

FUNDAMENTAL THEOREM OF PREHOMOGENEOUS
VECTOR SPACES OF CHARACTERISTIC p

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For a local field of characteristic 0, the functional equations of zeta distributions of prehomogeneous vector spaces have been obtained by M. Sato, Shintani, Igusa, F. Sato and Gyoja. In this paper, we shall consider the case of local fields of characteristic $p > 0$.

1. K -REGULAR P.V.'S

We fix a local field K of characteristic $p > 0$. Let G be a connected linear algebraic group, ρ its rational representation of G on a finite-dimensional vector space V , all defined over an algebraic closure \bar{K} of K . We call a triplet (G, ρ, V) a prehomogeneous vector space (abbreviated P.V.) if V has a Zariski-dense G -orbit Y .

Any point of Y is called a generic point and the isotropy subgroup $G_y = \{g \in G; \rho(g)y = y\}$ of a generic point y is called a generic isotropy subgroup. Note that we have $\dim G_y = \dim G - \dim V$ if and only if $y \in Y$. A non-zero rational function $f(x)$ on V is called a relative invariant of (G, ρ, V) if $f(\rho(g)x) = \chi(g)f(x)$ holds for any $g \in G$ and $x \in Y$ where $\chi: G \rightarrow GL_1$ is a rational character of G .

The complement S of Y is a Zariski-closed set which is called the singular set of the P.V. (G, ρ, V) . Now we assume that (G, ρ, V) is defined over K , that is, G, ρ, V are all defined over K . Let $S_i = \{x \in V; f_i(x) = 0\}$ ($i = 1, \dots, l$) be the K -irreducible component of the K -rational points S_K of S of codimension one defined by a K -irreducible (not necessarily absolutely irreducible) polynomial $f_i(x)$ ($i = 1, \dots, l$). Then $f_1(x), \dots, f_l(x)$ are algebraically independent relative invariants and any relative invariant $f(x)$ in $K(V)$ is of the form $f(x) = c \cdot f_1(x)^{m_1} \cdots f_l(x)^{m_l}$ ($c \in K^\times$, $(m_1, \dots, m_l) \in \mathbf{Z}^l$). We call $f_1(x), \dots, f_l(x)$ the basic K -relative invariants of (G, ρ, V) . Let χ_i be the rational character of G corresponding to f_i ($i = 1, \dots, l$). Let $X(G)_K$ be the group of K -rational characters of G , $X_1(G)_K$ its subgroup corresponding to K -relative invariants. Then $X_1(G)_K$ is a free Abelian group of rank l generated by χ_1, \dots, χ_l .

Received 9th December, 1996

We would like to express our hearty thanks to Professor M. Kashiwara and Professor A. Gyoja who gave us invaluable suggestions for our research.

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Let G_1 be a subgroup of G generated by the commutator subgroup $[G, G]$ and a generic isotropy subgroup. This does not depend on the choice of a generic point. For $\chi \in X(G)_K$, it is in $X_1(G)_K$ if and only if $\chi|_{G_1} = 1$. For a relative invariant $f(x)$ of (G, ρ, V) , we can define a rational map $\varphi_f : Y \rightarrow V^*$ by

$$\varphi_f(x) = {}^t \left(\frac{1}{f(x)} \cdot \frac{\partial f}{\partial x_1}(x), \dots, \frac{1}{f(x)} \cdot \frac{\partial f}{\partial x_n}(x) \right)$$

where V^* is the dual vector space of V and we identify V and V^* with \overline{K}^n by taking a pair of dual bases. This definition does not depend on the choice of bases. We sometimes denote $\varphi_f(x)$ by $\text{grad log } f(x)$. By a direct calculation, we have

- (1) $\varphi_f(\rho(g)x) = \rho^*(g)\varphi_f(x)$ for $g \in G$ and $x \in Y$ where ρ^* denotes the contragredient representation of ρ , and
- (2) $\langle d\rho(A)x, \varphi_f(x) \rangle = d\chi(A)$ for $x \in Y$ and $A \in \text{Lie}(G)$ where $d\rho$ (respectively $d\chi$) is the infinitesimal representation of ρ (respectively the infinitesimal character of χ) of the Lie algebra $\text{Lie}(G)$ of G .

