Proceedings of the Edinburgh Mathematical Society (1983) **26**, 353–360 ©

ON COMPACT ACTION IN *JB*-ALGEBRAS

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(Received 16th August 1982)

1. Introduction

A real Jordan algebra which is also a Banach space with a norm which satisfies

$$\|a^2-b^2\| \leqq \max(\|a^2\|, \|b^2\|), \quad \|a^2\| = \|a\|^2,$$

for each pair a, b of elements, is said to be a JB-algebra. A JB-algebra which is also a Banach dual space is said to be a JBW-algebra.

Important examples of JB-algebras include the JC-algebras, these are by definition the uniformly closed Jordan subalgebras of the Jordan algebra of all bounded self-adjoint operators on a complex Hilbert space, and also the algebra M_3^8 of hermitian 3×3 matrices over the octonions. A JC-algebra which is closed in the weak operator topology is a JBW-algebra said to be a JW-algebra.

The reader is referred to [**2–3, 7–9, 11, 14–15, 18**] for the relevant background on JB algebras and to [**10, 16–17**] for that on JC algebras. A detailed account of the general theory of Jordan algebras is to be found in [**12**].

In the JB-algebra A, the Jordan triple product $\{a,b,c\}$, of elements a, b and c in A, is defined by $\{a,b,c\}=(a\circ b)\circ c + a\circ(b\circ c)-(a\circ c)\circ b$, and for each element a in A, the operators on A, U_a and L_a are defined by $U_a(b)=\{a,b,a\}$, $L_a(b)=a\circ b$ for each element b in A. When A is a JC-algebra these operations reduce to

$$L_a(b)=\tfrac{1}{2}(ab+ba), \quad U_a(b)=aba, \quad \{a,b,c\}=\tfrac{1}{2}(abc+cba).$$

For the JB-algebra A, A^+, the set of squares of elements of A, is a positive cone which generates A. A JB-subalgebra B of A is said to be an hereditary JB-subalgebra of A if whenever $0\leqq a\leqq b$ with $a\in A$ and $b\in B$, then $a\in A$. Also, a linear subspace, J, of A, is said to be a Jordan (resp: quadratic) ideal of A if $L_x(A)$ (resp: $U_x(A)$) is contained in A for every element x of J. A fact that will frequently be used is that the uniformly closed quadratic ideals of A are precisely the hereditary JB-subalgebras of A [**8**, Theorem 2.3]. The JB-algebra A will habitually be considered to be a JB-subalgebra of the JBW-algebra A^{**}, the second dual of A (see [**11, 14–15**]), and by $\tilde{A}$ will be meant the JB-subalgebra of A^{**} generated by A and the unit, 1, of A^{**}.

The purpose of this note is to identify, in an arbitrary JB-algebra A, those sets of elements

$$x_1 \text{ (resp: } x_2\text{) of } A \text{ for which } U_{x_1} \text{ (resp: } L_{x_2}\text{) are weakly compact on } A. \qquad \text{(I)}$$

and,

$$x_3 \text{ (resp: } x_4\text{) of } A \text{ for which } U_{x_3} \text{ (resp: } L_{x_4}\text{) are compact on } A. \tag{II}$$

By way of justification, we might mention the fact that though in a C^* algebra A it is easily seen, by elementary analysis, that the set of elements x of A for which the operator on A, $a \mapsto axa$, is compact is a two-sided ideal of A and, in view of the results of [**1**], consequently recognisable as the largest dual (in the sense of [**5**, 4.7.20]) two-sided ideal in A, the corresponding method of insight is not available for JB algebras. Indeed, the former of the sets of (II), above, is not, at times, even a linear space. A similar difficulty arises with the latter set of (II).

