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Polynomials biorthogonal to Appell's polynomials

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We present the solution of a long-standing problem, namely, the determination of a set of polynomials in two independent variables which are biorthogonal over a triangular region to a set of polynomials previously introduced by Appell. Some elementary properties of our polynomials are investigated.

1. Introduction

Appell [3] introduced the polynomials $F_{m,n}(\alpha, \gamma, \gamma'; x, y)$ defined by

(1) $F_{m,n}(\alpha, \gamma, \gamma'; x, y)$

$$= (1-x-y)^{\gamma+\gamma'-\alpha} \frac{x^{1-\gamma}y^{1-\gamma'}}{(\gamma)_m(\gamma')_n} \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[x^{\gamma+m-1}y^{\gamma'+n-1}(1-x-y)^{\alpha+m+n-\gamma-\gamma'}\right]$$

in connection with an analysis of polynomials orthogonal with respect to the weight function

$$w(x, y) = x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'} \tag{2}$$

in the triangle T defined by

$$x \geq 0 \;,\quad y \geq 0 \;,\quad 1-x-y \geq 0 \;. \tag{3}$$

Here we have used Pochhammer's symbol $(\gamma)_m = \Gamma(\gamma+m)/\Gamma(\gamma)$, and we shall confine our attention to those cases where w is integrable on T, so that $\mathrm{Re}\gamma > 0$, $\mathrm{Re}\gamma' > 0$ and $\mathrm{Re}(\alpha-\gamma-\gamma') > -1$.

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Certain properties of Appell's polynomials are discussed in the treatise by Appell and Kampé de Feriet [5] and also in the Bateman Manuscript Project [Erdélyi, Magnus, Oberhettinger, Tricomi [7, 8]]. Unfortunately the $F_{m,n}$ do not form an orthogonal system for the weight function w and the region T defined in equations (2) and (3). However, in the special case where $\alpha = \gamma + \gamma'$ so that the weight function is

$$t(x, y) = x^{\gamma-1}y^{\gamma'-1}, \tag{4}$$

Appell [4] showed that the polynomials $F_{mn}(\gamma, \gamma', x, y)$ and $E_{mn}(\gamma, \gamma', x, y)$ defined by

$$F_{mn}(\gamma, \gamma', x, y) = F_{m,n}(\gamma+\gamma', \gamma, \gamma'; x, y) \tag{5}$$

and

$$E_{mn}(\gamma, \gamma', x, y) = F_2(\gamma+\gamma'+m+n, -m, -n, \gamma, \gamma'; x, y) \tag{6}$$

form a biorthogonal system, the basic formula being

$$\iint x^{\gamma-1}y^{\gamma'-1}F_{mn}(\gamma, \gamma', x, y)E_{kl}(\gamma, \gamma', x, y)\,dx\,dy = \frac{\delta_{mk}\delta_{nl}}{\gamma+\gamma'+2m+2n}\,\frac{m!n!(m+n)!\Gamma(\gamma)\Gamma(\gamma')}{(\gamma)_m(\gamma')_n\Gamma(\gamma+\gamma'+m+n)}. \tag{7}$$

Here δ_{mk} is the usual Kronecker symbol defined by

$$\delta_{mk} = 0, \quad m \neq k; \quad \delta_{mk} = 1, \quad m = k, \tag{8}$$

and the Appell hypergeometric function $F_2(a, b, b', c, c'; x, y)$ is defined for $|x| + |y| < 1$ by the double series (Appell [1, 2])

$$F_2(a, b, b', c, c'; x, y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_{j+k}(b)_j(b')_k x^j y^k}{(c)_j(c')_k j!k!}. \tag{9}$$