DEFINITION 1.1: A relative invariant is called non-degenerate if $\varphi_f : Y \rightarrow V^*$ is dominant. If there exists a non-degenerate relative invariant $f(x)$ in $K(V)$, we say that (G, ρ, V) is a K -regular P.V. In this case, $\varphi_f(Y)$ is a Zariski-dense G -orbit in V^* , and hence the dual triplet (G, ρ^*, V^*) is also a P.V. For a generic point $y \in Y^*$, the dominant morphism $\psi : G \rightarrow V^*$ defined by $\psi(g) = \rho^*(g)y$ is called an open orbit morphism. If there is a non-degenerate relative invariant f such that $\varphi_f : Y \rightarrow V^*$ is separable and also an open orbit morphism $\psi : G \rightarrow V^*$ is separable, we say that (G, ρ, V) is a separable K -regular P.V. In this case, the Hessian

$$F(x) = \det \left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}(x) \right)$$

of $\log f$ is a non-zero relative invariant corresponding to the character χ_0^{-1} with $\chi_0(g) = \det \rho(g)^2$ for $g \in G$. Hence we have $\det \rho(g)^2 \in X_1(G)_K$ for a separable K -regular P.V. In general, we denote by Y_K, S_K , et cetera, K -rational points of Y, S , et cetera. We write $X_1^*(G)_K$ (respectively $X^*(G)_K, Y^*, S^*$, et cetera) for (G, ρ^*, V^*) which corresponds to $X_1(G)_K$ (respectively $X(G)_K, Y, S$, et cetera) for (G, ρ, V) .

PROPOSITION 1.2. Assume that (G, ρ, V) and (G, ρ^*, V^*) are K -regular P.V.'s. Then we have the following assertion.

- (1) $X_1(G)_K = X_1^*(G)_K$.
- (2) For a non-degenerate K -relative invariant f , the map $\varphi = \text{grad log } f : Y \rightarrow Y^*$ is bijective.

PROOF: Since $\varphi(Y)$ is a Zariski-dense G -orbit in V^* , we have $\varphi(Y) = Y^*$, that is, φ is surjective. Since $\rho^*(g)\varphi(x) = \varphi(\rho(g)x)$, we have $G_x \subset G_{\varphi(x)}$ for $x \in Y$. Now let f^* be a non-degenerate relative invariant in $K(V^*)$, and put $\varphi^* = \text{grad log } f^* : Y^* \rightarrow Y$. Similarly we have $G_y \subset G_{x'}$ for $y = \varphi(x)$ and $x' = \varphi^*(y)$, and hence $G_x \subset G_y \subset G_{x'}$. Since $x' = \rho(g_0)x$ for some $g_0 \in G$, we have $G_{x'} = g_0G_xg_0^{-1} \supset G_x$. Since $\dim G_{x'} = \dim G_x$, the algebraic group $G_{x'}$ and G_x have the same connected component H of the identity. Since $G_{x'} = g_0G_xg_0^{-1}$ is isomorphic to G_x , the numbers of their connected components coincide, that is, $[G_{x'} : H] = [G_x : H]$ with $G_{x'} \supset G_x$. This implies $G_{x'} = G_x$, and hence $G_x = G_y$ with $y = \varphi(x)$.

Thus we have $G_1 = G_1^*$ and hence $X_1(G)_K = X_1^*(G)_K$. Note that $X_1(G)_K = \{\chi \in X(G)_K; \chi|_{G_1} = 1\}$. Now assume that $\varphi(x_1) = \varphi(x_2)$ with $x_2 = \rho(g)x_1$ for some $g \in G$. Then we have $\varphi(x_1) = \varphi(x_2) = \varphi(\rho(g)x_1) = \rho^*(g)\varphi(x_1)$ and hence $g \in G_{\varphi(x_1)} = G_{x_1}$, that is, $x_2 = \rho(g)x_1 = x_1$. Thus φ is injective. □

PROPOSITION 1.3. *Assume that (G, ρ, V) and (G, ρ^*, V^*) are K -regular P.V.'s. Then we have $\# \rho(G)_K \backslash Y_K = \# \rho^*(G)_K \backslash Y_K^*$.*

PROOF: Let f be a non-degenerate relative invariant in $K(V)$ and put $\varphi = \text{grad log } f$. Then for any $x \in Y_K$, we have $\varphi(\rho(G)_K \cdot x) = \rho^*(G)_K \cdot \varphi(x) \subset Y_K^*$, that is, φ maps an orbit in Y_K to an orbit in Y_K^* .

