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Index estimates of compact hypersurfaces in smooth metric measure spaces

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In this article, we investigate the spectra of the stability and Hodge–Laplacian operators on a compact manifold immersed as a hypersurface in a smooth metric measure space, possibly with singularities. Using ideas developed by A. Ros and A. Savo, along with an ingenious computation, we have obtained a comparison between the spectra of these operators. As a byproduct of this technique, we have deduced an estimate of the Morse index of such hypersurfaces.

Keywords: minimal hypersurfaces; Morse index; free boundary; weighted manifold; singular manifolds

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1. Introduction

Given Ω, a domain in $\mathbb{R}^{n+1}$ with a smooth boundary, and a smooth function $f:\Omega\to\mathbb{R}$ that plays the role of a density for a new measure obtained by f and the Riemannian volume μ. We shall be concerned here with free-boundary hypersurfaces M within Ω that are stationary for the weighted area functional when the boundary is subject to the sole constraint $\partial M\subset\partial\Omega$. Such extremal hypersurfaces are interesting in many settings, e.g. minimal or constant mean curvature hypersurfaces, partitioning problems for convex bodies, capillarity problems of fluids, and others; see, for instance, [**1**–**5**, **7**–**11**, **13**, **14**, **16**], and references therein.

Our results pertain to the comparison between the eigenvalues of the stability and Hodge–Laplacian operators on stationary free-boundary hypersurfaces of the weighted area functional. As is well known, we have the notion of the Morse index, which is a nonnegative integer measuring the maximal number of distinct deformations that locally decrease the weighted area up to the second order. Consequently, as a byproduct of the comparison, we obtain an estimate of the Morse index based on the topology of the hypersurfaces. Following some ideas in [**5**, **11**] and [**18**], we obtain several results in the setting of free-boundary compact hypersurfaces, possibly with singularities; see § 2 for details about the notations used in the next results.

Our first main result is as follows:

Theorem A. *Let Ω be a domain in $(\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\,\mathrm{d}\mu)$ with non-empty boundary. Let M^n be a compact f-minimal orientable hypersurface with free boundary in Ω. Assume that M is smooth or has a singular set satisfying $\mathcal{H}^{n-2}(\mathrm{sing}(M)) = 0$, where $\mathcal{H}^{n-2}$ is the $(n-2)$-Hausdorff measure; and that the tensor $\mathrm{Ric}_f^{\Omega} = \mathrm{Hess}\, f$ is bounded from below by a nonnegative constant α. Then,*

(1) *for Ω a convex domain and $k \in \mathbb{N}$:*

$$\lambda_k(L_f) \leqslant -2\alpha + \lambda_{d(k+1)}(\Delta_{fN}^{[1]}),$$

where $d(k+1) = \binom{n+1}{2}k + 1$, $\Delta_{fN}^{[1]}$ is the Hodge f-Laplacian acting on the 1-forms $\omega \in \Omega^1(M)$ satisfying the absolute condition on the boundary, and L_f is the Jacobi operator of the immersion.

(2) *for Ω a f-mean convex domain (i.e. f-mean curvature of $\partial\Omega$ is non-positive), and $k \in \mathbb{N}$:*

$$\lambda_k(L_f) \leqslant -2\alpha + \lambda_{d(k+1)}(\Delta_{fT}^{[1]}),$$

where $d(k) = \binom{n+1}{2}k + 1$ and $\Delta_{fT}^{[1]}$ is the Hodge f-Laplacian acting on the 1-forms $\omega \in \Omega^1(M)$ satisfying the relative condition on the boundary.

The next result compares the Morse index with the genus and the number of boundary components. The result is as follows:

Theorem B. *Let Ω^3 be a f-mean convex domain in $(\mathbb{R}^3, g_{can}, \mathrm{e}^{-f}\,\mathrm{d}\mu)$ with non-empty boundary. Let M^2 be a compact f-minimal orientable surface with r boundary components, genus g, and free boundary in Ω. Assume that M is smooth, and that the tensor $\mathrm{Ric}_f^{\Omega} = \mathrm{Hess}\, f$ is bounded from below by a nonnegative constant α. Then,*

(1)

$$\mathrm{Ind}_f(M) \geqslant \frac{1}{3}\left(2g + r - 1 + \Gamma^{+}_{\Delta_{fT}^{[1]}}(2\alpha)\right),$$

where $\Gamma^{+}_{\Delta_{fT}^{[1]}}(2\alpha)$ is the number of positive eigenvalues of $\Delta_{fT}^{[1]}$ less than 2α;

(2)

$$\mathrm{Ind}_f(M) \geqslant \frac{1}{3}(2g + r - 1) + \Gamma^{-}_{L_f}(-2\alpha),$$

where $\Gamma^{-}_{L_f}(-2\alpha)$ is the number of negative eigenvalues of L_f greater than -2α.

Applying the previous result to the case of free-boundary self-shrinkers of the mean curvature flow in the half-space $\mathbb{R}^3_+$, we obtain the following result:

THEOREM C. *Let M^2 be a free-boundary self-shrinker in the half-space $\mathbb{R}^3_+$. Then,*

$$\mathrm{Ind}_{|x|^2/2}(M) \geqslant \frac{1}{3}(2g + r + 5).$$

The paper is organized as follows: in § 2, we introduce the necessary concepts and basic results used in the paper. In § 3, we present several computations and results about the topology and (co)homology on manifolds. In § 4, we present the main results of the paper. Finally, in § 5, we present two direct applications for self-shrinkers with free boundary.

2. Notations and preliminaries

Here, we establish the notations used to compute and prove the main results of the paper.

2.1. Morse index of f-minimal hypersurfaces with free boundary

In this subsection, we establish the notion of the Morse index in the setting of two-sided f-minimal hypersurfaces with free boundary; for more details, refer to [7].

A hypersurface M^n in $(\overline{M},\, g,\, \mathrm{e}^{-f}\mathrm{d}\mu)$ with boundary $\partial M \subset \partial\overline{M}$ is considered a free-boundary hypersurface if M intersects $\partial\overline{M}$ orthogonally. In other words, if η denotes the unitary conormal vector field of ∂M at $\overline{M}$, pointing outwards, then M is considered a free-boundary hypersurface when η is orthogonal to $T(\partial\overline{M})$.

Given a normal variation M_t associated with the variational field uN, $u \in C^\infty$, the formula for the first variation of the f-volume is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Vol}_f(M_t) = -\int_M uH_f\mathrm{e}^{-f}\mathrm{d}\mu + \int_{\partial M} ug(\eta, N)\mathrm{e}^{-f}\mathrm{d}\sigma,$$

where $H_f = H + \langle N, \overline{\nabla} f\rangle$ is the f-mean curvature of M in $\overline{M}$. Therefore, M is critical (or stationary) for the f-volume if, and only if, M is f-minimal with free boundary. Let M be a f-minimal hypersurface with free boundary, the quadratic form associated with the second variation of the f-volume of M in the direction of the normal field uN is given by

$$\begin{aligned} Q_f(u,u) &:= \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{Vol}_f(M) \\ &= -\int_M uL_f u\mathrm{e}^{-f}\mathrm{d}\mu + \int_{\partial M} u\left(\eta(u) + h^{\partial\overline{M}}(N,N)u\right)\mathrm{e}^{-f}\mathrm{d}\sigma \\ &= \int_M \left(|\nabla u|^2 - (\mathrm{Ric}_f(N,N) + |A|^2)u^2\right)\mathrm{e}^{-f}\mathrm{d}\mu + \int_{\partial M} u^2 h^{\partial\overline{M}}(N,N)\mathrm{e}^{-f}\mathrm{d}\sigma, \end{aligned}$$

where $h^{\partial\overline{M}}(N,\, N) = -\langle\bar{\nabla}_N\nu,\, N\rangle$ is the second fundamental form of $\partial\overline{M}$ in $\overline{M}$ with respect to the outward normal vector field ν, $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f$, $|A|$ is the norm of the second fundamental form of M, and $L_f = \Delta_f + \mathrm{Ric}_f(N,\, N) + |A|^2$ is the

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Jacobi operator of the immersion. We say that $\lambda(L_f)$ is an eigenvalue of L_f with eigenfunction $u \in C^\infty(M)$ if

$$\begin{cases} L_f u + \lambda u = 0 & \text{on } M, \\ \eta(u) + h^{\partial \overline{M}}(N,N)u = 0 & \text{in } \partial M. \end{cases}$$

Notice that the boundary condition makes the Jacobi operator self-adjoint. Therefore, it follows from the classical partial differential equation theory that there is a non-decreasing sequence of eigenvalues $\{\lambda_k\}_{k=1}^\infty$ associated with an orthonormal basis $\{u_k\}_{k=1}^\infty$ of $L^2(M, \mathrm{e}^{-f}\mathrm{d}\mu)$. Recall the Morse index of M, $\mathrm{Ind}_f(M)$, is given by the number of negative eigenvalues of L_f counting with multiplicity.