2. On associative subalgebras in a *JB* algebra

Recall that an associative JB-algebra can be realised as the self-adjoint part of a commutative C^*-algebra [**1**, Proposition 2.3, **14**, Lemma 2.2], and that, in particular, for each element a in the JB algebra A the (associative) JB-subalgebra, $C(a)$, of A generated by a can be identified with the algebra of all continuous real valued functions vanishing at infinity on the locally compact Hausdorff space $\sigma(a)$, of all real numbers λ for which $a-\lambda 1$ is not invertible in $\tilde{A}$. Recall also that for each pair of elements a, b in the JB algebra A, $C(a, b)$, the JB subalgebra of A generated by a and b can be realised as a JC algebra, by [**18**, Proposition 2.1]. It is said that the pair a, b operator commute in A if $L_a L_b = L_b L_a$ on A. The set of all elements of A which operator commute with every element in A is said to be the centre, $Z(A)$, of A.

In this preliminary section two lemmas are proved, the first of which is a slight variation of a result of Youngson [**19**, Theorem 5]. The result is known for JC-algebras [**16**, Proposition 1]. The following Jordan identities will be needed (see [**12**, page 37]) in Lemma 2 and after.

$$U_{U_a(b)} = U_a U_b U_a \tag{2.1}$$

$$(U_a(b))^2 = U_a U_b(a^2). \tag{2.2}$$

Lemma 1. *Let A be a JB algebra and let a, b be elements of A. Then the following are equivalent:*

(i) $U_a(b) = a^2 \circ b$.
(ii) *a and b operator commute in A.*
(iii) *$C(a, b)$ is associative.*

Proof. From [**16**, Proposition 1] and [**18**, Proposition 2.1] it follows that (i) and (iii) are equivalent and that (ii) implies (i). It remains to prove that (i) implies (ii). In order to achieve this it is enough to suppose that A equals M_3^8, as can be seen on application of [**16**, Proposition 1] together with the fact that A has a faithful family of representations each member of which maps A onto M_3^8 or a JC-algebra. But then $C(a, b)$ is finite dimensional and so, in particular, a is a finite linear combination of projections of $C(a,b)$. The desired conclusion now follows on application of (i)⇔(ii) and [**2**, Lemma 2.11].

A projection p in the JB-algebra A is said to be a *one-dimensional projection* of A if $U_p(A)$ has dimension one.

Lemma 2. *Let A be a JB-algebra and let B be a maximal associative JB-subalgebra of A. Then every one-dimensional projection of B is also a one-dimensional projection of A.*

Proof. In any associative algebra one has the identity

$$(xyx)z+z(xyx)=(xz+zx)yx+xy(xz+zx)-x(yz+zy)x.$$

By Macdonald's Theorem [**12**, page 41] the corresponding identity

$$U_x(y)\circ z=2\{x\circ z,y,x\}-U_x(y\circ z) \tag{2.3}$$

will hold in every Jordan algebra.

Let $a\in A$, $b\in B$ and let e be a one-dimensional of B. Observe that $e\circ b=U_e(b)=\lambda e$, for some real number λ, and that e and b operator commute in A, by [**2**, Lemma 2.11] or Lemma 1, which in turn means that

$$U_e(a\circ b)=U_e(a)\circ b=\lambda U_e(a) \tag{2.4}$$

on applying (2.3). From (2.1), (2.2), it follows that

$$(U_e(a))^2\circ b=(U_eU_a(e))\circ b=\lambda U_eU_a(e)=U_{U_a(e)}(b),$$

where the second equality is obtained on replacing a with $U_a(e)$ in (2.4). From Lemma 1 and the assumption on B it follows that $U_e(A)\subset B$, as thus does the desired result.

3. Compact action in JB algebras

A JW-algebra is said to be a JW factor of Type I_n, $n=\infty,1,2,3,\ldots$, if it has trivial centre and contains a family, with cardinality n, of mutually orthogonal minimal projections with sum 1. Recall [**2**, §7] and [**17**], that the *spin factors* are precisely the JW factors of Type I_2 and that for each spin factor V,

$$V=\mathbb{R}1\oplus N(V),\quad N(V)\circ N(V)=\mathbb{R}1,$$

where $N(V)$ is the closed linear span of the non-trivial symmetries of V (an element s is said to be a non-trivial symmetry if $s\neq\pm1$, $s^2=1$). Let us write, for each spin factor V,

$$E(V)=\cup\{\mathbb{R}e;e^2=e\neq 0,1\}.$$

A JB-algebra A is said to be *dual*, [**4**], if $(J^0)^0=J$ for every hereditary JB-subalgebra J of A (where given a subset S of A, S^0 represents the annihilator of S in A). The following discussion relies heavily upon the theory of dual JB algebras and thus, for the convenience of the reader, some of the more immediately relevant properties are collected below in Theorem 3.

