to Theorem 1]). Therefore $\{K_\alpha\}_{\alpha<\omega_1}$ is a closure-preserving, closed compatible cover of Y by paracompact subspaces. Applying a result due to Morita [**2**, Theorem 1], we conclude that Y is paracompact.

2. Domains of local compactness. In this section we characterize the $\mathscr{F}$-resolvant of the class of locally compact spaces, where $\mathscr{F}$ represents either the class of continuous maps, closed maps or hereditarily quotient maps (with Hausdorff range). The continuous case can be dispensed with immediately, as a corollary to the following theorem.

THEOREM 2.1 [**1**]. *Every continuous Hausdorff image of* X *is regular if and only if* X *is compact.*

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COROLLARY 2.2. *Every continuous Hausdorff image of X is locally compact if and only if X is compact.*

Given a closed subset F of a space X, we say for convenience that a subset U of X is an *almost compact neighbourhood of* F if it is a neighbourhood of F and every closed subset of X that is contained in $U - F$ is compact. It is easy to see that whenever U is an almost compact neighbourhood of F and V is a neighbourhood of F such that $F \subset V \subset U$, then V is likewise an almost compact neighbourhood of F. Moreover, every compact subset of a locally compact Hausdorff space has an almost compact (in fact, compact) neighbourhood base.

A mapping f from a space X onto a space Y is said to be *hereditarily quotient* if and only if for every $y \in Y$ and every neighbourhood U of $f^{-1}y$, $y \in \text{Int} f(U)$. Note that the class of hereditarily quotient mappings contains both the class of open mappings and the class of closed mappings.

The following lemma may be of some independent interest, inasmuch as it provides both an internal and an external characterization of those normal topological spaces having finite remainder in their Stone-Čech compactification.

LEMMA 2.3. *Consider the following assertions about a Hausdorff space X:*

(a) *every hereditarily quotient Hausdorff image of X is locally compact;*

(b) *every continuous closed Hausdorff image of X is locally compact;*

(c) *X is a normal, countably compact space with the property that every closed subset has an almost compact neighbourhood;*

(d) *X is a normal space in which every system of pairwise disjoint closed non-compact subsets is finite;*

(e) *X is a normal space in which, for some positive integer n, there are at most n pairwise disjoint closed non-compact subsets;*

(f) *X is normal and $\beta X - X$ is finite.*

Then (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) $\Rightarrow$ (a) $\Rightarrow$ (b), *and* (b) *implies*

(c′) *X is a normal space with the property that every closed subset has an almost compact neighbourhood, and* acc X *is countably compact.*

Proof. The required implications will be proved in the following sequence: (c) $\Rightarrow$ (d) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (c) and (e) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c′).

(c) $\Rightarrow$ (d): Suppose X satisfies (c). Then if (d) does not hold, there exists a countably infinite collection $\{F_i\}_{i=1}^{\infty}$ of pairwise disjoint closed non-compact subsets of X. We shall first establish (by induction) the existence of a sequence of pairwise disjoint closed subsets of X, each of which is a neighbourhood of some closed non-compact subset of X. To this end, let $F_1' = F_1$. Choose G_1 to be an almost compact, open neighbourhood of F_1' and choose K_1 to be a closed neighbourhood of F_1' contained in G_1. Now suppose that for each $i = 1, \ldots, n$, we have a closed non-compact subset F_i' of F_i, a closed neighbourhood K_i of F_i', and an almost compact, open neighbourhood G_i of F_i' satisfying

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$F_i \subset K_i \subset G_i$ and $K_i \cap K_j = \emptyset$ whenever $i \neq j$. Define

$$F_{n+1}' = F_{n+1} \cap \left(X - \bigcup_{i=1}^{n} \operatorname{Int}(K_i) \right).$$

One then deduces that F_{n+1}' is a non-empty closed, non-compact subset of X. Choose G_{n+1} to be an almost compact, open neighbourhood of F_{n+1}' whose closure misses $\bigcup_{i=1}^{n} K_i$, and choose K_{n+1} to be a closed neighbourhood of F_{n+1}' contained in G_n. We can thus inductively construct a sequence $\{K_i\}_{i=1}^{\infty}$ of pairwise disjoint closed subsets, each K_i being a neighbourhood of a closed non-compact subset F_i'.

To see that the existence of such a collection contradicts (c), define C to be the set of all points x such that every neighbourhood of x meets F_i' for infinitely many i's. Although C is a closed set disjoint from each F_i', every neighbourhood of C must contain all but finitely many F_i', since X is countably compact. But this contradicts the hypothesis that C possesses an almost compact neighbourhood, thus completing the proof that (c) implies (d).

(d) $\Rightarrow$ (f): Suppose X satisfies (d). Since X is normal, βX is just the Wallman compactification of X, so that the points of $\beta X - X$ are in one-to-one correspondence with the free closed ultrafilters on X. Now if $\beta X - X$ is infinite, there are infinitely many distinct, free closed ultrafilters on X. One can then inductively construct a sequence $\{S_i\}_{i=1}^{\infty}$, where $S_i = \{F_{ij} : j = 1, \ldots, i\}$ is a collection of i pairwise disjoint closed non-compact subsets of X and $F_{ij} \subset F_{i-1,j}$. The collection $\{F_{i,i-1}\}_{i=2}^{\infty}$ is then a system whose existence contradicts (d). Thus $\beta X - X$ must be finite.

(f) $\Rightarrow$ (e): If X were a normal space in which there were n pairwise disjoint closed non-compact subsets for each positive integer n, then according to the remarks in the above proof there would be n distinct, free closed ultrafilters on X, hence n points in $\beta X - X$, for each positive integer n. But this contradicts the hypothesis that $\beta X - X$ is finite.