2.2. f-Harmonic 1-forms on manifolds with boundary

Throughout this paper, we denote the inclusion mapping of ∂M into M by i, and i^* denotes its pullback; d is the outer derivative operator; and $\delta := (-1)^{n(p+1)+1} \star d \star$ is the inner derivative operator, where $\star : \Omega^p(M) \to \Omega^{n-p}(M)$ is the Hodge star operator. In our setting, we have the weighted interior derivative operator defined by $\delta_f = \delta + \iota_{\nabla f}$, where ι_X is the contraction operator from the left hand by X, see [11]. Finally, we have the Hodge f-Laplacian acting on p-forms that is denoted by $\Delta_f^{[p]}$, and defined naturally as

$$\Delta_f^{[p]} := d\delta_f + \delta_f d.$$

Following the standard notation in literature, a p-form ω is called f-harmonic if $d\omega = 0$ and $\delta_f \omega = 0$. We note that on manifolds with boundary the set of p-forms that $\Delta_f^{[p]}\omega = 0$ may be different from the set of f-harmonic p-forms.

Regarding the behaviour of a p-form on the boundary, we say that a p-form ω is normal on the boundary whether $i^*\omega = 0$ or, equivalently, if $\eta \wedge \omega = 0$ on ∂M. Furthermore, ω is said to be tangential on the boundary whether $i^*(\star\omega) = 0$, that is, if $\iota_\eta \omega = 0$ on ∂M. We denote the spaces of the tangent and normal f-harmonic p-forms on the boundary, respectively, by

$$H^p_{Nf}(M) = \{\omega \in \Omega^p(M) : d\omega = 0, \delta_f\omega = 0 \text{ and } \iota_\eta\omega = 0 \text{ on } \partial M\},$$
$$H^p_{Tf}(M) = \{\omega \in \Omega^p(M) : d\omega = 0, \delta_f\omega = 0 \text{ and } i^*\omega = 0 \text{ on } \partial M\}.$$

A p-form ω satisfies the **relative boundary condition** if both ω and $\delta_f\omega$ are normal on the boundary. If ω and $d\omega$ are tangential on the boundary, we say that ω satisfies the **absolute boundary condition**. For the case where $f = 0$, refer to [5].

LEMMA 2.1. *Let $(M, g, \mathrm{e}^{-f}\mathrm{d}\mu)$ be a compact with possible non-empty boundary smooth metric measure space. Given $\omega \in \Omega^p(M)$, we have*

$$\int_M \left(\langle \Delta_f^{[p]}\omega, \omega\rangle - |d\omega|^2 - |\delta_f\omega|^2 \right) \mathrm{e}^{-f}\mathrm{d}\mu = \int_{\partial M} \left(\langle i^*\delta_f\omega, \iota_\eta\omega\rangle - \langle i^*\omega, \iota_\eta \mathrm{d}\omega\rangle\right) \mathrm{e}^{-f}\mathrm{d}\sigma.$$

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Proof. Consider the forms $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^{p+1}(M)$. From [**17**, Chapter 8], we have

$$\langle d(\mathrm{e}^{-f}\alpha), \beta\rangle - \langle \mathrm{e}^{-f}\alpha, \delta\beta\rangle = -\delta(\beta\llcorner(\mathrm{e}^{-f}\alpha)),$$

and using integration by parts, we get:

$$\int_M \langle d(\mathrm{e}^{-f}\alpha), \beta\rangle - \langle \mathrm{e}^{-f}\alpha, \delta\beta\rangle \mathrm{d}\mu = \int_{\partial M} \langle \mathrm{e}^{-f} i^*\alpha, \iota_\eta\beta\rangle \mathrm{d}\sigma = \int_{\partial M} \langle i^*\alpha, \iota_\eta\beta\rangle \mathrm{e}^{-f}\mathrm{d}\sigma.$$

On the contrary, a direct computation yields that

$$\begin{aligned}\int_M \langle d(\mathrm{e}^{-f}\alpha), \beta\rangle - \langle \mathrm{e}^{-f}\alpha, \delta\beta\rangle \mathrm{d}\mu &= \int_M \langle \mathrm{e}^{-f} d\alpha, \beta\rangle + \langle \alpha \wedge \mathrm{d}(\mathrm{e}^{-f}), \beta\rangle - \langle \mathrm{e}^{-f}\alpha, \delta\beta\rangle \mathrm{d}\mu \\ &= \int_M \langle \mathrm{e}^{-f} d\alpha, \beta\rangle - \langle \mathrm{e}^{-f}\alpha, \iota_{\nabla f}\beta\rangle - \langle \mathrm{e}^{-f}\alpha, \delta\beta\rangle \mathrm{d}\mu \\ &= \int_M (\langle d\alpha, \beta\rangle - \langle \alpha, \delta_f\beta\rangle)\, \mathrm{e}^{-f}\mathrm{d}\mu.\end{aligned}$$

Combining the last two equalities, we obtain that

$$\int_M (\langle d\alpha, \beta\rangle - \langle \alpha, \delta_f\beta\rangle)\, \mathrm{e}^{-f}\mathrm{d}\mu = \int_{\partial M} \langle i^*\alpha, \iota_\eta\beta\rangle \mathrm{e}^{-f}\mathrm{d}\sigma.$$

Therefore,

$$\begin{aligned}\int_M \langle \Delta_f^{[p]}\omega, \omega\rangle \mathrm{e}^{-f}\mathrm{d}\mu &= \int_M \langle \delta_f d\omega, \omega\rangle \mathrm{e}^{-f}\mathrm{d}\mu + \int_M \langle d\delta_f\omega, \omega\rangle \mathrm{e}^{-f}\mathrm{d}\mu \\ &= \int_M |d\omega|^2 \mathrm{e}^{-f}\mathrm{d}\mu - \int_{\partial M} \langle i^*\omega, \iota_\eta d\omega\rangle \mathrm{e}^{-f}\mathrm{d}\sigma \\ &\quad + \int_M |\delta_f\omega|^2 \mathrm{e}^{-f}\mathrm{d}\mu + \int_{\partial M} \langle i^*\delta_f\omega, \iota_\eta\omega\rangle \mathrm{e}^{-f}\mathrm{d}\sigma,\end{aligned}$$

for any $\omega \in \Omega^p(M)$ as desired. □

As a direct consequence of lemma 2.1, we can also characterize the spaces $H^p_{Nf}(M)$ and $H^p_{Tf}(M)$ respectively as follows:

$$H^p_{Nf}(M) = \{\omega \in \Omega^p(M) : \Delta_f^{[p]}\omega = 0 \text{ and } \omega \text{ satisfies the absolute boundary condition}\},$$
$$H^p_{Tf}(M) = \{\omega \in \Omega^p(M) : \Delta_f^{[p]}\omega = 0 \text{ and } \omega \text{ satisfies the relative boundary condition}\}.$$

From the Hodge decomposition, it is well known that $H^p_N(M) \cong H^p(M;\mathbb{R})$. Furthermore, the Hodge star operator $\star$ induces an isomorphism between $H^p_N(M)$ and $H^{n-p}_T(M)$. In general, the following isomorphisms hold:

$$H^p_T(M) \cong H^{n-p}_N(M) \cong H^{n-p}(M;\mathbb{R}) \cong H_p(M, \partial M;\mathbb{R}).$$