By Proposition 1.2, this map φ is injective, and hence $\# \rho(G)_K \backslash Y_K \leq \# \rho^*(G)_K \backslash Y_K^*$. Similarly we have $\# \rho^*(G)_K \backslash Y_K^* \leq \# \rho(G)_K \backslash Y_K$. □

PROPOSITION 1.4. *Assume that (G, ρ, V) is a separable K -regular P.V. Then its dual (G, ρ^*, V^*) is also a separable K -regular P.V.*

PROOF: Let f be a separable non-degenerate relative invariant in $K(V)$ and put $\varphi = \text{grad log } f : Y \rightarrow Y^*$. First we show that φ is injective. Assume that $\varphi(x) = \varphi(x')$. Since $d\chi(A) = \langle d\rho(A)x, \varphi(x) \rangle = -\langle x, d\rho^*(A)\varphi(x) \rangle$, we have $\langle x - x', d\rho^*(A)\varphi(x) \rangle = 0$ for all $A \in \text{Lie}(G)$. Since $\psi : G \rightarrow V^*$ with $\psi(g) = \rho^*(g)\varphi(x)$ is separable, we have $\{d\rho^*(A)\varphi(x); A \in \text{Lie}(G)\} = V^*$, and hence $x - x' = 0$, that is, $x = x'$. For any $g \in G_{\varphi(x)} (\supset G_x)$, we have $\varphi(\rho(g)x) = \rho^*(g)\varphi(x) = \varphi(x)$. As φ is injective, we have $\rho(g)x = x$, that is, $g \in G_x$. This implies that $G_x = G_{\varphi(x)}$ and hence $X_1(G)_K = X_1^*(G)_K$. A rational character χ corresponding to f is in $X_1(G)_K$ and hence $\chi^{-1} \in X_1^*(G)_K$. This implies that there exists a relative invariant f^* in $K(V^*)$ satisfying $f^*(\rho^*(g)y) = \chi(g)^{-1}f^*(y)$ for $g \in G$ and $y \in Y^*$.

Put $\varphi^* = \text{grad log } f^*$. Then we have $\langle \varphi^*(y), d\rho^*(A)y \rangle = -d\chi(A)$. Since $d\chi(A) = \langle d\rho(A)x, \varphi(x) \rangle = -\langle x, d\rho^*(A)\varphi(x) \rangle$, we have $\langle x - \varphi^*(y), d\rho^*(A)y \rangle = 0$ for $y = \varphi(x)$ and all $A \in \text{Lie}(G)$.

Since the open orbit morphism ψ is separable, we have $\{d\rho^*(A)y; A \in \text{Lie}(G)\} = V^*$, and hence $\varphi^*(y) = x \in Y$, that is, $\varphi^*(Y^*) = Y$. This implies that Y and Y^*

are biregularly isomorphic. Hence f^* is separable non-degenerate, and an open orbit morphism $\psi^* : G \rightarrow V$ is separable, that is, (G, ρ^*, V^*) is a separable K -regular P.V. □

Note that in the case of $ch(K) = 0$, a K -regular P.V. is automatically a separable K -regular P.V. and the proof of Proposition 1.4 gives the equivalence between K -regularity of (G, ρ, V) and that of (G, ρ^*, V^*) .

Now we shall consider a sufficient condition to ensure that $\# \rho(G)_K \setminus Y_K$ is finite.

LEMMA 1.5. *Let X be a homogeneous space of the connected algebraic group G over \bar{K} where the action is defined over K . Let K_{sep} be the separable closure of K in \bar{K} . Then $X(K_{\text{sep}})$ is a homogeneous space of $G(K_{\text{sep}})$.*

PROOF: Since $X(K_{\text{sep}})$ is dense in $X = X(\bar{K})$ (see Borel [2]), we may take $x_0 \in X(K_{\text{sep}})$. Since the map $f : G \rightarrow X$ with $f(g) = g \cdot x_0$ is defined over K_{sep} , the fibre $f^{-1}(x)$ of each $x \in X(K_{\text{sep}})$ is also defined over K_{sep} . Since the set $X(\bar{K})$ is homogeneous by $G(\bar{K})$, we have $f^{-1}(x)(\bar{K}) \neq \emptyset$. $f^{-1}(x)(K_{\text{sep}})$ is dense in $f^{-1}(x)(\bar{K})$ and $f^{-1}(x)(K_{\text{sep}}) \subset G(K_{\text{sep}})$ which implies $g \cdot x_0 = x$. Thus $X(K_{\text{sep}})$ is a homogeneous space of $G(K_{\text{sep}})$. □

By this lemma, we can use Galois cohomology to investigate $X(K)$ as a $G(K)$ -space.

Professor J.P. Serre kindly told us of the following theorem with the proof which was explained by Tits to him.