The main purpose of this paper is to present and to discuss a system of polynomials $E_{m,n}(\alpha, \gamma, \gamma', x, y)$ which is biorthogonal to the system $F_{m,n}(\alpha, \gamma, \gamma', x, y)$ for the weight function w and the region T defined by equations (2) and (3). Explicitly, our polynomials $E_{m,n}(\alpha, \gamma, \gamma'; x, y)$ are given by

$$E_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y) = F_2(\alpha+m+n,\ -m,\ -n,\ \gamma,\ \gamma';\ x,\ y)\ , \tag{10}$$

which of course reduce to Appell's $E_{mn}(\gamma,\ \gamma';\ xy)$ in the special case where $\alpha = \gamma + \gamma'$. Our principal result is the biorthogonality relation

$$\iint_T x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}F_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y) \times$$
$$\times\ E_{k,l}(\alpha,\ \gamma,\ \gamma';\ x,\ y)dxdy = \frac{\delta_{mk}\delta_{nl}\Gamma(\gamma)\Gamma(\gamma')\Gamma(\alpha-\gamma-\gamma'+m+n+1)m!n!}{(\alpha+2m+2n)(\gamma)_m(\gamma')_n\Gamma(\alpha+m+n)}\ . \tag{11}$$

A secondary purpose of this paper is to draw attention to the construction by Karlin and McGregor [9] of a system of polynomials orthogonal with respect to the weight function w and the region T of equations (2) and (3). In this apparently little known paper Karlin and McGregor first solved the rather more general problem of obtaining a system of discrete orthogonal polynomials in a triangular region and then by a limiting process obtained the corresponding continuous orthogonal polynomials. We shall give a direct derivation of these orthogonal polynomials by using a procedure suggested by the work of Kimura [10] for the solution of a diffusion equation arising in mathematical genetics.

2. Biorthogonality of $F_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$ and $E_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$

The first step in proving the biorthogonality of the polynomial sequences $\{F_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y)\}$ and $\{E_{k,l}(\alpha,\ \gamma,\ \gamma';\ x,\ y)\}$ is to show that both $F_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$ and $E_{k,l}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$ satisfy the partial differential equation

$$x(1-x)z_{xx} - 2xyz_{xy} + y(1-y)z_{yy} + [\gamma-(\alpha+1)x]z_x + [\gamma'-(\alpha+1)y]z_y + L(L+\alpha)z = 0\ , \tag{12}$$

where $L = m + n$ for $F_{m,n}$, $L = k + l$ for $E_{k,l}$ and we have used the convenient abbreviations $z_x = \partial z/\partial x$, $z_{xy} = \partial^2 z/\partial x\partial y$. In the case of the polynomials

$$E_{k,l}(\alpha,\ \gamma,\ \gamma';\ x,\ y) = F_2(\alpha+k+l,\ -k,\ -l,\ \gamma,\ \gamma';\ x,\ y)$$

this property follows almost immediately from the well-known fact ([7], Section 5.9, equation (10)) that the function

$$z = F_2(a,\ b,\ b',\ c,\ c';\ x,\ y)$$

satisfies the simultaneous partial differential equations

$$x(1-x)z_{xx} - xyz_{xy} + [c-(a+b+1)x]z_x - byz_y - abz = 0$$

and

$$y(1-y)z_{yy} - xyz_{xy} + [c'-(a+b+1)y]z_y - b'xz_x - ab'z = 0\ ,$$

and therefore also satisfies their sum, namely,

$$(13)\quad x(1-x)z_{xx} - 2xyz_{xy} + y(1-y)z_{yy} + [c-(a+b+b'+1)x]z_x + [c'-(a+b+b'+1)y]z_y - a(b+b')z = 0\ .$$