An important fact is that the Hodge decomposition is still valid on smooth metric measure spaces, see [**6**]. For the case of 1-forms, we have the following isomorphisms:

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- $H^1_N(M) \cong H^1_{Nf}(M)$ via isomorphism $\omega \mapsto \omega + \mathrm{d}u$ for $\omega \in H^1_N(M)$, where $u \in C^\infty(M)$ is a solution to the boundary problem:

$$\begin{cases} \Delta_f u = -\iota_{\nabla f}\omega & \text{on } M, \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{in } \partial M. \end{cases}$$

- $H^1_T(M) \cong H^1_{Tf}(M)$ via isomorphism $\omega \mapsto \omega + \mathrm{d}u$ for $\omega \in H^1_T(M)$, where $u \in C^\infty(M)$ is a solution to the boundary problem:

$$\begin{cases} \Delta_f u = -\iota_{\nabla f}\omega & \text{on } M, \\ \dfrac{\partial u}{\partial \eta} = -\iota_\eta \omega, u = 0 & \text{in } \partial M. \end{cases}$$

In particular, the dimension of $H^1_{Nf}(M)$ is equal to $\dim H^1(M;\mathbb{R})$. Throughout the paper, we use the isomorphisms $H^1_{Nf}(M) \cong H^1(M;\mathbb{R})$ and $H^1_{Tf}(M) \cong H^{n-1}(M;\mathbb{R})$.

2.3. Singular manifolds

In this subsection, we will set up some terminology inspired by the discussion found in [**18**, Sections 1 and 2]. Let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional connected, compact, orientable Riemannian manifold and $M \subset \overline{M}$ a closed subset. The regular part of M is defined as

$$\operatorname{reg}(M) := \{x \in M : M \text{ is a smooth embedded hypersurface near } x\},$$

and the singular part is $\operatorname{sing}(M) := M \setminus \operatorname{reg}(M)$. Clearly, the regular part $\operatorname{reg}(M)$ is an open subset of M.

By a singular hypersurface with a singular set of Hausdorff codimension no less than k, $k \in \mathbb{N}$ and $k < n$, we mean a closed subset M of $\overline{M}$ with finite n-dimensional Hausdorff measure $\mathcal{H}^n(M) < \infty$ and the $(n-k+\varepsilon)$-dimensional Hausdorff measure $\mathcal{H}^{n-k+\varepsilon}(\operatorname{sing}(M)) = 0$, for all $\varepsilon > 0$. Later on, we will denote $M = \operatorname{reg}(M)$ and also call M a singular hypersurface; see [**18**] and references therein for more details.

Definition 2.2.

(1) *A singular minimal hypersurface M (with $\dim \operatorname{sing}(M) \leqslant n-7$) is called connected if its regular part is connected.*

(2) *A singular hypersurface M is called orientable (or non-orientable) if the regular part is orientable (or non-orientable).*

(3) *A singular hypersurface M is called two-sided if the normal bundle $N(M)$ of the regular part M inside $\overline{M}$ is trivial.*

Lemma 2.3 ([**18**], Lemma 2.6). *Let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional, connected, compact, orientable manifold, and $M \subset \overline{M}$ a connected, singular hypersurface with*

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$\dim \operatorname{sing}(M) \leqslant n-2$, *and with compact closure* $\bar{M}$. *Then* M *is orientable if and only if* M *is two-sided.*

To study the Morse index on hypersurfaces with singularities, we must use test functions that allow us to deal with singularities in the moment of integrate. For this purpose, it will be necessary to present the cut-off functions given in the following proposition, which was originally created by Morgan and Ritoré in [**12**], and reproduced by Zhu in [**19**].

PROPOSITION 2.4. *Let* M^n *be a smooth submanifold embedded in* $\mathbb{R}^N$ *with bounded mean curvature and compact closure* $\bar{M}$. *If* $\operatorname{sing}(M) = \bar{M} \setminus M$ *satisfies* $\mathcal{H}^{n-2}(\operatorname{sing}(M)) = 0$, *then for every* $\varepsilon > 0$, *there is a smooth function* $\rho_\varepsilon : \bar{M} \to [0, 1]$ *supported in* M *such that:*

(1) $\mathcal{H}^n(\{\rho_\varepsilon \neq 1\}) < \varepsilon$;

(2) $\int_M |\nabla \rho_\varepsilon|^2 < \varepsilon$;

(3) $\int_M |\Delta \rho_\varepsilon| < \varepsilon$.

3. Main lemmas

In this section, we recall some fundamental formulas for our computations. You can refer to [**11**, **16**] and [**15**] for these well-known formulas.

Let M be a hypersurface in the smooth metric measure space $(\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\,\mathrm{d}\mu)$. From now on, we denote the set of parallel vector fields on $\mathbb{R}^{n+1}$ as $\overline{\mathbf{P}}$. Given $\overline{V} \in \overline{\mathbf{P}}$, consider the orthogonal decomposition:

$$\overline{V} = V + \langle V, N\rangle\, N,$$

where V is the orthogonal projection of $\overline{V}$ onto TM.

For each pair of parallel vector fields $\overline{V}, \overline{W} \in \overline{\mathbf{P}}$, define a vector field on M by the following expression: $X_{\overline{V},\overline{W}} := \left\langle \overline{V}, N\right\rangle W - \left\langle \overline{W}, N\right\rangle V$. The test functions used here are obtained by taking the inner product of $X_{\overline{V},\overline{W}}$ with appropriate vector fields ξ on M. That is,

$$u := \left\langle X_{\overline{V},\overline{W}}, \xi \right\rangle. \tag{3.1}$$

In general, ξ will be chosen as a f-harmonic vector field or an eigenvector field of the Hodge f-Laplacian.

LEMMA 3.1 [**11**]. *Let* $f \in C^\infty(\mathbb{R}^{n+1})$ *and let* $x : M^n \to \mathbb{R}^{n+1}$ *be an* f*-minimal hypersurface. Let* $\xi \in TM$ *be a vector field on* M *and* u *the function defined in* (3.1). *Then*

$$L_f u = -u \operatorname{Hess} f(N, N) - \operatorname{Hess} f(X_{\overline{V},\overline{W}}, \xi) + \left\langle X_{\overline{V},\overline{W}}, \Delta_f^{[1]} \xi \right\rangle + v,$$

where $v = 2(\langle \nabla_{AV}\xi, W\rangle - \langle \nabla_{AW}\xi, V\rangle) - \langle W, \xi\rangle \operatorname{Hess} f(V, N) + \langle V, \xi\rangle \operatorname{Hess} f(W, N)$.

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Let $\overline{\mathbf{U}} = \{\overline{V} \in \overline{P} : |\overline{V}| \equiv 1\}$. Then, $\overline{\mathbf{U}}$ can be identified with $\mathbb{S}^n$, and we endow it with the measure $\mu := ((n+1)/\mathrm{Vol}(\mathbb{S}^n))\mathrm{d}V_{\mathbb{S}^n}$. A direct computation yields us:

LEMMA 3.2. *For any $\overline{X}$, $\overline{Y} \in \mathbb{R}^{n+1}$:*

$$\int_{\overline{U}} \left\langle \overline{V}, \overline{X} \right\rangle \left\langle \overline{V}, \overline{Y} \right\rangle \mathrm{d}\mu(\overline{V}) = \left\langle \overline{X}, \overline{Y} \right\rangle.$$

In the next two lemmas, we assume that M is an f-minimal hypersurface with free boundary in a domain $\Omega \subset (\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\mathrm{d}\mu)$ with a non-empty boundary.