The following terminology and notation is used below. The JB-(∞) sum of a family, (A_λ), of JB algebras, written $(\Sigma A_\lambda)_0$, is the JB-algebra of all generalised sequences (x_λ), $(x_\lambda \in A_\lambda)$, vanishing at infinity, with norm $\|(x_\lambda)\| = \operatorname{Sup}\|x_\lambda\|$; in addition, if $F_\lambda \subset A_\lambda$, for each λ, defines a family of subsets, then $(\Sigma F_\lambda)_0$ will denote the set of the (x_λ) in $(\Sigma A_\lambda)_0$ for which each x_λ lies in F_λ. A JB-algebra is said to be *simple* if it contains no non-trivial norm closed Jordan ideals. Finally, given an arbitrary JB-algebra A, let $C(A)$ denote the closed linear span of the one-dimensional projections of A, and further, let

$$C_1(A) = \{x \in A; U_x : A \to A \text{ is weakly compact}\},$$

$$C_2(A) = \{x \in A; L_x : A \to A \text{ is weakly compact}\},$$

$$C_3(A) = \{x \in A; U_x : A \to A \text{ is compact}\},$$

$$C_4(A) = \{x \in A; L_x : A \to A \text{ is compact}\}.$$

Theorem 3. ([**4**, Sections 3–4])

(a) *The simple dual JB-algebras are (up to isomorphism) precisely* M_3^8, *the spin factors, and the simple JC-algebras consisting of compact operators.*

(*b*) *The following are equivalent for the JB-algebra A:*

- (i) *A is a dual JB-algebra,*
- (ii) *Each element of A is of the form* $\Sigma \lambda_n e_n$ *(norm convergent), where* (e_n) *is a sequence of mutually orthogonal one-dimensional projections of A,*
- (iii) *A is the JB-*(∞) *sum of a family of simple dual JB-algebras,*
- (iv) *A is an hereditary JB-subalgebra of* A^{**},
- (v) $U_x : A \to A$ *is weakly compact for each x in A,*
- (vi) $L_x : A \to A$ *is weakly compact for each x in A.*

Moreover, each of the above six conditions is equivalent to the condition (vii), *below, if and only if A has no representations onto an infinite dimensional spin factor:*

- (vii) $U_x : A \to A$ *is compact for each x in A.*

A JB-algebra which satisfies condition (vii) of Theorem 3 is said to be a *compact JB-algebra.*

Lemma 4. *Let A be a JB-algebra. Then C(A) is a Jordan ideal of A and is the largest hereditary dual JB-subalgebra of A. It is also the largest hereditary JB-subalgebra of* A^{**} *contained in A.*

Proof. Given an element a of A, $U_a(C(A))$ is contained in $C(A)$, since for each one-dimensional projection e of A there exists a one-dimensional projection f of A for which $U_a(e) = \|U_a(e)\| f$, by the argument of [**4**, Proposition 3.1], for example. It therefore follows, from [**8**, Lemma 2.4], that $C(A)$ is a Jordan ideal and, moreover, by [**8**, Theorem 2.3] and of Theorem 3 (b) (i)⇔(ii), it is seen to be the largest hereditary dual JB-subalgebra of A. Finally, since any hereditary JB-subalgebra, J, of A^{**} which is contained in A is, via the canonical embedding, an hereditary JB-subalgebra of J^{**}, the last statement in the lemma follows from Theorem 3 (b) (i)⇔(iv) and the first part of the proof.