The required result for $v = E_{k,l}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$, namely,

$$(14)\quad x(1-x)v_{xx} - 2xyv_{xy} + y(1-y)v_{yy} + [\gamma-(\alpha+1)x]v_x + [\gamma'-(\alpha+1)y]v_y + (k+l)(\alpha+k+l)v = 0\ ,$$

follows directly on making the identifications $a = \alpha + k + l$, $b = -k$, $b' = -l$, $c = \gamma$, $c' = \gamma'$. In order to prove the corresponding result for $F_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$, we note that provided $|x| + |y| < 1$ and $\gamma + \gamma' - \alpha - m - n < 0$,

$$(1-x-y)^{\alpha+m+n-\gamma-\gamma'} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (\gamma+\gamma'-\alpha-m-n)_{r+s} \frac{x^r y^s}{r!s!}\ ,$$

so that from equation (1),

$$F_{m,n}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$$

$$= (1-x-y)^{\gamma+\gamma'-\alpha} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (\gamma+\gamma'-\alpha-m-n)_{r+s} \frac{(\gamma+m)_r}{(\gamma)_r} \frac{(\gamma'+n)_s}{(\gamma')_s} \frac{x^r y^s}{r!s!}$$

$$= (1-x-y)^{\gamma+\gamma'-\alpha} F_2(\gamma+\gamma'-\alpha-m-n,\ \gamma+m,\ \gamma'+n,\ \gamma,\ \gamma';\ x,\ y)\ .$$

Since $w = F_2(\gamma+\gamma'-\alpha-m-n,\ \gamma+m,\ \gamma'+n,\ \gamma,\ \gamma';\ x,\ y)$ satisfies the partial differential equation

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$$x(1-x)w_{xx} - 2xyw_{xy} + y(1-y)w_{yy} + [\gamma-(2\gamma+2\gamma'-\alpha+1)x]w_x$$
$$+ [\gamma'-(2\gamma+2\gamma'-\alpha+1)y]w_y + (\alpha+m+n-\gamma-\gamma')(m+n+\gamma+\gamma')w = 0 ,$$

$u = F_{m,n}(\alpha, \gamma, \gamma'; x, y)$ is easily shown to satisfy the equation

$$(15) \quad x(1-x)u_{xx} - 2xyu_{xy} + y(1-y)u_{yy} + [\gamma-(\alpha+1)x]u_x + [\gamma'-(\alpha+1)y]u_y$$
$$+ (m+n)(m+n+\alpha)u = 0 ,$$

thus completing the proof of our first assertion.

On multiplying equation (14) by u and subtracting equation (15) multiplied by v, we obtain

$$(16) \quad x(1-x)\left(uv_{xx}-vu_{xx}\right) - 2xy\left(uv_{xy}-vu_{xy}\right) + y(1-y)\left(uv_{yy}-vu_{yy}\right) +$$
$$[\alpha-(\alpha'+1)x]\left(uv_x-vu_x\right) + [\alpha'-(\alpha+1)y]\left(uv_y-vu_y\right) = (m+n-k-l)(m+n+k+l+\alpha)uv .$$

We now define functions P, Q and H by

$$P = uv_x - vu_x ,$$

$$Q = uv_y - vu_y ,$$

and

$$H = x^{\gamma}y^{\gamma'}(1-x-y)^{\alpha-\gamma-\gamma'}(P-Q) .$$

We find that equation (16) may be written as

$$\frac{\partial}{\partial x}\left[x^{\gamma}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'+1}P\right] + \frac{\partial}{\partial y}\left[x^{\gamma-1}y^{\gamma'}(1-x-y)^{\alpha-\gamma-\gamma'}Q\right] + \frac{\partial H}{\partial x} - \frac{\partial H}{\partial y}$$
$$= (m+n-k-l)(m+n+k+l+\alpha)x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}uv .$$

When this equation is integrated over the region T, we find, using an obvious notation, that

$$(17) \quad (m+n-k-l)(m+n+k+l+\alpha)\iint_T x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}F_{m,n}E_{k,l}dxdy = 0 .$$

Thus if $m + n \neq k + l$, equation (17) requires that

$$(18) \qquad \iint_T x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}F_{m,n}E_{k,l}dxdy = 0 ,$$

since by the integrability requirements the only case in which $m + n + k + l + \alpha$ can vanish is $m = n = k = l = \alpha = 0$, which violates the assumption that $m + n \neq k + l$.

