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BIRKHOFF INTERPOLATION AT THE nth ROOTS OF UNITY: CONVERGENCE

S. D. RIEMENSCHNEIDER AND A. SHARMA

The first investigations on this type of problem were carried out by O. Kiš [2]. Kiš considered the problem of interpolating a function and its second derivative at the nth roots of unity (the $(0, 2)$ problem) by $2n - 1$ degree polynomials, and the convergence of such approximates. Later, Sharma [8], [9] extended the existence and uniqueness results to $(0, m)$ interpolation, and essentially to $(0, m_1, m_2)$ interpolation. In the latter case, Sharma established the convergence results for $(0, 2, 3)$, $(0, 1, 3)$ and $(0, 1, 4)$ interpolation as well. Although some further special cases were considered [10] these were the essential results until very recently. Now Cavaretta, Sharma and Varga [1] have established the existence and uniqueness theorem for all possible interpolations of this type (see Theorem 2.1 below). Motivated by the work of Cavaretta, Sharma and Varga and the earlier work of Sharma [9], a different proof of this result is provided, and this proof is used to establish a convergence theorem in the general case.

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1. Preliminaries. The Birkhoff interpolation problem can be stated as follows: For an $m \times (N + 1)$ interpolation matrix $E = (e_{j,k})$ (a matrix of zeros and ones with $N + 1$ ones), for a set of knots $Z = \{z_1, \ldots, z_m\}$ in the complex plane, and for arbitrary (complex) numbers $\{\gamma_{j,k}\}_{j=k,k=0}^{m\ \ N}$, does there exist a polynomial of degree N satisfying

$$(1.1) \qquad p^{(k)}(z_j) = \gamma_{j,k}, \quad (e_{jk} = 1)?$$

The pair (E, Z) is called *regular* if the equations (1.1) always have a unique solution.

The regularity of a pair (E, Z) is equivalent to the following: If $P(z)$ satisfies (1.1) for homogeneous data, $\gamma_{j,k} = 0$, (P is said to be *annihilated by* (E, Z)), then $P(z) \equiv 0$.

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We are interested in a particular case of this problem. Let $\bar{m} = (m_0, m_1, \ldots, m_q)$ where $0 = m_0 < m_1 < \ldots < m_q$ are integers. The interpolation matrix $E_{\bar{m}}$ will be the $n \times (q+1)n$ interpolation matrix which has ones in columns $m_0, m_1, \ldots, m_q$ and zeros elsewhere. Let $Z_n = \{z_k = e^{2\pi ik/n}: k = 1, \ldots, n\}$ denote the nth roots of unity. In the next section we give a proof of the regularity of $(E_{\bar{m}}, Z_n)$ which is more direct than the one given in [**1**]. We first give some lemmas about special determinants arising in the proofs later on.

LEMMA 1.1. [**12**, **1**, **5**] *For fixed integers* $0 < m_1 < \ldots < m_q$, $m_j \leqq jn$, $j = 1, \ldots, q$, *and for any non-negative integer* ν, *the determinant*

$$(1.2) \qquad M(\nu) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (\nu)_{m_1} & (\nu+n)_{m_1} & \cdots & (\nu+qn)_{m_1} \\ \vdots & \vdots & & \vdots \\ (\nu)_{m_q} & (\nu+n)_{m_q} & \cdots & (\nu+qn)_{m_q} \end{vmatrix}$$

is positive (>0). (*Here* $(a)_m = a(a-1)\ldots(a-m+1)$).

LEMMA 1.2. *For integers* $0 < m_1 < \ldots < m_q$, *the generalized Vandermonde determinant*

$$(1.3) \qquad V(\alpha) = \begin{vmatrix} 1 & 1 & & 1 \\ \alpha^{m_1} & (\alpha+1)^{m_1} & \cdots & (\alpha+q)^{m_1} \\ \vdots & \vdots & & \vdots \\ \alpha^{m_q} & (\alpha+1)^{m_q} & \cdots & (\alpha+q)^{m_q} \end{vmatrix}$$

is positive (>0) *for* $0 \leqq \alpha \leqq 1$, *and is a continuous and increasing function of* α.

These two determinants are connected by the following:

LEMMA 1.3. *Let* $M(\nu)$ *be the determinant* (1.2) *then*

$$(1.4) \qquad n^{\Sigma_1^q m_l}/M(\nu) = g(\nu) + O(1/n)$$

where $g(\nu) = 1/V(\nu/n)$ *is bounded and decreasing for* $0 \leqq \nu \leqq n$.