LEMMA 3.3. *Suppose ω is an 1-form satisfying the absolute condition on the boundary. Then, at a point $p \in \partial M$:*

$$\left\langle \nabla_\eta \omega^\sharp, \omega^\sharp \right\rangle = h^{\partial\Omega}(\omega^\sharp, \omega^\sharp).$$

Proof. See Lemma 3.2 of [1]. □

LEMMA 3.4. *Suppose ω is a co-closed 1-form ($\delta_f \omega = 0$) satisfying the relative condition on the boundary. Then, at a point $p \in \partial M$:*

$$\left\langle \nabla_\eta \omega^\sharp, \omega^\sharp \right\rangle = H_f^{\partial M} |\omega^\sharp|^2.$$

Proof. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal frame of ∂M. The hypothesis that ω satisfies the relative boundary condition implies that $\omega^\sharp = \alpha\eta$, for some function $\alpha : \partial M \to \mathbb{R}$. Then, on ∂M:

$$\begin{aligned} 0 = \delta_f \omega &= \sum_{j=1}^{n-1} \left\langle \nabla_{e_j} \omega^\sharp, e_j \right\rangle - \left\langle \nabla_\eta \omega^\sharp, \eta \right\rangle + \left\langle \nabla f, \omega^\sharp \right\rangle \\ &= \alpha H^{\partial M} - \left\langle \nabla_\eta \omega^\sharp, \eta \right\rangle + \alpha \left\langle \nabla f, \eta \right\rangle = \alpha H_f^{\partial M} - \left\langle \nabla_\eta \omega^\sharp, \eta \right\rangle. \end{aligned}$$

Therefore, $\left\langle \nabla_\eta \omega^\sharp, \omega^\sharp \right\rangle = \alpha^2 H_f^{\partial M} = H_f^{\partial M} |\omega^\sharp|^2$. □

Furthermore, we have the following information about the relative homology group:

LEMMA 3.5. *Let M^n be a compact, orientable (connected) n-dimensional manifold with non-empty boundary ∂M, $n \geqslant 2$. If ∂M has $r \geqslant 1$ boundary components, then*

$$\dim H_1(M, \partial M; \mathbb{R}) = r - 1 + (\dim H_1(M; \mathbb{R}) - \dim Im(i_*)),$$

where $i_* : H_1(\partial M; \mathbb{R}) \to H_1(M; \mathbb{R})$ denotes the map between first homology groups induced by the inclusion $i : \partial M \to M$.

And

LEMMA 3.6. *Let M^2 be a compact, orientable surface with non-empty boundary ∂M. If M has genus g and $r \geqslant 1$ boundary components, then*

$$\dim H_1(M, \partial M; \mathbb{R}) = 2g + r - 1.$$

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4. Main results

Let Ω be a domain, not necessarily compact, in $(\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\mathrm{d}\mu)$ with a non-empty boundary. Assume that $\partial\Omega$ is smooth, and let ν be the unitary normal vector field of $\partial\Omega$ in Ω, pointing outwards. We recall that the second fundamental form and the mean curvature of $\partial\Omega$ are defined as follows:

$$h^{\partial\Omega}(X,Y) = \langle AX, Y\rangle, X, Y \in T\partial\Omega, \text{ and } H^{\partial\Omega} = trh^{\partial\Omega},$$

where $A = -D\nu$ is the shape operator. We define Ω as a convex domain if $h^{\partial\Omega}(X, X) \leqslant 0$ for all $X \in T(\partial\Omega)$. Similarly, we will define Ω as an f-mean convex domain if $H_f^{\partial\Omega} \leqslant 0$.

Before stating the main theorems, we prove the following lemma:

Lemma 4.1. *Let $\Omega \subset M$ be a bounded domain, $\varphi \in C_0^\infty(\Omega)$ and $u \in C^\infty(\Omega)$. Let $L_f = \Delta_f + T$ be a Schrodinger operator, where $T \in C^\infty(\Omega)$ is a potential function, then*

$$\int_\Omega \varphi u L_f(\varphi u)\mathrm{e}^{-f}\mathrm{d}\mu = \int_\Omega \left(\varphi^2 u L_f(u) + u^2|\nabla\varphi|^2\right)\mathrm{e}^{-f}\mathrm{d}\mu - \int_{\partial M\cap\Omega} u^2\varphi\eta(\varphi)\mathrm{e}^{-f}\mathrm{d}\sigma.$$

Proof. It follows directly from the identity $\Delta_f(\varphi u) = \varphi\Delta_f(u) + u\Delta_f(\varphi) + 2\langle\nabla\varphi, \nabla u\rangle$ that

$$\int_\Omega \varphi u L_f(\varphi u)\mathrm{e}^{-f}\mathrm{d}\mu = \int_\Omega \left(\varphi^2 u L_f(u) + u^2\varphi\Delta_f(\varphi) + 2\varphi u\langle\nabla\varphi, \nabla u\rangle\right)\mathrm{e}^{-f}\mathrm{d}\mu.$$

Consider the function $h := f - \ln(v^2)$, where $v^2 = u^2 + c$, for $c > 0$. Using the divergence theorem, we obtain

$$\begin{aligned}
\int_\Omega v^2|\nabla\varphi|^2\mathrm{e}^{-f}\mathrm{d}\mu &= \int_\Omega |\nabla\varphi|^2\mathrm{e}^{-h}\mathrm{d}\mu \\
&= \int_\Omega \varphi\Delta_h(\varphi)\mathrm{e}^{-h}\mathrm{d}\mu + \int_{\partial M\cap\Omega}\varphi\eta(\varphi)\mathrm{e}^{-h}\mathrm{d}\sigma \\
&= \int_\Omega \varphi\left(\Delta_f(\varphi) + \left\langle\nabla\ln(v^2), \nabla\varphi\right\rangle\right)v^2\mathrm{e}^{-f}\mathrm{d}\mu \\
&\quad + \int_{\partial M\cap\Omega}\varphi\eta(\varphi)v^2\mathrm{e}^{-f}\mathrm{d}\sigma \\
&= \int_\Omega \left(v^2\varphi\Delta_f(\varphi) + 2\varphi u\langle\nabla u, \nabla\varphi\rangle\right)\mathrm{e}^{-f}\mathrm{d}\mu \\
&\quad + \int_{\partial M\cap\Omega} v^2\varphi\eta(\varphi)\mathrm{e}^{-f}\mathrm{d}\sigma,
\end{aligned}$$

and making c goes to zero, and plugging that in the previous equality we conclude the desired result. □

The main result of this section is as follows:

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Theorem 4.2. *Let Ω^{n+1} be a domain in $(\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\mathrm{d}\mu)$ with non-empty boundary. Let M^n be a compact f-minimal orientable hypersurface with free boundary in Ω. Assume that M is smooth or has a singular set satisfying $\mathcal{H}^{n-2}(\mathrm{sing}(M)) = 0$; and that the tensor $\mathrm{Ric}_f^{\Omega} = \mathrm{Hess}\, f$ is bounded from below by a nonnegative constant α. Then,*

(1) *for Ω a convex domain and $k \in \mathbb{N}$:*

$$\lambda_k(L_f) \leqslant -2\alpha + \lambda_{d(k)}(\Delta_{fN}^{[1]}),$$

where $d(k) = \binom{n+1}{2}(k-1)+1$ and $\Delta_{fN}^{[1]}$ is the Hodge f-Laplacian acting on the 1-forms $\omega \in \Omega^1(M)$ satisfying the absolute condition on the boundary.

(2) *for Ω a f-mean convex domain and $k \in \mathbb{N}$:*

$$\lambda_k(L_f) \leqslant -2\alpha + \lambda_{d(k)}(\Delta_{fT}^{[1]}),$$

where $d(k) = \binom{n+1}{2}(k-1)+1$ and $\Delta_{fT}^{[1]}$ is the Hodge f-Laplacian acting on the 1-forms $\omega \in \Omega^1(M)$ satisfying the relative condition on the boundary.