THEOREM 1.6. *Let K be a local field of characteristic $p > 0$ (or more generally let K be a field complete with respect to a discrete valuation, and with the residue field k of type (F) in the sense of Serre [18]). Let G be a connected smooth reductive group over K . Then $H^1(K, G)$ is finite.*

PROOF: (After Serre’s letter on 9th September 1992.) Let K' be the maximal unramified extension of K . The field K' is known to be of dimension at most 1. (in the sense of [18, II, Section 3]). By a theorem of Steinberg (for K' perfect) and of Borel-Springer (for K' imperfect - see [1, II, p.761] we have $H^1(K', G) = 0$. Hence the Galois cohomology of G over K is killed by K' , that is, it is equal to $H^1(K'/K, G)$. We may now apply a theorem of Bruhat-Tits [3, Theorem 3.12]; this says that $H^1(K'/K, G)$ is contained in a finite union of cohomology sets $H^1(k, G_i)$, where the G_i ’s are algebraic linear groups (not necessarily connected) over k . Since k is type (F), each $H^1(k, G_i)$ is finite (see [1, II, p.404, Theorem 6.2], or [18, III, Theorem 4]). Hence $H^1(K, G)$ is finite. □

Let B be a subgroup of $GL_n(K_{\text{sep}})$ and A a normal subgroup of B . Then $G = Gal(K_{\text{sep}}/K)$ acts on them. For a 1-cocycle $b \in Z^1(G, B)$, let ${}_bA$ be the twist of A by $b = \{b_s\}$, that is, ${}_bA \cong A$, but G acts on ${}_bA$ by $s \circ a = b_s \cdot s(a) \cdot b_s^{-1}$ for $s \in G, a \in A$.

We learned the following lemma from Professor A. Gyoja.

LEMMA 1.7. *There exists a bijection between $H^1(G, A)$ and $H^1(G, {}_bA)$.*

PROOF: Since $H^1(G, GL_n) = \{1\}$, there exists $a \in GL_n$ satisfying $b_s = a^{-1} \cdot s(a)$ ($s \in G$). Define $f : GL_n \rightarrow GL_n$ by $f(x) = a^{-1}xa$. Then we have

$$s \circ f(x) = b_s \cdot s(a^{-1}xa) \cdot b_s^{-1} = a^{-1} \cdot s(x) \cdot a = f(s(x)),$$

and hence we can identify ${}_bA = f(A)$. For $\{a_s\} \in Z^1(G, A)$, we make correspond $\{f(a_s)\} \in Z^1(G, {}_bA)$. This gives a bijection between $H^1(G, A)$ and $H^1(G, {}_bA)$. □

THEOREM 1.8. *Let G be a smooth reductive group over K , not necessarily connected. Then $H^1(K, G)$ is finite.*

PROOF: Let G^0 be a connected component of G . Then G/G^0 is a finite group. Since $1 \rightarrow G^0 \rightarrow G \rightarrow G/G^0 \rightarrow 1$ is exact, we have $H^1(K, G^0) \rightarrow H^1(K, G) \rightarrow H^1(K, G/G^0)$ is exact as distinguished point sets. By [18, Chapter III, Section 4, Proposition 8], we have $\#H^1(K, G/G^0) < +\infty$. By Theorem 1.6, we have $\#H^1(K, G^0) < +\infty$. By Lemma 1.7 and [18, Proposition 35], we have $\#H^1(K, G) < +\infty$. □

PROPOSITION 1.9. *Let (G, ρ, V) be a P.V. defined over K with a reductive generic isotropy subgroup. Then $\#\rho(G)_K \backslash Y_K$ is finite.*

PROOF: Let H be a generic isotropy subgroup of a point in Y_K . Then there exists a bijection between $\rho(G)_K \backslash Y_K$ and $\text{Ker}(H^1(K, H) \rightarrow H^1(K, G))$ (see Serre [18]). By Theorem 1.8, $H^1(K, H)$ is finite, and hence $\rho(G)_K \backslash Y_K$ is a finite set. □

EXAMPLE 1.10. Let G be the subgroup of GL_n consisting of all lower triangular matrices. Let V be the totality of symmetric $n \times n$ matrices and define ρ by $\rho(g)x = gx^t g$ for all $g \in G$ and $x \in V$. Since $\dim G = \dim V$, a generic isotropy subgroup is a finite subgroup and hence we have $\#\rho(G)_K \backslash Y_K = \nu < +\infty$ by Proposition 1.9.

Moreover $\det x$ is a non-degenerate K -relative invariant. By $\text{tr}(xy)$ ($x, y \in V$), we identify V with its dual V^* . Then (G, ρ, V) and (G, ρ^*, V^*) are K -regular P.V.'s. Hence, by Proposition 1.3, we have $\#\rho^*(G)_K \backslash Y_K^* = \nu < +\infty$.