Proof. The result follows easily from the observation

$$(\nu + \lambda n)_{m_l} = n^{m_l}\left(\frac{\nu}{n} + \lambda\right)\left(\frac{\nu}{n} + \lambda - 1/n\right)\ldots\left(\frac{\nu}{n} + \lambda - \frac{m_l - 1}{n}\right)$$
$$= n^{m_l}\left\{\left(\frac{\nu}{n} + \lambda\right)^{m_l} + O\left(\frac{1}{n}\right)\right\}$$

where the $O\left(\frac{1}{n}\right)$ term depends only on λ and m_l.

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2. The regularity theorem. In this section we give a different approach to the theorem of [**1**] concerning the regularity of $(E_{\bar{m}}, Z_n)$ interpolation.

THEOREM 2.1. [**1**]. *The pair* $(E_{\bar{m}}, Z_n)$ *is regular if and only if*

$$(2.1)\qquad m_j \leqq jn, \quad j = 0, \ldots, q.$$

Proof. The condition (2.1) is required so that the matrix $E_{\bar{m}}$ satisfies the Pólya condition. The Pólya condition is well known to be necessary for polynomial interpolation at any set of knots Z (see [**4**]).

Thus, assume that $\bar{m} = (0, m_1, \ldots, m_q)$ satisfies (2.1) and suppose that $P(z)$ is a polynomial of degree $(q + 1)n - 1$ which is annihilated by $(E_{\bar{m}}, Z_n)$. Then $P(z)$ may be written in the form

$$(2.2)\qquad P(z) = P_0(z) + z^n P_1(z) + \ldots + z^{qn} P_q(z)$$

where each polynomial

$$P_\lambda(z) = \sum_{\nu=0}^{n-1} a_{\lambda,\nu} z^\nu$$

is of degree $n - 1$.

From Leibnitz's formula for differentiation of a product, we have

$$(z^{\lambda n} P_\lambda(z))^{(m_j)}|_{z=z_k} = \sum_{l=0}^{m_j} \binom{m_j}{l} (\lambda n)_l z_k{}^{\lambda n - l} D^{m_j - l} P_\lambda(z_k)$$

$$= z_k{}^{-m_j} \sum_{l=0}^{m_j} \binom{m_j}{l} (\lambda n)_l z_k{}^{m_j - l} D^{m_j - l} P_\lambda(z_k)$$

$$= z_k{}^{-m_j} G_{j,\lambda n}(D) P_\lambda(z)|_{z=z_k}$$

where $G_{j,\lambda n}(D)$ is the Euler differential operator

$$(2.3)\qquad G_{j,\lambda n}(D) = \sum_{l=0}^{m_j} \binom{m_j}{l} (\lambda n)_l z^{m_j - l} D^{m_j - l}, \quad 0 \leqq j, \lambda \leqq q.$$

(Recall that $(a)_l = a(a - 1) \ldots (a - l + 1)$, $(a)_0 = 1$). Therefore the interpolatory conditions $P^{(m_j)}(z_k) = 0$, $0 \leqq j \leqq q$, $1 \leqq k \leqq n$ when applied to (2.2) give the equations

$$(2.4)\qquad P_0{}^{(m_j)}(z_k) + z_k{}^{-m_j} \sum_{\lambda=1}^{q} G_{j,\lambda n}(D) P_\lambda(z)|_{z=z_k} = 0$$

$k = 1, \ldots, n$, $j = 0, \ldots, q$. Since $G_{j,\lambda n}(D) P_\lambda(z)$ is again a polynomial of degree $n - 1$, the equations (2.4) yield that the polynomials $P_\lambda(z)$, $\lambda = 0, 1, \ldots, q$, satisfy the system of differential equations

$$(2.5)\qquad z^{m_j} P_0{}^{(m_j)}(z) + \sum_{\lambda=1}^{q} G_{j,\lambda n}(D) P_\lambda(z) = 0, \quad j = 0, \ldots, q.$$