Proof. Let's argue for the first item. Consider $\{\psi_j\}$ an orthonormal basis of $L^2(M, \mathrm{e}^{-f}\mathrm{d}\mu)$ formed by eigenfunctions of the Jacobi operator L_f, where ψ_j is associated with the eigenvalue $\lambda_j(L_f)$. For each $d \in \mathbb{N}$, consider the direct sum:

$$E^d(\Delta_{fN}^{[1]}) = \bigoplus_{j=1}^{d} V_{\lambda_j(\Delta_{fN}^{[1]})},$$

where $V_{\lambda_j(\Delta_{fN}^{[1]})}$ is the space of the eigenforms of $\Delta_{fN}^{[1]}$ associated with the eigenvalue $\lambda_j(\Delta_{fN}^{[1]})$. For each $\varepsilon > 0$, consider the cut-off function ρ_ε given in proposition 2.4. We define the functions:

$$u = \left\langle X_{\overline{V},\overline{W}}, \omega^\sharp \right\rangle = \langle \overline{V}, N\rangle \langle W, \omega^\sharp\rangle - \langle \overline{W}, N\rangle \langle V, \omega^\sharp\rangle$$

where $\omega \in E^d(\Delta_{fN}^{[1]})$, $\overline{V}, \overline{W} \in \overline{\mathbf{P}}$ and V, W are their projections on TM. Then, for each $\varepsilon > 0$, consider the family of functions:

$$\{v_\varepsilon\} = \{\rho_\varepsilon u\}.$$

Note that each function of this family can be used as a test function for the stability operator. Initially, for each $\varepsilon > 0$, we desire to find $d = d(k)$ and some $\omega_\varepsilon \in E^d(\Delta_{fN}^{[1]})$, $\omega_\varepsilon \not\equiv 0$, in order that the function $u_\varepsilon := \rho_\varepsilon \left\langle X_{\overline{V},\overline{W}}, \omega_\varepsilon^\sharp \right\rangle$ perform the orthogonality conditions:

$$\int_M \rho_\varepsilon \left\langle X_{\overline{V},\overline{W}}, \omega_\varepsilon^\sharp \right\rangle \psi_1 \mathrm{e}^{-f}\mathrm{d}\mu = \cdots = \int_M \rho_\varepsilon \left\langle X_{\overline{V},\overline{W}}, \omega_\varepsilon^\sharp \right\rangle \psi_{k-1} \mathrm{e}^{-f}\mathrm{d}\mu = 0, \tag{4.1}$$

for all $\overline{V}$, $\overline{W} \in \overline{P}$. Note that $X_{\overline{V},\overline{W}}$ is a skew symmetric bilinear function of $\overline{V}$, $\overline{W}$ and $\dim \overline{P} = \dim \mathbb{R}^{n+1} = n+1$. Then, equation (4.1) is a system composed by

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$\binom{n+1}{2}(k-1)$ homogeneous linear equations in the unknown $\omega_\varepsilon \in E^d(\Delta^{[1]}_{fN})$. Thus, if $d = d(k) = \binom{n+1}{2}(k-1)+1$, we can find a non-trivial 1-form $\omega_\varepsilon \in E^d(\Delta^{[1]}_{fN})$ such that $u_\varepsilon = \rho_\varepsilon \left\langle X_{\overline{V},\overline{W}}, \omega_\varepsilon^\sharp \right\rangle$ is $L^2(M, \mathrm{e}^{-f}\mathrm{d}\mu)$-orthogonal with the first $k-1$ eigenfunctions of L_f for all $\overline{V}$, $\overline{W} \in \overline{P}$. By the min-max principle, it follows that:

$$\lambda_k(L_f)\int_M \rho_\varepsilon^2 u_\varepsilon^2 \mathrm{e}^{-f}\mathrm{d}\mu \leqslant \int_M \rho_\varepsilon u_\varepsilon L_f(\rho_\varepsilon u_\varepsilon)\mathrm{e}^{-f}\mathrm{d}\mu \\ + \int_{\partial M} \rho_\varepsilon u_\varepsilon \left(\eta(\rho_\varepsilon u_\varepsilon) + h^{\partial\Omega}(N,N)\rho_\varepsilon u_\varepsilon\right)\mathrm{e}^{-f}\mathrm{d}\sigma,$$

for all $u_\varepsilon = \left\langle X_{\overline{V},\overline{W}}, \omega_\varepsilon^\sharp \right\rangle$.

Applying lemma 4.1 with $\Omega = M$, $\varphi = \rho_\varepsilon$, and $T = |A|^2 + \operatorname{Hess} f(N, N)$, together lemma 3.1, we get:

$$\begin{aligned}\lambda_k(L_f)\int_M \rho_\varepsilon^2 u_\varepsilon^2 \mathrm{e}^{-f}\mathrm{d}\mu &\leqslant \int_M \left(\rho_\varepsilon^2 u_\varepsilon L_f(u_\varepsilon) + u_\varepsilon^2|\nabla\rho_\varepsilon|^2\right)\mathrm{e}^{-f}\mathrm{d}\mu \\ &\quad + \int_{\partial M} \rho_\varepsilon^2\left(u_\varepsilon\eta(u_\varepsilon) + h^{\partial\Omega}(N,N)u_\varepsilon^2\right)\mathrm{e}^{-f}\mathrm{d}\sigma \\ &= \int_M \rho_\varepsilon^2\left(-u_\varepsilon^2\operatorname{Hess} f(N,N) - u_\varepsilon \operatorname{Hess} f(X_{\overline{V},\overline{W}}, \omega_\varepsilon^\sharp)\right)\mathrm{e}^{-f}\mathrm{d}\mu \\ &\quad + \int_M \rho_\varepsilon^2\left(u_\varepsilon\left\langle X_{\overline{V},\overline{W}}, (\Delta^{[1]}_{fN}\omega_\varepsilon)^\sharp\right\rangle + u_\varepsilon v_\varepsilon\right)\mathrm{d}\mu \\ &\quad + \int_M u_\varepsilon^2|\nabla\rho_\varepsilon|^2\mathrm{e}^{-f}\mathrm{d}\mu \\ &\quad + \int_{\partial M} \rho_\varepsilon^2\left(u_\varepsilon\eta(u_\varepsilon) + h^{\partial\Omega}(N,N)u_\varepsilon^2\right)\mathrm{e}^{-f}\mathrm{d}\sigma,\end{aligned}$$

for all $\overline{V}, \overline{W} \in \overline{\mathbf{P}}$, where $v_\varepsilon = 2(\left\langle\nabla_{AV}\omega_\varepsilon^\sharp, W\right\rangle - \left\langle\nabla_{AW}\omega_\varepsilon^\sharp, V\right\rangle) - \left\langle W, \omega_\varepsilon^\sharp\right\rangle \operatorname{Hess} f(V,N) + \left\langle V, \omega_\varepsilon^\sharp\right\rangle \operatorname{Hess} f(W, N)$. Furthermore,

$$\begin{aligned}\eta(u_\varepsilon) &= \eta\left(\left\langle\overline{V}, N\right\rangle\left\langle W, \omega_\varepsilon^\sharp\right\rangle - \left\langle\overline{W}, N\right\rangle\left\langle V, \omega_\varepsilon^\sharp\right\rangle\right) \\ &= \left\langle\overline{V}, D_\eta N\right\rangle\left\langle W, \omega_\varepsilon^\sharp\right\rangle + \left\langle\overline{V}, N\right\rangle\left(\left\langle D_\eta W, \omega_\varepsilon^\sharp\right\rangle + \left\langle W, D_\eta\omega_\varepsilon^\sharp\right\rangle\right) \\ &= -\left\langle\overline{W}, D_\eta N\right\rangle\left\langle V, \omega_\varepsilon^\sharp\right\rangle - \left\langle\overline{W}, N\right\rangle\left(\left\langle D_\eta V, \omega_\varepsilon^\sharp\right\rangle + \left\langle V, D_\eta\omega_\varepsilon^\sharp\right\rangle\right).\end{aligned}$$

Integrating with respect to $\overline{V}$ and $\overline{W}$ and using lemma 3.2 we obtain the following equalities for each $p \in M$:

$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u^2\mathrm{d}\overline{V}\mathrm{d}\overline{W} = 2|\omega^\sharp|^2;$$

$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u\operatorname{Hess} f(X_{\overline{V},\overline{W}}, \omega^\sharp)\mathrm{d}\overline{V}\mathrm{d}\overline{W} = 2\operatorname{Hess} f(\omega^\sharp, \omega^\sharp);$$