On the other hand, if $k + l = m + n$, we may use equations (1) and (10) directly to show that

$$\iint_T x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}F_{m,n}E_{k,l}\,dxdy$$

$$= \frac{1}{(\gamma)_m(\gamma')_n}\iint_T \frac{\partial^{m+n}}{\partial x^m \partial y^n}\left[x^{\gamma+m-1}y^{\gamma'+n-1}(1-x-y)^{\alpha+m+n-\gamma-\gamma'}\right]E_{k,l}\,dxdy$$

$$= \frac{(-1)^{m+n}}{(\gamma)_m(\gamma')_n}\iint_T x^{\gamma+m-1}y^{\gamma'+n-1}(1-x-y)^{\alpha+m+n-\gamma-\gamma'}\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left(E_{k,l}\right)dxdy$$

$$= \frac{\delta_{mk}\delta_{nl}\Gamma(\gamma)\Gamma(\gamma')\Gamma(\alpha-\gamma-\gamma'+m+n+1)m!n!}{(\alpha+2m+2n)(\gamma)_m(\gamma')_n\Gamma(\alpha+m+n)} ,$$

since the only term of total degree $m + n = k + l$ in $E_{k,l}(\alpha, \gamma, \gamma'; x, y)$ is

$$\frac{(\alpha+k+l)_{k+l}(-k)_k(-l)_l x^k y^l}{(\gamma)_k(\gamma')k!l!} = (-1)^{k+l}\frac{\Gamma(\alpha+2k+2l)}{\Gamma(\alpha+k+l)}\frac{x^k y^l}{(\gamma)_k(\gamma')_l} .$$

We have also used the result

$$\iint_T x^{a-1}y^{b-1}(1-x-y)^{c-1}dxdy = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)} ,$$

which is valid for $\mathrm{Re}\,a > 0$, $\mathrm{Re}\,b > 0$ and $\mathrm{Re}\,c > 0$. This completes the proof of the fundamental biorthogonality relation (11).

3. Recurrence relations for $E_{m,n}(\alpha, \gamma, \gamma'; x, y)$

The aim of this section is to prove the existence of recurrence relations of the form

$$xE_{m,n} = aE_{m+1,n} + b_1E_{m+1,n-1} + b_2E_{m,n} + c_1E_{m+1,n-2} + c_2E_{m,n-1} + c_3E_{m-1,n}$$

and

$$yE_{m,n} = dE_{m,n+1} + e_1E_{m-1,n+1} + e_2E_{m,n} + f_1E_{m-2,n+1} + f_2E_{m-1,n} + f_3E_{m,n-1} ,$$

where a, b_1, b_2, c_1, c_2, c_3, d, e_1, e_2, f_1, f_2 , and f_3 are certain constants whose precise values will be determined later. Similar recurrences for *orthogonal* polynomials in two independent variables are discussed in general terms in [8], Section 12.3.

It is convenient to present the discussion by means of a series of lemmas.

LEMMA 3.1. *Given any monomial* $x^m y^n$, *there exists a unique set of constants* c_{kl}, *such that*

$$x^m y^n = \sum_{k+l \le m+n} \sum c_{kl} E_{k,l} . \tag{19}$$

Proof. We note first of all that there is only one term of total degree $k + l$ in $E_{k,l}$, and that this one term is of precise degree k in x and l in y. All other terms in $E_{k,l}$ are of lower total degree than $k + l$. Because of this, the various constants c_{kl} may be determined in a unique recursive fashion. c_{mn} is the reciprocal of the coefficient of the unique term in $x^m y^n$ in $E_{m,n}$. If $k + l = m + n$ but $k \ne m$ and so $l \ne n$, $c_{kl} = 0$, so that we may rewrite equation (19) as

$$x^m y^n - c_{mn} E_{m,n} = \sum_{k+l \le m+n-1} \sum c_{kl} E_{k,l} . \tag{20}$$

The coefficients c_{kl} with $k + l = m + n - 1$ may now be determined uniquely from the terms in $x^k y^l$ with $k + l = m + n - 1$ in the left-hand-side of equation (20). It is clear that this process may be repeated until all of the coefficients have been determined uniquely.