Solving this system of differential equations, we see that each polynomial $P_\lambda(z)$ satisfies

$$(2.6)\qquad H_{\bar m}(D)P_\lambda(z) \equiv \begin{vmatrix} 1 & 1 & \dots & 1 \\ z^{m_1}D^{m_1} & G_{1,n}(D) & \dots & G_{1,qn}(D) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z^{m_q}D^{m_q} & G_{q,n}(D) & \dots & G_{q,qn}(D) \end{vmatrix} P_\lambda(z) = 0.$$

From (2.3) we have

$$(2.7)\qquad G_{j,\lambda n}(D)z^\nu = z^\nu \sum_{l=0}^{m_j} \binom{m_j}{l} (\lambda n)_l(\nu)_{m_j-l}$$
$$= z^\nu(\nu + \lambda n)_{m_j}, \quad 0 \leqq j, \lambda \leqq q.$$

Therefore, by the linearity of $H_{\bar m}(D)$, (2.6) can be written in the form

$$(2.8)\qquad H_{\bar m}(D)P_\lambda(z) = \sum_{\nu=0}^{n-1} a_{\lambda,\nu}M(\nu)z^\nu \equiv 0$$

where the $M(\nu)$ are the determinants of Lemma 1.1. Since $M(\nu) > 0$, equation (2.8) implies that $a_{\lambda,\nu} = 0$ for $\nu = 0, 1, \dots, n - 1$ and $\lambda = 0, 1, \dots, q$. Therefore by (2.2), $P(z) \equiv 0$. This shows that $(E_{\bar m}, Z_n)$ is regular.

The fundamental polynomials for $(E_{\bar m}, Z_n)$ interpolation can be obtained in a similar way. For this we shall need the Lagrange polynomials of degree $n - 1$ for interpolation at the roots of unity:

$$(2.9)\qquad l_k(z) = \frac{1}{n}\sum_{\nu=0}^{n-1} z_k^{-\nu}z^\nu = \frac{z_k}{n}\cdot\frac{z^n - 1}{z - z_k}, \quad k = 1, \dots, n.$$

These are the unique polynomials of degree $n - 1$ which satisfy $l_k(z_k) = 1$, $l_k(z_l) = 0$, $k \neq l$, $1 \leqq k, l \leqq n$.

The *fundamental polynomials* $\alpha_{\bar m,k,j}(z)$ for $(E_{\bar m}, Z_k)$ interpolation are the unique polynomials of degree $(q + 1)n - 1$ satisfying

$$(2.10)\qquad \alpha_{\bar m,k,j}^{(m_r)}(z_l) = \begin{cases} 1 & r = j, l = k \\ 0 & \text{otherwise} \end{cases}, \quad 0 \leqq r, j \leqq q; 1 \leqq k, l \leqq n.$$

Let k and j be fixed. If we ask for $\alpha_{\bar m,k,j}(z)$ of the form (2.2), then exactly as was the case for (2.4), the conditions (2.13) give rise to the system of equations

$$(2.11)\qquad P_0^{(m_r)}(z_l) + z_l^{-m_r}\sum_{\lambda=1}^{q} G_{r,\lambda n}(D)P_\lambda(z)|_{z=z_l} = \begin{cases} l_k(z_l), & r = j \\ 0\quad, & \text{otherwise}\end{cases}$$

$1 \leqq l \leqq n$. Solving this system we obtain that the polynomials $P_\lambda(z)$ must satisfy

$$(2.12)\qquad H_{\bar m}(D)P_\lambda(z) = (-1)^{\lambda+j}H_{j,\lambda}(D)\ (z_k^{m_j}l_k(z))$$

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where $H_{j,\lambda}(D)$ is the differential operator defined by the $(j+1, \lambda+1)$-minor of the determinant in (2.6) which defines $H_{\bar{m}}(D)$.

3. Estimates on the fundamental polynomials. The fundamental polynomials can be estimated by using the representation

$$(3.1) \qquad \alpha_{\bar{m},k,j}(z) = \sum_{\lambda=0}^{q} z^{\lambda n} P_{\lambda,k,j}(z)$$

where the polynomials $P_{\lambda,k,j}(z)$ are determined by the differential equations (2.12).