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$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u^2 \operatorname{Hess} f(N,N) \mathrm{d}\overline{V}\mathrm{d}\overline{W} = 2\operatorname{Hess} f(N,N)|\omega^\sharp|^2;$$

$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u \left\langle X_{\overline{V},\overline{W}}, (\Delta^{[1]}_{fN}\omega)^\sharp \right\rangle \mathrm{d}\overline{V}\mathrm{d}\overline{W} = 2\left\langle \omega^\sharp, (\Delta^{[1]}_{fN}\omega)^\sharp \right\rangle;$$

$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} uv \mathrm{d}\overline{V}\mathrm{d}\overline{W} = 0;$$

$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u \left\langle \overline{V}, D_\eta N \right\rangle \left\langle \overline{W}, \omega^\sharp \right\rangle \mathrm{d}\overline{V}\mathrm{d}\overline{W} = 0;$$

$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u \left\langle \overline{V}, N \right\rangle \left\langle D_\eta\overline{W}, \omega^\sharp \right\rangle \mathrm{d}\overline{V}\mathrm{d}\overline{W} = -\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u \left\langle \overline{W}, N \right\rangle \left\langle D_\eta\overline{V}, \omega^\sharp \right\rangle \mathrm{d}\overline{V}\mathrm{d}\overline{W};$$

$$\int_{\overline{\mathbf{U}}\times\overline{\mathbf{U}}} u \left\langle \overline{V}, N \right\rangle \left\langle \overline{W}, D_\eta\omega^\sharp \right\rangle \mathrm{d}\overline{V}\mathrm{d}\overline{W} = \left\langle \omega^\sharp, D_\eta\omega^\sharp \right\rangle = \frac{1}{2}\eta\left(|\omega^\sharp|^2\right).$$

Using the equalities above and the Fubini's theorem in the later inequality, we get:

$$\begin{aligned}\lambda_k(L_f)\int_M \rho_\varepsilon^2|\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu \leqslant &-\int_M \rho_\varepsilon^2\left(\operatorname{Hess} f(\omega_\varepsilon^\sharp,\omega_\varepsilon^\sharp)+\operatorname{Hess} f(N,N)|\omega_\varepsilon^\sharp|^2\right)\mathrm{e}^{-f}\mathrm{d}\mu \\ &+\int_M \rho_\varepsilon^2\left\langle \omega_\varepsilon^\sharp, (\Delta^{[1]}_{fN}\omega_\varepsilon)^\sharp\right\rangle \mathrm{e}^{-f}\mathrm{d}M + \int_M |\nabla\rho_\varepsilon|^2|\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu \\ &+\int_{\partial M}\rho_\varepsilon^2\left(\eta(|\omega_\varepsilon^\sharp|^2)+2h^{\partial\Omega}(N,N)|\omega_\varepsilon^\sharp|^2\right)\mathrm{e}^{-f}\mathrm{d}\sigma.\end{aligned}$$

Finally, note that:

- Each $\rho_\varepsilon : M \to [0,\,1]$ satisfies $|\nabla\rho_\varepsilon| \leqslant \varepsilon$.
- $\operatorname{Hess} f(\omega_\varepsilon^\sharp,\,\omega_\varepsilon^\sharp) + \operatorname{Hess} f(N,\,N)|\omega_\varepsilon^\sharp|^2 \geqslant 2\alpha|\omega_\varepsilon^\sharp|^2$.
- ω_ε being the linear combination of the first $d(k)$ eigenforms of $\Delta^{[1]}_{fN}$ implies that

$$\int_M \rho_\varepsilon^2\left\langle \omega_\varepsilon^\sharp, (\Delta^{[1]}_{fN}\omega_\varepsilon)^\sharp\right\rangle \mathrm{e}^{-f}\mathrm{d}\mu \leqslant \lambda_{d(k)}(\Delta^{[1]}_{fN})\int_M \rho_\varepsilon^2|\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu.$$

- By lemma 3.3:

$$\begin{aligned}&\int_{\partial M}\rho_\varepsilon^2\left(\eta(|\omega_\varepsilon^\sharp|^2)+2h^{\partial\Omega}(N,N)|\omega_\varepsilon^\sharp|^2\right)\mathrm{e}^{-f}\mathrm{d}\sigma \\ &\quad=\int_{\partial M}\rho_\varepsilon^2\left(2h^{\partial\Omega}(\omega_\varepsilon^\sharp,\omega_\varepsilon^\sharp)+2h^{\partial\Omega}(N,N)|\omega_\varepsilon^\sharp|^2\right)\mathrm{e}^{-f}\mathrm{d}\sigma,\end{aligned}$$

  and this integral is negative, since $h^{\partial\Omega}(Y,\,Y) \leqslant 0$ for every vector Y tangent to $\partial\Omega$.

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Therefore, for each $\varepsilon > 0$, we have that:

$$\lambda_k(L_f)\int_M \rho_\varepsilon^2|\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu \leqslant \left(-2\alpha+\lambda_{d(k)}(\Delta_{fN}^{[1]})\right)\int_M \rho_\varepsilon^2|\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu + \varepsilon^2\int_M |\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu,$$

and so

$$\lambda_k(L_f) \leqslant -2\alpha+\lambda_{d(k)}(\Delta_{fN}^{[1]})+\varepsilon^2\frac{\int_M |\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu}{\int_M \rho_\varepsilon^2|\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu}.$$

To complete the proof, it suffices to show that:

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2\int_M |\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu}{\int_M \rho_\varepsilon^2|\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu}=0. \tag{4.2}$$

Note that, for each $\varepsilon > 0$, we may choose $\omega_\varepsilon \in E^d(\Delta_{fN}^{[1]})$ satisfying equation (4.1), and with $\omega_\varepsilon^\sharp$ unitary in the $L^2(M,\, \mathrm{e}^{-f}\mathrm{d}\mu)$-norm, that is:

$$\int_M |\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu = 1.$$

For each $\varepsilon > 0$, consider the set $A_\varepsilon := \{x \in M : \rho_\varepsilon(x) \neq 1\}$. The ρ_ε cut-off functions are constructed so that $\mathcal{H}^n(A_\varepsilon) < \varepsilon$. Defining $M_\varepsilon := M - A_\varepsilon$, we declare the existence of constants $\delta > 0$ and $R > 0$ such that:

$$\int_{M_\delta} |\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu \geqslant R > 0, \quad \forall\varepsilon > 0. \tag{4.3}$$

Indeed, if we assume that inequality (4.3) does not occur, then we have that:

$$\inf_{\varepsilon>0}\int_{M_{1/n}} |\omega_\varepsilon^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu = 0, \quad \forall n > 0.$$

So, for each $n \in \mathbb{N}$, we can find an 1-form $\omega_n \in E^d(\Delta_{fN}^{[1]})$ satisfying the inequality:

$$\int_{M_{1/n}} |\omega_n^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu < \frac{1}{n}.$$

Thus, there is a sequence $\{\omega_n\}_{n=1}^\infty$ such that:

$$\lim_{n\to\infty}\int_{M_{1/n}} |\omega_n^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu = 0. \tag{4.4}$$

We will show that the limit in (4.4) cannot occur. For this, consider the norms:

$$\|\omega\|_M^2 := \int_M |\omega^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu, \quad \|\omega\|_{M_\delta}^2 := \int_{M_\delta} |\omega^\sharp|^2\mathrm{e}^{-f}\mathrm{d}\mu.$$

By the compactness of $\mathbb{S}^{d-1} := \{\omega \in E^d(\Delta_{fN}^{[1]}) : \|\omega\|_M^2 = 1\}$, there is a subsequence $\{\omega_{n_j}\}_{j=1}^\infty$ converging to an 1-form $\omega \in \mathbb{S}^{d-1}$ in the $L^2(M,\, \mathrm{e}^{-f}\mathrm{d}\mu)$ sense, when