Two functions u and v defined in the region T of equation (3) are said to be orthogonal if $\iint_T w(x, y) uv dx dy = 0$.

LEMMA 3.2. $E_{m,n}$ *is orthogonal to* $E_{k,l}$ *if* $m + n \ne k + l$.

Proof. This result follows by a trivial modification of the proof in the previous section of the biorthogonality of $E_{m,n}$ and $E_{k,l}$ for $m + n \ne k + l$. For, on redefining u as $E_{m,n}$ and keeping $v = E_{k,l}$, the argument from equation (16) to equation (18) inclusive still obtains, thus proving the lemma.

LEMMA 3.3. *$E_{m,n}$ is orthogonal to every polynomial $P(x, y)$ of total degree less than $m + n$, that is, to every polynomial*

$$P(x, y) = \sum_{i+j<m+n}\sum A_{ij}x^i y^j .$$

Proof. By Lemma 3.1, $P(x, y)$ can be written as $\sum_{i+j<m+n}\sum B_{ij}E_{i,j}$ for appropriate constants B_{ij}. Each term in this finite sum is then orthogonal to $E_{m,n}$ by Lemma 3.2.

LEMMA 3.4. *$F_{m,n}$ is orthogonal to every polynomial $P(x, y)$ of degree k in x and l in y provided either $k < m$ or $l < n$.*

Proof. This result follows immediately on writing down the relevant integral, using equation (1) and integrating by parts.

With these lemmas at our disposal it is easy to prove

THEOREM 3.1. *There exist constants a, b_1, b_2, c_1, c_2, and c_3 such that*

$$xE_{m,n} = aE_{m+1,n} + b_1E_{m+1,n-1} + b_2E_{m,n} + c_1E_{m+1,n-2} + c_2E_{m,n-1} + c_3E_{m-1,n} .$$

Proof. It follows from Lemma 3.1 that we may write

$$xE_{m,n} = \sum_{k+l\le m+n+1}\sum a_{kl}E_{k,l} . \tag{21}$$

Indeed we may write

$$xE_{m,n} = \sum_{k=0}^{m+1}\sum_{l=0}^{n} a_{kl}E_{k,l} , \tag{22}$$

because if we multiply both sides of equation (21) by $w(x, y)_{k,l}$ and integrate over T, Lemma 3.4 implies that $a_{kl} = 0$ if either $k > m + 1$ or $l > n$. Moreover, if $k + l < m + n - 1$, the same operation of multiplying by $w(x, y)F_{i,j}$ with $i + j < m + n - 1$ and integrating over T gives, on the left-hand-side, the single term $\iint_T w(x, y)xF_{i,j}E_{m,n}dxdy$, where $xF_{i,j}$ is a polynomial of total degree less than $m+n$, so that by Lemma 3.3 the left-hand-side vanishes. However, by the biorthogonality of $F_{i,j}$ and $E_{k,l}$, the right-hand-side is a non-zero multiple of a_{ij}, so that a_{ij} must vanish for $i+j < m+n-1$. Consequently, the only non-vanishing a_{kl} occur for integral k and l satisfying simultaneously

$$m+n-1 \le k+l \le m+n+1 \,,$$

$$k \le m+1 \,,$$

and

$$l \le n \,;$$

that is, for (k, l) being one of the six combinations $(m+1, n)$, $(m+1, n-1)$, (m, n), $(m+1, n-2)$, $(m, n-1)$ or $(m-1, n)$. The actual values of the surviving a_{kl} may be computed by the same procedure that was outlined in the proof of Lemma 3.1. After some tedious algebra one obtains the result