(e) $\Rightarrow$ (c): Suppose X is a normal space in which there are at most n pairwise disjoint closed non-compact subsets. Then every closed non-compact subset of X possesses a closed, almost compact neighbourhood. For otherwise, normality allows us to choose a collection of $n + 1$ pairwise disjoint closed non-compact subsets, contrary to hypothesis. That X is countably compact follows easily, since the existence of a sequence without a cluster point would imply the existence of $n + 1$ pairwise disjoint closed non-compact subsets, a contradiction.

(e) $\Rightarrow$ (a): Suppose X satisfies (e) and possesses at most n pairwise disjoint closed non-compact subsets. Let f be an hereditarily quotient mapping of X onto a space Y. Given an arbitrary point $y \in Y$, we must show that y has a compact neighbourhood. Now by the previous implication, the closed set $f^{-1}(y)$ has a closed, almost compact neighbourhood V. Since $f(V)$ is a neighbourhood of y, the proof will be complete once we show that $f(V)$ is compact. But this follows easily, since if H is any open neighbourhood of y, then $V - f^{-1}(H)$ is compact, whence $f(V) - H$ is compact. Therefore $f(V)$ is compact.

(a) $\Rightarrow$ (b): This implication is obvious, since every continuous closed mapping is hereditarily quotient.

(b) $\Rightarrow$ (c′): Suppose every continuous closed image of X is locally compact. Then X is locally compact and Hausdorff, hence regular. That X must be normal is clear, for otherwise there exist two disjoint closed subsets which cannot be separated by disjoint open sets. By identifying one of these to a point, we obtain a non-regular Hausdorff image of X under a closed mapping, a contradiction.

If acc X is not countably compact, there is an infinite closed subset $A \subset \operatorname{acc} X$ with no accumulation points. Let $f : X \rightarrow Y$ be the quotient mapping obtained by identifying A to a point. It will be shown that Y is not locally compact at the point $f(A) = a$. We may assume $A = \{a_1, a_2, \ldots\}$.

Since X is regular, we may select neighbourhoods $U_1, U_2, \ldots$ of $a_1, a_2, \ldots$ such that $\bar{U}_i \cap \bar{U}_j = \emptyset$ whenever $i \neq j$. Let $U = f(\bigcup_{i=1}^{\infty} U_i)$. Then U is a neighbourhood of a in Y and so contains a compact neighbourhood V of a. But then $f^{-1}V$ is a neighbourhood of A in X, and $f^{-1}V = V_1 \cup V_2 \cup \ldots$ where $V_i = f^{-1}V \cap U_i$. Now V_i is a neighbourhood of a_i for each i, and we can pick a neighbourhood T_i of a_i such that $\bar{T}_i \subsetneq V_i$ for each i. Now let $T = f(\bigcup_{i=1}^{\infty} T_i)$ and let $R_i = f(V_i - \{a_i\})$. Then T and R_i are open subsets of V. Moreover, $V = T \cup \bigcup_{i=1}^{\infty} R_i$ and no subcover of this cover can be found, contradicting the assumption that V is compact. Thus acc X must be countably compact.

Finally, that every closed subset F of X has an almost compact neighbourhood becomes evident if one considers the closed mapping obtained by identifying F to a point.

Since statements (c) and (c′) of Lemma 2.3 are equivalent for a dense-in-itself Hausdorff space X, the following corollary is immediate.

COROLLARY 2.4. *If X is a dense-in-itself Hausdorff space, then all the assertions* (a)–(f) *in Lemma* 2.3 *are equivalent.*

The following theorem has a trivial analog for continuous open maps: every continuous open Hausdorff image of X is locally compact if and only if X is locally compact. In other words, the $\mathscr{F}$-resolvant of the class of locally compact (Hausdorff) spaces is simply the class of locally compact spaces, where $\mathscr{F}$ is the class of continuous open maps.

THEOREM 2.5. *The following statements are equivalent for any Hausdorff space X:*

(a) *every hereditarily quotient Hausdorff image of X is locally compact;*

(b) *every continuous closed Hausdorff image of X is locally compact;*

(c) *X is a normal space which is the topological sum of two spaces X_0 and D, where $\beta X_0 - X_0$ is finite and D consists of points isolated in X.*

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c): Suppose every continuous closed Hausdorff image of X is locally compact. Then by Lemma 2.3, X is normal, acc X is countably compact, and acc X has an almost compact neighbourhood X_0. Easily, X_0 is also normal and countably compact. Now X_0 is open-closed, and so is an almost compact neighbourhood of each of its closed subsets. Hence X_0 satisfies property (c) of Lemma 2.3, implying that $\beta X_0 - X_0$ is finite. Thus X is the topological sum of X_0 and $(X - X_0)$, where $\beta X_0 - X_0$ is finite and $(X - X_0)$ consists of points isolated in X, as required.

(c) $\Rightarrow$ (a): Suppose X satisfies (c) and let f be an hereditarily quotient mapping of X onto a space Y. That the restriction of f to X_0 is also hereditarily quotient can be readily verified. It then follows from the equivalence of statements (a) and (f) in Lemma 2.3 that $f(X_0)$ is locally compact. Now notice that every point $y \in Y - f(X_0)$ is an isolated point in Y, since $f^{-1}y \subset D$ and f is a quotient mapping. Thus Y is the topological sum of a locally compact space and a discrete space, whence Y is locally compact and we are done.

That condition (c) of Theorem 2.5 is necessary for every Hausdorff quotient of X to be locally compact is clear. The author conjectures that it is also sufficient.

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