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$j \to \infty$. Observe that:

$$\begin{aligned}\lim_{j\to\infty}\int_{M_{1/n_j}} |\omega^\sharp_{n_j}|^2 \mathrm{e}^{-f}\mathrm{d}\mu &= \lim_{j\to\infty}\int_{M_{1/n_j}} \left(|\omega^\sharp_{n_j}|^2 - |\omega^\sharp|\right)\mathrm{e}^{-f}\mathrm{d}\mu \\ &\quad + \lim_{j\to\infty}\int_{M_{1/n_j}} |\omega^\sharp|\mathrm{e}^{-f}\mathrm{d}\mu \\ &= \lim_{j\to\infty}\int_{M_{1/n_j}} \left(|\omega^\sharp_{n_j}|^2 - |\omega^\sharp|\right)\mathrm{e}^{-f}\mathrm{d}\mu + 1.\end{aligned}$$

Furthermore, for every pair of unit vectors v, w in an inner product space, the next inequality holds:

$$\left|\|v\|^2 - \|w\|^2\right| \leqslant 2\,\|v-w\|\,.$$

Thus,

$$\begin{aligned}\lim_{j\to\infty}\left|\int_{M_{1/n_j}} \left(|\omega^\sharp_{n_j}|^2 - |\omega^\sharp|\right)\mathrm{e}^{-f}\mathrm{d}\mu\right| &= \lim_{j\to\infty}\left|\left\|\omega^\sharp_{n_j}\right\|^2_{M_{1/n_j}} - \|w\|^2_{M_{1/n_j}}\right| \\ &\leqslant 2\lim_{j\to\infty}\left\|\omega^\sharp_{n_j} - \omega^\sharp\right\|^2_{M_{1/n_j}} \\ &\leqslant 2\lim_{j\to\infty}\left\|\omega^\sharp_{n_j} - \omega^\sharp\right\|^2_{M} = 0.\end{aligned}$$

Thereby, we get:

$$\lim_{j\to\infty}\int_{M_{1/n_j}} |\omega^\sharp_{n_j}|^2 \mathrm{e}^{-f}\mathrm{d}\mu = \lim_{j\to\infty}\int_{M_{1/n_j}} \left(|\omega^\sharp_{n_j}|^2 - |\omega^\sharp|\right)\mathrm{e}^{-f}\mathrm{d}\mu + 1 = 1,$$

contradicting (4.4). Now, the equality in (4.2) is a direct consequence of the inequalities:

$$R \leqslant \int_{M_\delta} |\omega^\sharp_\varepsilon|^2\mathrm{e}^{-f}\mathrm{d}\mu \leqslant \int_M \rho_\varepsilon^2|\omega^\sharp_\varepsilon|^2\mathrm{e}^{-f}\mathrm{d}\mu \leqslant \int_M |\omega^\sharp_\varepsilon|^2\mathrm{e}^{-f}\mathrm{d}\mu = 1,$$

for all $\varepsilon > 0$.

For the second item, we follow step-by-step the previous computation making the appropriate substitutions of $\Delta^{[1]}_{fN}$ by $\Delta^{[1]}_{fT}$. Furthermore, we should note that lemma 3.4 implies:

$$\begin{aligned}&\int_{\partial M} \rho_\varepsilon^2\left(\eta(|\omega^\sharp_\varepsilon|^2) + 2h^{\partial\Omega}(N,N)|\omega^\sharp_\varepsilon|^2\right)\mathrm{e}^{-f}\mathrm{d}\sigma \\ &\quad = \int_{\partial M} \rho_\varepsilon^2\left(2H_f^{\partial M} + 2h^{\partial\Omega}(N,N)\right)|\omega^\sharp_\varepsilon|^2\mathrm{e}^{-f}\mathrm{d}\sigma \\ &\quad = \int_{\partial M} 2\rho_\varepsilon^2 H_f^{\partial\Omega}|\omega^\sharp_\varepsilon|^2\mathrm{e}^{-f}\mathrm{d}\sigma < 0.\end{aligned}$$

□

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As byproduct of the computation in the previous theorem we have the following result:

THEOREM 4.3. *Let Ω^{n+1} be a domain in $(\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\mathrm{d}\mu)$ with non-empty boundary. Let M^n be a compact f-minimal orientable hypersurface with free boundary in Ω. Assume that M is smooth or has a singular set satisfying $\mathcal{H}^{n-2}(\mathrm{sing}(M)) = 0$; and that the tensor $\mathrm{Ric}_f^\Omega = \mathrm{Hess}\, f$ is bounded from below by a nonnegative constant α.*

(1) *If Ω is a convex domain in $\mathbb{R}^{n+1}$, then*
   (a)

$$\mathrm{Ind}_f(M) \geqslant \frac{2}{n(n+1)} \left(\Gamma^+_{\Delta^{[1]}_{fN}}(2\alpha) + \dim H^1(M;\mathbb{R}) \right),$$

   where $\Gamma^+_{\Delta^{[1]}_{fN}}(2\alpha)$ is the number of positive eigenvalues of $\Delta^{[1]}_{fN}$ less than 2α;

   (b)

$$\mathrm{Ind}_f(M) \geqslant \frac{2}{n(n+1)} \dim H^1(M;\mathbb{R}) + \Gamma^-_{L_f}(-2\alpha),$$

   where $\Gamma^-_{L_f}(-2\alpha)$ is the number of negative eigenvalues of L_f greater than -2α.

(2) *If Ω is a f-mean convex domain in $\mathbb{R}^{n+1}$, then*
   (a)

$$\mathrm{Ind}_f(M) \geqslant \frac{2}{n(n+1)} \left(\Gamma^+_{\Delta^{[1]}_{fT}}(2\alpha) + \dim H^{n-1}(M;\mathbb{R}) \right).$$

   where $\Gamma^+_{\Delta^{[1]}_{fT}}(2\alpha)$ is the number of positive eigenvalues of $\Delta^{[1]}_{fT}$ less than 2α;

   (b)

$$\mathrm{Ind}_f(M) \geqslant \frac{2}{n(n+1)} \dim H^{n-1}(M;\mathbb{R}) + \Gamma^-_{L_f}(-2\alpha),$$

   where $\Gamma^-_{L_f}(-2\alpha)$ is the number of negative eigenvalues of L_f greater than -2α.

Proof. For the first item, consider the number:

$$\beta := \#\{\text{eigenvalues of } \Delta^{[1]}_{fN} \text{ that are less than } 2\alpha\}.$$

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Let k be the largest integer such that $d(k) = ((n(n+1))/2)(k-1) + 1 \leqslant \beta$. It follows directly from theorem 4.2 with the definitions of β and k that

$$\begin{aligned}\operatorname{Ind}_f(M) \geqslant k \geqslant \frac{2}{n(n+1)}\beta &= \frac{2}{n(n+1)}\left(\Gamma_{\Delta_{fN}^{[1]}}(2\alpha) + \dim H^1_{Nf}(M)\right)\\ &= \frac{2}{n(n+1)}\left(\Gamma_{\Delta_{fN}^{[1]}}(2\alpha) + \dim H^1(M;\mathbb{R})\right).\end{aligned}$$

Moreover, if k is the largest integer such that $d(k) = ((n(n+1))/2)(k-1) + 1 \leqslant \dim H^1(M;\mathbb{R})$, then

$$\lambda_k(L_f) \leqslant -2\alpha, \text{ and } k \geqslant \frac{2}{n(n+1)} \dim H^1(M;\mathbb{R}).$$

The second part follows the same steps considering 1-forms satisfying the relative boundary condition. □

As a consequence, we have the following result:

Corollary 4.4. *Let Ω^{n+1} be a domain in $(\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\mathrm{d}\mu)$ with non-empty boundary. Let M^n be a compact f-minimal orientable hypersurface with free boundary in Ω. Assume that M is smooth or has a singular set satisfying $\mathcal{H}^{n-2}(\operatorname{sing}(M)) = 0$; and that the tensor $\operatorname{Ric}^{\Omega}_f = \operatorname{Hess} f$ is bounded from below by a nonnegative constant α. If Ω is a f-mean convex domain in $\mathbb{R}^{n+1}$, then*

$$\operatorname{Ind}_f(M) \geqslant \frac{2}{n(n+1)}\left(Eig_{\Delta_f}(2\alpha) + \dim H^{n-1}(M;\mathbb{R})\right).$$

where $Eig_{\Delta_f}(2\alpha)$ is the number of positive Neumann eigenvalues of Δ_f less than 2α.