PROPOSITION 1.11. *Let (G, ρ, V) be an irreducible regular P.V. defined over K . Then we have $\#\rho(G)_K \backslash Y_K < +\infty$.*

PROOF: By a classification of irreducible P.V.'s (see Chen [5]), we know that a generic isotropy subgroup is reductive. □

2. ZETA DISTRIBUTIONS

Let K be a local field of characteristic $p > 0$. Assume that (G, ρ, V) and its dual (G, ρ^*, V^*) are K -regular P.V.'s. Moreover we shall assume that $Y_K = Y_1 \cup \dots \cup Y_\nu$

decomposes into a finite union of $\rho(G)_K$ -orbits Y_i ($1 \leq i \leq \nu$), that is, $\# \rho(G)_K \setminus Y_K = \nu < +\infty$. Then by Proposition 1.3, we have $Y_K^* = Y_1^* \cup \dots \cup Y_\nu^*$.

Let $f_1(x), \dots, f_l(x)$ (respectively $f_1^*(y), \dots, f_l^*(y)$) be basic K -relative invariants of (G, ρ, V) (respectively (G, ρ^*, V^*)). Let χ_i (respectively χ_i^*) be the corresponding character of f_i (respectively f_i^*). Then we have $X_1(G)_K = \langle \chi_1, \dots, \chi_l \rangle$ and $X_1^*(G)_K = \langle \chi_1^*, \dots, \chi_l^* \rangle$.

By Proposition 1.2, we have $X_1(G)_K = X_1^*(G)_K$ so that there exists a unique matrix $U = (u_{ij}) \in GL_l(\mathbf{Z})$ satisfying $\chi_i = \prod_{j=1}^l \chi_j^{*u_{ij}}$. Since $\det \rho(g)^2 \in X_1(G)_K$, we have $\det \rho(g)^2 = \chi_1^{2\lambda_1} \dots \chi_l^{2\lambda_l}$ for some $\lambda = (\lambda_1, \dots, \lambda_l) \in ((1/2)\mathbf{Z})^l$ and $\det \rho^*(g)^2 = \chi_1^{*2\lambda_1^*} \dots \chi_l^{*2\lambda_l^*}$ for some $\lambda^* = (\lambda_1^*, \dots, \lambda_l^*) \in ((1/2)\mathbf{Z})^l$. Since $\det \rho^*(g) = \det \rho(g)^{-1}$, we have $\lambda^* = -\lambda U$.

EXAMPLE 2.1. For simplicity, we deal with the case $n = 2$ in Example 1.10. Then we have

$$G = \left\{ g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}; ab \neq 0 \right\}$$

and

$$V = \left\{ X = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \right\}$$

The basic K -relative invariants of (G, ρ, V) (respectively (G, ρ^*, V^*)) are $f_1(X) = x$ and $f_2(X) = \det X$ (respectively $f_1^*(X) = z$, $f_2^*(X) = \det X$) corresponding to $\chi_1(g) = a^2$, $\chi_2(g) = a^2b^2$ (respectively $\chi_1^*(g) = b^{-2}$, $\chi_2^*(g) = a^{-2}b^{-2}$) for

$$g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$$

in G . Hence $\chi_1 = \chi_1^* \chi_2^{*-1}$ and $\chi_2 = \chi_2^{*-1}$ so that we have

$$U = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Since

$$\det \rho \left(\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \right) = a^3b^3,$$

we have $\lambda = \lambda^* = (0, 3/2)$.

Let $\{\varepsilon_1, \dots, \varepsilon_\nu\}$ be the complete set of representatives of $K^\times / K^{\times 2}$ in K^\times . Then we have $Y_K = Y_1 \cup \dots \cup Y_\nu$ with $Y_i = \{y \in Y_K; f_2(y) \equiv \varepsilon_i \pmod{K^{\times 2}}\}$ ($i = 1, \dots, \nu$).

Let $\omega^{(i)} : K^\times \rightarrow \mathbf{C}^\times$ ($i = 1, \dots, l$) be a quasicharacter, that is, a continuous homomorphism.

For $\omega = (\omega^{(1)}, \dots, \omega^{(l)})$ and the basic K -relative invariants $f(x) = (f_1(x), \dots, f_l(x))$, we write $\omega(f(x))$ instead of $\prod_{i=1}^l \omega^{(i)}(f_i(x))$ for simplicity of notation.

Let $|\cdot|$ be the absolute value of K normalised by $|\pi| = q^{-1}$ for a prime element π where q is the module of K . For $s = (s_1, \dots, s_l)$, we write $\omega_s = (|\cdot|^{s_1}, \dots, |\cdot|^{s_l})$ so that $\omega_s(f(x)) = \prod_{i=1}^l |f_i(x)|^{s_i}$.