We shall need a lemma generalizing the inequality of M. Riesz [**7**]:

LEMMA. *If $P(z)$ is a polynomial of degree n, then*

$$(3.2) \qquad |P^{(m)}(z)| \leqq n^m \max\nolimits_{|\omega|\leqq 1}|P(w)|, \quad |z| \leqq 1.$$

LEMMA 3.1. *There exists a constant $C(\bar{m})$ so that*

$$(3.3) \qquad |H_{j,\lambda}(D)P(z)| \leqq C(m)n^{\Sigma_{l\neq j} m_l} \max\nolimits_{|w|\leqq 1} |P(w)|$$

for $|z| \leqq 1$, $0 \leqq j, \lambda \leqq q$ and for any polynomial $P(z)$ of degree n.

Proof. By (2.3) and (3.2) it follows easily that

$$(3.4) \qquad |G_{l,\lambda n}(D)P(z)| \leqq C(m_l, \lambda)n^{m_l} \max\nolimits_{|w|\leqq 1} |P(w)|$$

for $0 \leqq l, \lambda \leqq q$. From the definition of $H_{j,\lambda}(D)$ as the $(j+1, \lambda+1)$ minor of $H_{\bar{m}}(D)$ and repeated application of (3.4), the estimate (3.3) follows easily.

The differential operator $H_{\bar{m}}(D)$ defined in (2.6) and appearing on the left side of (2.12) is invertible as an operator on polynomials. In fact, if

$$P(z) = \sum_{\nu=0}^{n-1} a_\nu z^\nu,$$

we have

$$(3.5) \qquad H_{\bar{m}}(D)^{-1}P(z) = \sum_{\nu=0}^{n-1} (a_\nu/M(\nu))z^\nu,$$

since $M(\nu) \neq 0$ by Lemma 1.

We observe the following: From (2.12),

$$(3.6) \qquad P_{\lambda,k,j}(z) = (-1)^{\lambda+j} \frac{1}{n} \sum_{\nu=0}^{n-1} \frac{M_{j,\lambda}(\nu)}{M(\nu)} z_k^{m_j-\nu} z^\nu$$
$$= (-1)^{\lambda+j} H_{j,\lambda}(D) \left[\frac{1}{n} \sum_{\nu=0}^{n-1} \frac{z_k^{m_j-\nu}}{M(\nu)} z^\nu\right]$$

where $M_{j,\lambda}(\nu)$ is the $(j+1, \lambda+1)$-minor of $M(\nu)$. That is, the operators

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$H_{j,\lambda}(D)$ and $H_{\bar{m}}(D)^{-1}$ commute in this equation, and we need only estimate $H_{\bar{m}}(D)^{-1}$ on $z_k^{m_j}l_k(z)$.

LEMMA 3.2. *If $\bar{m}$ satisfies (2.1), then there is a constant, depending only on $\bar{m}$, such that*

$$(3.7)\qquad \sum_{k=1}^{n} |H_{\bar{m}}(D)^{-1}(z_k^{m_j}l_k(z))| \leqq C(\bar{m})n^{-\Sigma m_l} \log n, \quad \textit{for } |z| \leqq 1.$$

Proof. From the summation by parts formula, we have

$$\frac{1}{n}\sum_{\nu=0}^{n-1} \frac{z_k^{m_j-\nu}}{M(\nu)} z^{\nu} = \frac{1}{n}\sum_{r=0}^{n-1}\left(\sum_{\nu=0}^{r} z_k^{m_j-\nu}z^{\nu}\right)\left(\frac{1}{M(r)} - \frac{1}{M(r+1)}\right) + \frac{1}{nM(n)}\sum_{\nu=0}^{n-1} z_k^{m_j-\nu}z^{\nu}.$$

Summing the geometric series and using Lemma 1.3 we obtain for $|z| = 1$

$$\sum_{k=1}^{n} |H_{\bar{m}}(D)^{-1}(z_k^{m_j}l_k(z))| \leqq C(\bar{m})n^{-\Sigma m_l} \sum_{k=1}^{n} \min\{1, n^{-1}|z - z_k|^{-1}\}.$$

But a straightforward geometric argument shows that

$$(3.8)\qquad \sum_{k=1}^{n} \min\{1, n^{-1}|z - z_k|^{-1}\} \leqq 3 + \log n \quad \text{for } |z| = 1.$$

THEOREM 3.4. *The fundamental polynomials $\alpha_{\bar{m},k,j}(z)$ of $(E_{\bar{m}}, Z_n)$ interpolation satisfy*

$$(3.9)\qquad \sum_{k=1}^{n} |\alpha_{\bar{m},k,j}(z)| \leqq C(\bar{m}) \log n/n^{m_j}$$

for $|z| \leqq 1$ and $jn \geqq m_j$, $j = 0, 1, \ldots, q$, where $C(\bar{m})$ is independent of n.