Proof. Recall that $\Gamma^+_{\Delta_{fT}^{[1]}}(2\alpha)$ denotes the number of positive eigenvalues of $\Delta_{fT}^{[1]}$ less than 2α. Let $\zeta = Eig_{\Delta_f}(2\alpha)$ denote the number of positive eigenvalues of the f-Laplacian that are less than 2α. Consider $u_1, \ldots, u_\zeta$ as a set of orthogonal eigenfunctions of the f-Laplacian associated with positive Neumann eigenvalues less than 2α. By Stokes' theorem, we observe that these functions have zero mean.

Next, by Stokes' theorem and the Neumann condition, we find that the set $\mathrm{d}u_1, \ldots, \mathrm{d}u_\zeta$ of differential 1-forms are orthogonal and non-trivial. Moreover, $\Delta_{fT}^{[1]}(\mathrm{d}u_i) = \mathrm{d}(\Delta u_i) = -\lambda_i \mathrm{d}u_i$, where λ_i is an eigenvalue of the f-Laplacian. Furthermore, $i_\eta(\mathrm{d}u_i) = 0$ due to the Neumann condition, and $i_\eta(\mathrm{d}(\mathrm{d}u_i)) = 0$. Therefore, summarizing, we conclude that $\Gamma^+_{\Delta_{fT}^{[1]}}(2\alpha) \geqslant \zeta$. The result now follows from the previous theorem. □

Remark 4.5. For the case that M has $r \geqslant 1$ boundary components, from lemma 3.5 and the fact that $H^1_{Tf}(M)$ and $H_1(M, \partial M;\mathbb{R})$ are isomorphic, we obtain that $\dim H^1_{Tf}(M) \geqslant r - 1$.

The next result follows directly from lemma 3.6 and theorem 4.3, item (2).

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THEOREM 4.6. *Let Ω^3 be a f-mean convex domain in $(\mathbb{R}^3, g_{can}, \mathrm{e}^{-f}\mathrm{d}\mu)$ with non-empty boundary. Let M^2 be a compact orientable f-minimal surface with r boundary components, genus g, and free boundary in Ω. Assume that M is smooth, and that the tensor $\mathrm{Ric}_f^{\Omega} = \mathrm{Hess}\, f$ is bounded from below by a nonnegative constant α. Then,*

(1)

$$\mathrm{Ind}_f(M) \geqslant \frac{1}{3}\left(2g + r - 1 + \Gamma^+_{\Delta^{[1]}_{fT}}(2\alpha)\right),$$

where $\Gamma^+_{\Delta^{[1]}_{fT}}(2\alpha)$ is the number of positive eigenvalues of $\Delta^{[1]}_{fT}$ less than 2α;

(2)

$$\mathrm{Ind}_f(M) \geqslant \frac{1}{3}(2g + r - 1) + \Gamma^-_{L_f}(-2\alpha),$$

where $\Gamma^-_{L_f}(-2\alpha)$ is the number of negative eigenvalues of L_f greater than -2α.

5. Applications

First of all, we recall that a self-shrinkers of the mean curvature flow are defined as connected, orientable, isometrically immersed hypersurfaces $x : M \to \mathbb{R}^{n+1}$ whose mean curvature function satisfies the equation:

$$H = -\langle x, N \rangle.$$

Notice that self-shrinkers of the mean curvature flow are f-minimal hypersurfaces in the Gaussian space, that is, in $(\mathbb{R}^{n+1}, g_{can}, \mathrm{e}^{-f}\mathrm{d}\mu)$ endowed with the weight function $f(x) = \frac{1}{2}|x|^2$. So, $\mathrm{Hess}\, f = g_{can}$ and $\alpha = 1$. We will focus on two cases as follows.

5.1. Free-boundary self-shrinkers in the half-space $\Omega = \mathbb{R}^{n+1}_+$

Consider the Euclidean half-space given by $\Omega = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geqslant 0\}$. Notice that $\partial\Omega = \mathbb{R}^n \times \{0\}$, and so its unit outward normal vector field is $\nu = -e_{n+1}$. In particular, the f-mean curvature of $\partial\Omega$ is equal to zero, and thus $\partial\Omega$ is f-mean convex.

Let M be a compact, orientable, free-boundary self-shrinker of mean curvature flow in Ω. Observe that a straightforward computation gives us that $\Delta_f x_i + x_i = 0$ and $\partial x_i/\partial\nu = 0$, for $n = 1, \ldots, n$. Hence, $\lambda = 1$ is eigenvalue of the f-Laplacian less than $2\alpha = 2$ with multiplicity n. Thus, we obtain:

THEOREM 5.1. *Let M^n be a free-boundary self-shrinker in the half-space $\mathbb{R}^{n+1}_+$. Then,*

$$\mathrm{Ind}_{|x|^2/2}(M) \geqslant \frac{2}{n(n+1)}\left(\dim H^{n-1}(M; \mathbb{R}) + n\right).$$

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Proof. The result follows from the previous computation and corollary 4.4. $\square$

Another strategy is consider the function $\langle e_{n+1}, N\rangle$. It is well known that the functions $\left\langle \overline{V},\, N\right\rangle$ satisfy the equation:

$$L_f \left\langle \overline{V}, N\right\rangle - \left\langle \overline{V}, N\right\rangle = 0, \quad \forall \overline{V} \in \overline{\mathbf{P}}.$$

So, $\langle \overline{V},\, N\rangle$ will be an eigenfunction of L_f associated with the eigenvalue $\lambda = -1 \in (-2, 0)$, if:

$$\eta(\left\langle \overline{V}, N\right\rangle) + h^{\partial\Omega}(N, N) \left\langle \overline{V}, N\right\rangle = 0,$$

that is:

$$0 = \eta(\langle \overline{V}, N\rangle) = -\langle \overline{V}, A^M\eta\rangle \text{ on } \partial M.$$

Since the hyperplane is totally geodesic and the hypersurface M is free boundary, a direct computation shows that $A^M\eta = k\eta$, for some function k along ∂M, and therefore the function $\langle \overline{V},\, N\rangle$ satisfies the Neumann condition for any $\overline{V}$ orthogonal to $\eta = -e_{n+1}$.

Hence, the space $Z := span\{\overline{V} \in \overline{\mathbf{P}} : \eta\langle \overline{V},\, N\rangle = 0 \text{ on } \partial M\}$ has dimension at least n, and so:

$$\Gamma^-_{L_f}(-2) \geqslant \dim Z \geqslant n.$$

THEOREM 5.2. *Let M^n be a free-boundary self-shrinker in the half-space $\mathbb{R}^{n+1}_+$. Then,*

$$\mathrm{Ind}_{|x|^2/2}(M) \geqslant \frac{2}{n(n+1)} \dim H^{n-1}(M; \mathbb{R}) + n.$$

In particular, for $n = 2$, we have

$$\mathrm{Ind}_{|x|^2/2}(M) \geqslant \frac{1}{3}(2g + r + 5).$$

Proof. The result follows from the previous computation and item (2.b) of theorem 4.3. $\square$

REMARK 5.3. The previous result improves theorem 5.1, but we presented it here because the strategy of its proof is different and could be useful in other settings.

5.2. Free-boundary self-shrinkers into a slab

Using the previous strategy we obtain:

THEOREM 5.4. *Let M^2 be a free-boundary self-shrinker in a slab $\mathbb{R}^2 \times [a, b]$, where $a < b$ are real numbers. Then,*

$$\mathrm{Ind}_{|x|^2/2}(M) \geqslant \frac{2g + r + 5}{3}.$$

Proof. We use similar computations as before, the convexity of the boundary of the ambient space, and item (1.b) of theorem 4.3. $\square$

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Notice that the Morse index equal to two implies a topological rigidity. Indeed:

COROLLARY 5.5. *Let M^2 be a free-boundary self-shrinker in a slab $\mathbb{R}^2 \times [a, b]$ with Morse index two, where $a < b$ are real numbers. Then, M is topologically a disk.*

Data

This manuscript has no associated data.

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