Let dx be the Haar measure on $V_K = K^n$ normalised by $\int_{R^n} dx = 1$ where R is the maximal compact subring of K . Since $d(\rho(g)x) = |\det \rho(g)| dx$ and $\omega_\lambda(f(\rho(g)x)) = |\det \rho(g)| \omega_\lambda(f(x))$, the measure $d_Y(x) = (dx)/(\omega_\lambda(f(x)))$ is a G -invariant measure on Y .

For $\Phi \in \mathfrak{S}(V_K)$ where $\mathfrak{S}(V_K)$ denotes the Schwartz-Bruhat space of V_K , we define the integral

$$Z_i(\omega, \Phi) = \int_{Y_i} \omega(f(x))\Phi(x) d_Y(x) \quad (i = 1, \dots, \nu).$$

Now any quasi-character $\omega^{(i)} : K^\times \rightarrow \mathbf{C}^\times = \{z \in \mathbf{C}; z \neq 0\}$ can be written uniquely as $\omega^{(i)} = |\cdot|^{s_i} \cdot \phi_i$ for some $s_i \in \mathbf{C}$ and $\phi_i : R^\times \rightarrow \mathbf{C}_1^\times = \{z \in \mathbf{C}; |z| = 1\}$ where R^\times is the set of units of R . Put $Re \omega^{(i)} = Re s_i$ ($i = 1, \dots, l$). The following lemma is easy to prove and we omit the proof (see Sato [15]).

LEMMA 2.2. *If $Re \omega^{(i)} > \lambda_i$ ($i = 1, \dots, l$), the integral $Z_i(\omega, \Phi)$ is absolutely convergent and holomorphic with respect to $s = (s_1, \dots, s_l) \in (\mathbf{C}/((2\pi i)/(\log q))\mathbf{Z})^l \cong \mathbf{C}^{\times l}$ for $\omega = (|\cdot|^{s_1} \cdot \phi_1, \dots, |\cdot|^{s_l} \cdot \phi_l)$.*

Let $\mathfrak{S}'(V_K) = \{z : \mathfrak{S}(V_K) \rightarrow \mathbf{C}, \text{ where } \mathbf{C} \text{ is a linear mapping}\}$ be the space of distributions on V_K . By Lemma 2.2, the mapping $\Phi \mapsto Z_i(\omega, \Phi)$ defines a distribution on V_K when $Re \omega^{(i)} > \lambda_i$ ($i = 1, \dots, l$).

For (G, ρ^*, V^*) , we can define a similar distribution $Z_j^*(\omega)$ ($j = 1, \dots, \nu$) given by

$$Z_j^*(\omega)(\Phi^*) = Z_j^*(\omega, \Phi^*) = \int_{Y_j^*} \omega(f^*(y))\Phi^*(y) d_{Y^*}(y).$$

Now we fix a non-trivial additive character $\psi : K \rightarrow \mathbf{C}_1^\times$ and define the Fourier transformation $\mathfrak{S}(V_K^*) \ni \Phi^* \mapsto \widehat{\Phi}^* \in \mathfrak{S}(V_K)$ by $\widehat{\Phi}^*(x) = \int_{V_K^*} \Phi^*(y)\psi(\langle x, y \rangle)dy$ where dy is the Haar measure on V_K^* dual to a fixed Haar measure on V_K .

For $\omega = (\omega^{(1)}, \dots, \omega^{(l)})$, put $\omega^* = \omega^U = \left(\prod_{i=1}^l \omega^{(i)u_{i1}}, \dots, \prod_{i=1}^l \omega^{(i)u_{il}} \right)$.

Our purpose is to show that $Z_i(\omega)$ and $Z_j^*(\omega)$ are continued analytically to all ω and satisfy the functional equation:

$$(2.1) \quad \widehat{Z}_i(\omega) = \sum_{j=1}^{\nu} \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}) \quad (i = 1, \dots, \nu)$$

under some additional conditions where $\widehat{Z}_i(\omega)(\Phi^*) = Z_i(\omega, \widehat{\Phi}^*)$. Recall that $\omega_{\lambda^*} = (|\lambda_1^*|, \dots, |\lambda_l^*|)$ with $\det \rho^*(g)^2 = \chi_1^{2\lambda_1^*} \cdots \chi_l^{2\lambda_l^*}$.

Actually when K is a local field with $ch(K) = 0$, then (2.1) is obtained under some conditions and it is called “the fundamental theorem of P.V. over K ”.