$$\begin{aligned}(23)\quad xE_{m,n} = &-\frac{(\gamma+m)(\alpha+m+n)}{A(A+1)}E_{m+1,n} \\ &-\frac{2n(\gamma+m)}{(A-1)(A+1)}E_{m+1,n-2}+\left[\frac{(\gamma+m)(m+1)}{A+1}-\frac{m(m+\gamma-1)}{A-1}\right]E_{m,n} \\ &-\frac{n(n-1)(\gamma+m)}{A(A-1)(\alpha+m+n-1)}E_{m+1,n-2}-\frac{n}{\alpha+m+n-1}\left[\frac{m(\gamma+m-1)}{A-1}+\frac{(m+1)(\gamma+m)(A-2)}{A(A+1)}\right]E_{m,n-1} \\ &-\frac{m}{\alpha+m+n-1}\left[\frac{m(m+\gamma-1)(A-2)}{A+1}-\frac{(m+1)(\gamma+m)(A-2)}{2A}-\frac{(m-1)(\gamma+m-2)}{2}\right]E_{m-1,n}\,,\end{aligned}$$

where

$$(24)\qquad A = \alpha + 2m + 2n \,.$$

Similarly we may prove

THEOREM 3.2.

$$(25)\quad yE_{m,n} = -\frac{(\gamma'+n)(\alpha+m+n)}{A(A+1)}E_{m,n+1}$$
$$-\frac{2m(\gamma'+n)}{(A-1)(A+1)}E_{m-1,n+1} + \left[\frac{(\gamma'+n)(n+1)}{A+1} - \frac{n(n+\gamma'-1)}{A-1}\right]E_{m,n}$$
$$-\frac{m(m-1)(\gamma'+n)}{A(A-1)(\alpha+m+n-1)}E_{m-2,n+2} - \frac{m}{\alpha+m+n-1}\left[\frac{n(\gamma'+n-1)}{A-1} + \frac{(n+1)(\gamma'+n)(A-2)}{A(A+1)}\right]E_{m-1,n}$$
$$-\frac{n}{\alpha+m+n-1}\left[\frac{n(n+\gamma')(A-2)}{A+1} - \frac{(n+1)(\gamma'+n)(A-2)}{2A} - \frac{(n-1)(\gamma'+n-2)}{2}\right]E_{m,n-1}\ .$$

4. Orthogonal polynomials in the triangle

As mentioned in the introduction, Karlin and McGregor [9] derived from a system of orthogonal polynomials for discrete variables a family of polynomials orthogonal with respect to the weight function w and triangular region T of equations (2) and (3). In this section we shall derive these orthogonal polynomials of Karlin and McGregor by the more conventional technique of examining those solutions of equation (12) that may be written as a product of two factors. To do this we follow Kimura's technique [10] of introducing variables

$$(26)\qquad \xi = x + y$$

and

$$(27)\qquad \eta = \frac{y}{x+y}\ ,$$

which is equivalent to

$$(28)\qquad x = \xi(1-\eta)$$

and

$$(29)\qquad y = \xi\eta\ .$$

Defining $u(\xi, \eta)$ by

$$(30)\qquad u(\xi, \eta) = z(x, y)\ ,$$

we find that equation (12) transforms to

$$(31)\quad \xi(1-\xi)\frac{\partial^2 u}{\partial\xi^2} + \frac{\eta}{\xi}(1-\eta)\frac{\partial^2 u}{\partial\eta^2} + [\gamma+\gamma'-(\alpha+1)\xi]\frac{\partial u}{\partial\xi}$$
$$+\frac{1}{\xi}[\gamma'-(\gamma+\gamma')\eta]\frac{\partial u}{\partial\eta} + \lambda u = 0\ ,$$