Before stating the main theorem of this note, let us observe that for each element x in the JB-algebra A, the norm closure, $A(x)$, of $U_x(A)$ is, by identity (2.1) and a simple limit argument, a quadratic ideal (and hence an hereditary JB-subalgebra) of A.

Writing $x=x_+-x_-$ (where x_+ and x_- lie in $C(x)^+$, $x_+\circ x_-=0$), it is easily seen, by spectral theory together with hereditary considerations, that both x_+ and x_- , and hence x, lie(s) in $A(x)$.

Theorem 5. *Let A be a JB-algebra. Then*

(i) *$C_1(A)$, $C_2(A)$ and the closed linear subspace generated by $C_3(A)$ are all equal to $C(A)$.*

Moreover, on identifying $C(A)$ with the JB-(∞) sum of the family, $\{A_\lambda;\lambda\in\Lambda\}$, of simple dual JB-algebras (justified by Theorem 3 and Lemma 4) and letting F (resp: G) denote the set of those $\lambda\in\Lambda$ for which A_λ is finite-dimensional (resp: an infinite-dimensional spin factor), one has, on retaining the above-mentioned identification,

(ii) $$C_3(A)=\Big(\sum_{\Lambda\backslash G}A_\lambda\Big)_0\oplus\Big(\sum_{G}E(A_\lambda)\Big)_0,$$

(iii) $$C_4(A)=\Big(\sum_{F}A_\lambda\Big)_0\oplus\Big(\sum_{G}N(A_\lambda)\Big)_0.$$

Proof. (i) Let $x\in A$ and suppose that $U_x:A\to A$ is compact (resp: weakly compact). Then from identity (2.1), a simple limit argument, and well-known properties of compact (resp: weakly compact operators), [**6**, Chapter 6], for example, one has that $U_y:A(x)\to A(x)$ is compact (resp: weakly compact) for every element y of $A(x)$. Therefore, $x\in A(x)\subset C(A)$, by Theorem 3 and Lemma 4.

If z lies in A and $L_z:A\to A$ is weakly compact, let B be a maximal associative subalgebra of A containing z. Then $L_{z^2}=L_z^2=U_z$ on B and so, by the above argument and the fact that $C(B)=C(A)\cap B$ (see Lemma 2), it follows that z lies in $C(A)$. Thus $C_3(A)\subset C_1(A)\subset C(A)$ and $C_2(A)\subset C(A)$.

Conversely, suppose that x lies in $C(A)$. By spectral theory, Lemma 4 and the equivalence of (i), (v) and (vi) of Theorem 3(b), $U_{x_+^{\frac12}}$, $U_{x_-^{\frac12}}$, $L_{x_+^{\frac12}}$, $L_{x_-^{\frac12}}$, L_{x_+}, L_{x_-} are all weakly compact operators on $C(A)$ which, being a Jordan ideal of A, by Lemma 4, implies that

$$U_x=(U_{x_+^{\frac12}})^2+(U_{x_-^{\frac12}})^2-2L_{x_+}L_{x_-},$$

and consequently

$$L_x = 2(L_{x_+^{\frac{1}{2}}})^2 - U_{x_+^{\frac{1}{2}}} + 2(L_{x_-^{\frac{1}{2}}})^2 - U_{x_-^{\frac{1}{2}}},$$

are both weakly compact operators on A. Thus, $C_3(A) \subset C_1(A) = C_2(A) = C(A)$, and since all one-dimensional projections of A must lie in $C_3(A)$, (i) results.

(ii) Certainly, $E(A_\lambda)$ is contained in $C(A_3)$ for each λ in G and from (i), together with Theorem 3(a), (b)(vii), one has that $A_\lambda = C_3(A_\lambda)$ for each λ in $\Lambda \backslash G$. Thus given λ in $\Lambda \backslash G$ and an element x in A_λ, writing $|x| = x_+ + x_-$, $x_n = (|x| + 1/n)^{-\frac{1}{2}}$, for each integer n, and operating in the associative JB-subalgebra $C(1, x)$ of $\tilde{A}$, one sees that $x_+^2 \circ x_n$ converges uniformly to x_+ (by Dini's Theorem, since $x_+^2 \circ x_n$ increases to x_+). Similarly, $x_-^2 \circ x_n$ converges uniformly to x_-. Hence,

$$U_x = \lim U_{(x_n \circ |x|) \circ x} = \lim U_x U_{|x|} U_{x_n}$$

acts compactly on the whole algebra A, since A_λ is (identified with) a Jordan ideal of A. So, $A_\lambda = C_3(A_\lambda) \subset C_3(A)$ for each λ in $\Lambda \backslash G$.