3. RATIONALITY FOR ALMOST ALL p

For a rational prime p , let K_p denote the local field with the constant field \mathbf{F}_p . For $f \in \mathbf{Z}[x_1, \dots, x_n]$, we denote $(f \bmod p) \in \mathbf{F}_p[x_1, \dots, x_n]$ by f_p . Then we have the following theorem which was suggested by Professor M. Kashiwara.

THEOREM 3.1. *For almost all p , the integral*

$$Z_p(s, \Phi_p) = \int_{K_p^n} |f_p(x)|_{K_p}^s \Phi_p(x) d_p x$$

is a rational function of $t = p^{-s}$ where $\Phi_p \in \mathfrak{S}(K_p^n)$ and $d_p x$ is a Haar measure on K_p^n .

PROOF: Let $K = \mathbf{Q}((t))$ be the field of formal power series over \mathbf{Q} , $X = \Omega^n$ the affine space and $X_K = K^n$. Let f denote the morphism $X \rightarrow \Omega$ defined by $f \in \mathbf{Z}[x_1, \dots, x_n]$; then there exists a nonsingular algebraic variety Y and a projective morphism $h : Y \rightarrow X$ both defined over K with the following property: let b denote an arbitrary point of Y_K , \mathfrak{D}_K the local ring of Y at b relative to K (consisting of “functions” defined over K), and \mathfrak{M}_K the ideal of non-units of \mathfrak{D}_K ; then there exists an ideal basis (y_1, \dots, y_n) of \mathfrak{M}_K , elements u, v of $\mathfrak{D}_K - \mathfrak{M}_K$, and integers $N_i \geq 0, \nu_i \geq 1$ for $1 \leq i \leq n$ such that

$$f \circ h = u \cdot \prod_{i=1}^n y_i^{N_i}, \quad h^*(dx) = v \cdot \prod_{i=1}^n y_i^{\nu_i - 1} dy.$$

The existence of such a pair (Y, h) is guaranteed by Hironaka’s theorem [7, p.109–326]. Then for almost all p , the reduction modulo p is well-defined and we have similar results for K_p, f_p, \dots et cetera. Then by arguments similar to those in the Appendix of Igusa [13], we obtain our result. □

REMARK 3.2. Let K be a number field. For $f \in \mathfrak{D}_K[x_1, \dots, x_n]$, we have a similar result to Theorem 3.1 for almost all prime ideals \mathfrak{P} of \mathfrak{D}_K .

4. FUNCTIONAL EQUATIONS

LEMMA 4.1. *Let G denote a locally compact totally disconnected group, H a closed subgroup of G , $X = H \backslash G$, and $\omega : G \rightarrow \mathbf{C}^\times$ a quasicharacter. Put $\xi_X(\omega) =$*

$\{T \in \mathfrak{S}(X)'; gT = \omega(g)^{-1}T \text{ for all } g \in G\}$. Then we have $\dim_{\mathbb{C}} \xi_X(\omega) \leq 1$. Moreover $\dim_{\mathbb{C}} \xi_X(\omega) = 1$ if and only if $\Delta_G \cdot \omega|_H = \Delta_H$ where Δ_G, Δ_H denotes the module of G, H respectively.

PROOF: See Igusa [11, p.1015]. □

Let (G, ρ, V) and its dual (G, ρ^*, V^*) be K -regular P.V.'s with $\# \rho(G)_K \backslash Y_K = \nu < +\infty$ where K is a local field of characteristic p . Then, by Proposition 1.3, we have $Y_K = Y_1 \cup \dots \cup Y_\nu$ and $Y_K^* = Y_1^* \cup \dots \cup Y_\nu^*$, that is, $\# \rho^*(G)_K \backslash Y_K^* = \nu$.

As in Section 2, we can define the zeta distribution $Z_i(\omega, \Phi)$ (respectively $Z_i^*(\omega, \Phi^*)$) which is convergent when $Re \omega^{(j)} > \lambda_j$ (respectively $Re \omega^{(j)} > \lambda_j^*$) ($1 \leq i \leq \nu, 1 \leq j \leq l$).

We denote by $Z_i(\omega)$ the distribution defined by $\Phi \mapsto Z_i(\omega, \Phi)$, et cetera.

PROPOSITION 4.2. We have

$$(1) \quad Z_j^*(\omega^* \omega_{\lambda^*}) \in \xi_{Y_j^*}(\omega^* \omega_{\lambda^*}) \text{ and}$$

$$(2) \quad \widehat{Z}_i(\omega) \in \xi_{Y_i^*}(\omega^* \omega_{\lambda^*})$$

($i, j = 1, \dots, \nu$).