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where

(32) $$\lambda = L(L+\alpha) .$$

If we assume that $u(\xi, \eta)$ may be written in the product form

(33) $$u(\xi, \eta) = X(\xi)Y(\eta) ,$$

we obtain by the usual procedure the two equations

(34) $$\eta(1-\eta) \frac{d^2Y}{d\eta^2} + [\gamma'-(\gamma+\gamma')\eta] \frac{dY}{d\eta} + KY = 0$$

and

(35) $$\xi^2(1-\xi) \frac{d^2X}{d\xi^2} + [(\gamma+\gamma')-(\alpha+1)\xi]\xi \frac{dX}{d\xi} + (\lambda\xi-K)X = 0 ,$$

where K is some constant. From the theory of Sturm-Liouville problems with singular boundary points it is known (*cf.* Courant and Hilbert [6], p. 328) that the only well-behaved solutions of equation (34) occur for

(36) $$K = n(n+\gamma+\gamma'-1) ,$$

where n is a non-negative integer. In this case equation (34) has as its only well-behaved solution the Jacobi polynomial

(37) $$Y(\eta) = G_n(\gamma+\gamma'-1, \gamma', n) ,$$

where we have used the notation of Courant and Hilbert [6], namely,

(38) $$G_n(p, q, x) = {}_2F_1(-n, p+n, q; x) .$$

With the value of K given by equation (36) we find, on making the substitution $X = \xi^n U$, that equation (35) transforms to

(39) $$\xi(1-\xi) \frac{d^2U}{d\xi^2} + [(\gamma+\gamma'+2n)-(\alpha+1+2n)\xi] \frac{dU}{d\xi} + [\lambda-n(\alpha+n)]U = 0 ,$$

for which well-behaved solutions occur only for

(40) $$\lambda = (n+l)(n+l+\alpha) ,$$

where l is a non-negative integer. The corresponding solutions of equation (39) are

(41) $$U(\xi) = G_l(\alpha+2n, \gamma+\gamma'+2n, \xi) ,$$

so that our solution of equation (12) is

$$(42)\quad z(x,\ y) = P_{ln}(\alpha,\ \gamma,\ \gamma;\ x,\ y)$$
$$= (x+y)^n G_l(\alpha+2n,\ \gamma+\gamma'+2n,\ x+y) G_n\left(\gamma+\gamma'-1,\ \gamma',\ \frac{y}{x+y}\right),$$

where l and n are arbitrary non-negative integers. It is clear that $P_{ln}(\alpha,\ \gamma,\ \gamma';\ x,\ y)$ is a polynomial, since the function $G_n\left(\gamma+\gamma'-1,\ \gamma',\ \frac{y}{x+y}\right)$ is precisely of degree n in $(x+y)^{-1}$.

We now prove that $\{P_{ln}\}$ is an orthogonal system for $w(x,\ y)$ in the triangular region T.

We have

$$\iint_T wP_{kl}P_{mn}dxdy$$
$$= \iint_T x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}P_{kl}P_{mn}dxdy$$
$$= \left\{\int_0^1 \eta^{\gamma'-1}(1-\eta)^{\gamma-1}G_l(\gamma+\gamma'-1,\ \gamma',\ \eta)G_n(\gamma+\gamma'-1,\ \gamma',\ \eta)d\eta\right\} \times$$
$$\times \left\{\int_0^1 \xi^{\gamma+\gamma'+l+n-1}(1-\xi)^{\alpha-\gamma-\gamma'}G_k(\alpha+2l,\ \gamma+\gamma'+2l,\ \xi)G_m(\alpha+2n,\ \gamma+\gamma'+2n,\ \xi)d\xi\right\}.$$

Now, if $l \neq n$, the first integral is zero. If $l = n$, the second integral vanishes unless $k = m$. Thus we have

$$(43)\qquad \iint_T w(x,\ y)P_{kl}(x,\ y)P_{mn}(x,\ y)dxdy = \delta_{km}\delta_{ln}R_{kl},$$

where R_{kl} is a certain constant. In fact it is easily shown that

$$(44)\quad R_{kl} = \frac{k!\,l!\,\Gamma(\gamma+l)\Gamma(\alpha-\gamma-\gamma'+1+k)\Gamma^2(\gamma')\Gamma^2(\gamma+\gamma'+2l)}{(\gamma+\gamma'-1+2l)(\alpha+2l+2k)\Gamma(\gamma'+l)\Gamma(\gamma+\gamma'-1+l)\Gamma(\alpha+2l+k)\Gamma(\gamma+\gamma'+2l+k)}.$$