On the other hand, given an element x of $C_3(A)$ and representing x as an element (x_λ) of the JB-(∞) sum of the family $\{A_\lambda; \lambda \in \Lambda\}$, using (i), it is immediate that each x_λ lies in $C_3(A_\lambda)$. Finally, since in a spin factor the norm and weak operator topologies coincide, [**17**, page 1060], and because $A(x_\lambda)$ is a compact hereditary JB-subalgebra of A_λ, by the first part of the proof of (i), it follows from Theorem 3(a), (b)(vii) together with [**7**, Theorem 2.3] that x_λ lies in $E(A_\lambda)$ for each λ in G. This completes the proof of (ii).

(iii) If λ belongs to G, so that $V = A_\lambda$ is an infinite dimensional spin factor, then $N(V) = C_4(V) \subset C_4(A)$. Indeed, the equality follows because $L_s(V) = \mathbb{R}1 + \mathbb{R}s$ for each symmetry $s \neq \pm 1$ in V, and the inclusion follows because (as one may check) $L_s^3 = L_s$ for each such s.

If, on the other hand, λ lies in F, then it is clear that $A = C_4(A_\lambda) \subset C_4(A)$, where the inclusion follows by an argument similar to that used in the last part of (i).

It is easy to see that if (x_λ) is an element of $C(A)$ lying in $C_4(A)$ then each x_λ lies in $C_4(A_\lambda)$. Thus the proof will be complete if, in these circumstances, it can be shown that whenever x_λ is non-zero and λ is not in G, then λ is in F. In order to prove this, in view of Theorem 3(a), suppose that B is a simple compact JC-algebra, contained in $B(H)$ for some complex Hilbert space H, for which there exists a non-zero element x in $C_4(B)$. Let us define $T_y: B \to B(H)(b \mapsto yb)$, for each y in B. Then, $T_x = 2T_x L_x - U_x: B \to B(H)$ is compact. It is seen therefore, from Theorem 3(b)(i)$\Leftrightarrow$(ii), that there exists a one-dimensional projection e of B such that $T_e: B \to B(H)$ is compact. Notice now that if (x_n) is any sequence of elements in B for which ex_n converges uniformly in $B(H)$, then

$$\lim ex_n = \lim e(x_n e + ex_n - ex_n e) = e \lim (x_n \circ e - U_e(x_n)),$$

which again lies in eB. It follows that eB is of finite dimension, since it is uniformly closed and has compact identity operator. The weak operator closure M, of B is a Type I JW-factor, by the results of [**4**], for example. Since eM must be of finite dimension, so also must be the space generated by $\{U_s(e); s$, a symmetry in $M\}$. It follows, therefore, from [2, Theorem 6.10, Proposition 8.3], that M, and hence B, is of finite dimension. This completes the proof.

In conclusion, the following corollary (of which (ii), it should be said, was announced without proof in [4] and is a generalisation of a result of Kaplansky [13], on C^*-algebras) may be deduced by inspection of the above proof.

Corollary 6. *Let A be a JB-algebra. Then*

(i) $C_i(J) = C_i(A) \cap J$, $i = 1, 2, 3, 4$, *for every hereditary JB-subalgebra J of A,*
(ii) $L_x: A \to A$ *is compact for every x in A if and only if A is the JB-*(∞) *sum of finite dimensional JB-algebras.*

Much of this work appears in the author's Ph.D. thesis, written at the University of Reading under the supervision of Professor J. D. M. Wright, and the author would like to acknowledge this fact.

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