PROOF: By a direct calculation, we obtain our results. □

PROPOSITION 4.3. Let K be a local field of characteristic $p > 0$ with the module q . For $\omega = (\omega^{(1)}, \dots, \omega^{(l)})$ with $\omega^{(i)} = \omega_{s_i} \cdot \phi_i$ ($\phi_i(\pi) = 1$ for a prime element π), assume that $Z_i(\omega, \Phi)$ and $Z_j^*(\omega, \Phi^*)$ are rational functions of $q^{-s_1}, \dots, q^{-s_l}$. Then for all $\Phi^* \in \mathfrak{S}(Y_K^*)$, we have

$$Z_i(\omega, \widehat{\Phi}^*) = \sum_j \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}, \Phi^*)$$

for $i, j = 1, \dots, \nu$.

PROOF: Since $Z_i(\omega, \Phi)$ and $Z_j(\omega, \Phi^*)$ are rational functions, it is defined for all ω except poles and hence by Lemma 4.1 and Proposition 4.2, we have our result. □

THEOREM 4.4. Let (G, ρ, V) be a K -regular P.V. satisfying the following conditions:

- (C1) its dual (G, ρ^*, V^*) is a K -regular P.V. such that $\# \rho^*(G)_K \backslash V_K^* < +\infty$,
- (C2) for $x \in S_K^*$, there exists $\chi \in X_1(G)_K$ satisfying $\chi(G_{x,K}) \not\subseteq R^\times$ where R^\times is the set of units of the maximal compact subring R of K , and
- (C3) $Z_j(\omega, \Phi)$ is a rational function of $q^{-s_1}, \dots, q^{-s_l}$ where $\omega = (\omega^{(1)}, \dots, \omega^{(l)})$ with $\omega^{(i)} = \omega_{s_i}$ ($1 \leq i \leq l$).

Then we have the functional equation

$$Z_i(\omega, \widehat{\Phi}^*) = \sum_j \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}, \Phi^*)$$

for all $\Phi^* \in \mathfrak{S}(V_K^*)$ for $i, j = 1, \dots, \nu$ where $\nu = \# \rho^*(G)_K \setminus Y_K^*$.

PROOF: The condition (C2) corresponds to Sato [15, Lemma 2.2, p.474] for the case of $ch(K) = 0$. Then the proof is similar to the case of $ch(K) = 0$ (using Proposition 4.3). (See Igusa [11] and Sato [15, p.477]). □

Now let (G, ρ, V) be a reductive \mathbf{Q} -regular P.V. Then for almost all p , we have the reduction modulo p and we obtain the K_p -regular P.V. (G_p, ρ_p, V_p) where K_p is a local field with the constant field \mathbf{F}_p .

ASSUMPTION A. Assume that $\# \rho_p(G)_{K_p} \setminus S_{K_p} < +\infty$ and for $x \in S_{K_p}$, there exists $\chi \in X_1(G_p)_{K_p}$ satisfying $\chi(G_{p,x,K_p}) \not\subseteq R_p^\times$ for almost all p .

Let (G, ρ, V) be a reductive \mathbf{Q} -regular P.V. with Assumption A. Let f_1, \dots, f_l be basic \mathbf{Q} -relative invariants with \mathbf{Z} -coefficients. Denote $|f_1 \bmod p|_{K_p}^{s_1} \cdots |f_l \bmod p|_{K_p}^{s_l}$ by $|f^{(p)}(x)|_{K_p}^s$ and put $Z_i^p(s, \Phi_p) = \int_{(Y_{K_p})_i} |f^{(p)}(x)|_{K_p}^s \Phi_p(x) d_{Y_p}(x)$ for $\Phi_p \in \mathfrak{S}(V_{K_p})$.

THEOREM 4.5. *Let (G, ρ, V) be a reductive \mathbf{Q} -regular P.V. with Assumption A. Then for almost all rational primes p , the integral $Z_i^p(s, \Phi_p)$ ($i = 1, \dots, \nu_p$, $Y_{K_p} = Y_1 \cup \dots \cup Y_{\nu_p}$) is a rational function and satisfies the functional equation:*

$$Z_i^p\left(s, \widehat{\Phi}_p\right) = \sum_{j=1}^{\nu_p} \Gamma_{ij}(s) Z_j^p(s^*, \Phi_p)$$

($i = 1, \dots, \nu_p$).

When $l = 1$, we have $s^* = (n/d) - s$ with $n = \dim V$ and $d = \deg f$. In general, for $\omega = \omega_s = \omega_{s_1} \cdots \omega_{s_l}$, we have $\omega_{s^*} = \omega^* \omega_{\lambda^*}$.

PROOF: By Theorem 4.4 and using the results of Section 1 and Section 3, we obtain our result. □

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