5. Other biorthogonal and orthogonal polynomials in the triangle

Other biorthogonal and orthogonal families of polynomials in the triangle given by equation (3) may be obtained if we note that both the weight function of equation (2) and the triangular region specified by

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equation (3) are invariant under the simultaneous permutation of the variables x, y and the parameters γ and γ' given explicitly by

$$x \to y\ ,$$

$$y \to 1 - x - y\ ,$$

$$\gamma \to \gamma'\ ,$$

$$\gamma' \to \alpha - \gamma - \gamma' - 1\ ,$$

which of course entails

$$1 - x - y \to x$$

and

$$\alpha \to \alpha\ .$$

Indeed, the complete group of substitutions under which the region and the weight function are invariant gives rise to the fact that $\{P_{ln}(\alpha,\ \gamma_1,\ \gamma_2;\ x_1,\ x_2\}$ is a set of orthogonal polynomials, and $\{E_{k,l}(\alpha,\ \gamma_1,\ \gamma_2;\ x_1,\ x_2\}$ and $\{F_{k,l}(\alpha,\ \gamma_1,\ \gamma_2;\ x_1,\ x_2\}$ are biorthogonal sets of polynomials for our weight function and region provided $\{\gamma_1,\ \gamma_2,\ x_1,\ x_2\}$ is identical with one of the six sets

$$\{\gamma,\ \gamma',\ x,\ y\}\ , \tag{45}$$

$$\{\gamma',\ \alpha-\gamma-\gamma'+1,\ y,\ 1-x-y\}\ , \tag{46}$$

$$\{\alpha-\gamma-\gamma'+1,\ \gamma,\ 1-x-y,\ x\}\ , \tag{47}$$

$$\{\gamma,\ \alpha-\gamma-\gamma'+1,\ x,\ 1-x-y\}\ , \tag{48}$$

$$\{\gamma',\ \gamma,\ y,\ x\}\ , \tag{49}$$

or

$$\{\alpha-\gamma-\gamma'+1,\ \gamma',\ 1-x-y,\ y\}\ . \tag{50}$$

In this connection there arises a sequence of interesting problems, such as the determination of coefficients C_{kl}^{ij} such that, for example,

$$E_{k,l}(\alpha,\ \gamma';\ \alpha-\gamma-\gamma'+1,\ y,\ 1-x-y) = \sum_{i,j} C_{kl}^{ij}\, E_{i,j}(\alpha,\ \gamma,\ \gamma';\ x,\ y)\ . \tag{51}$$

From equation (11) we find that

$$(52)\quad C_{kl}^{ij} = \frac{(\alpha+2i+2j)(\gamma)_i(\gamma')_j\Gamma(\alpha+i+j)}{\Gamma(\gamma)\Gamma(\alpha')\Gamma(\alpha-\gamma-\gamma'+i+j+1)i!j!} \iint_T x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}F_{i,j}(\alpha, \gamma, \gamma', x, y) \times$$
$$\times E_{k,l}(\alpha, \gamma', \alpha-\gamma-\gamma'+1, y, 1-x-y)dxdy .$$

However, we shall not pursue such problems in this paper.

6. Conclusion

In equations (1), (10) and (11) we have obtained an explicit answer to the problem raised in Volume 2, p. 271 of [8], namely, to find systems of polynomials biorthogonal in the triangle $0 \le x, y, 1-x-y \le 1$ with respect to the weight function $x^{\gamma-1}y^{\gamma'-1}(1-x-y)^{\alpha-\gamma-\gamma'}$. Though some properties of these biorthogonal polynomials have been discussed in the present paper, the investigation of these polynomials opens up a wide field of research.

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