Proof. For fixed z_0, $|z_0| \leqq 1$, choose ϵ_k, $|\epsilon_k| = 1$, such that

$$\epsilon_k P_{\lambda,k,j}(z_0) = |P_{\lambda,k,j}(z_0)|.$$

Then from (3.6) we have

$$\sum_{k=1}^{n} |P_{\lambda,k,j}(z_0)| = (-1)^{j+\lambda} H_{j,\lambda}(D)\left\{\sum_{k=1}^{n} \epsilon_k H_{\bar{m}}(D)^{-1}(z_k^{m_j}l_k(z))\right\}\Bigg|_{z=z_0}.$$

But $\sum_{k=1}^{n} \epsilon_k H_{\bar{m}}(D)^{-1}(z_k^{m_j} l_k(z))$ is a polynomial of degree $(n - 1)$ with bound given by (3.7). Consequently, by Lemma 3.1

$$(3.10)\qquad \sum_{k=1}^{n} |P_{\lambda,k,j}(z_0)| \leqq C(\bar{m}) \log n/n^{m_j}.$$

The estimate (3.9) then follows immediately from (3.1).

4. Convergence. A first step in the convergence theorems for interpolation at the roots of unity is an approximation by polynomials whose

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derivatives can be estimated in terms of the order of approximation. The basic ideas and construction for the convergence theorems are due to Kiš [**2**].

The basic function class for which the convergence theorems are valid will be composed of functions which are analytic in $|z| < 1$ and have certain continuity properties on the boundary. If $f(z)$ is analytic for $|z| < 1$ and continuous for $|z| = 1$, then let $\omega(f, \delta)$ be the modulus of continuity of $f(e^{ix})$. The function $f(z)$ is said to belong to the Dini–Lipschitz class, $\mathscr{D}$, if

$$(4.1) \qquad \omega(f, \delta) \log 1/\delta \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Let $\bar{m} = (0, m_1, \ldots, m_q)$ be given. The linear interpolation operator for $(E_{\bar{m}}, Z_n)$ interpolation, $Q_{\bar{m},n}(f, z)$, can be written in the form

$$(4.2) \qquad Q_{\bar{m},n}(f, z) = \sum_{j=0}^{q} \sum_{k=1}^{n} f^{(m_j)}(z_k)\alpha_{\bar{m},k,j}(z)$$

where $\alpha_{\bar{m},k,j}(z)$ are the fundamental polynomials for $(E_{\bar{m}}, Z_n)$ interpolation. Instead of the operator (4.2), we shall consider the operator

$$(4.3) \qquad \tilde{Q}_{\bar{m},n}(f, z) = \sum_{k=1}^{n} \left\{ f(z_k)\alpha_{\bar{m},k,0}(z) + \sum_{j=1}^{q} \beta_{k,j}(n)\alpha_{\bar{m},k,j}(z) \right\}$$

which applies to wider classes of functions.

The following convergence theorem contains the earlier results of [**2**] and [**8**], [**9**] as special cases.

THEOREM 4.1. *For fixed* $\bar{m}$, *let* $\{\tilde{Q}_{\bar{m},n}\}$ *be a sequence of polynomial interpolation operators* (4.3) *for which*

$$(4.4) \qquad \max_{1 \leqq k \leqq n} |\beta_{k,j}(n)| = o(n^{m_j}/\log n) \text{ as } n \rightarrow \infty$$

$j = 1, \ldots, q$. *If* $f \in \mathscr{D}$, *then* $\tilde{Q}_{\bar{m},n}(f, z)$ *converges uniformly to* $f(z)$ *on* $|z| \leqq 1$ *as* $n \rightarrow \infty$.

Proof. The proof uses the Jackson polynomials and ideas introduced in [**2**], modified in the obvious way. We omit the details.

COROLLARY 4.2. *If* f *is analytic in* $|z| < 1$ *and* $f(e^{ix})$ *belongs to* $C^{m_q}[0, 2\pi]$, *then* $Q_{\bar{m},n}(f, z)$ *converges uniformly on* $|z| \leqq 1$ *to* $f(z)$ *as* $n \rightarrow \infty$.

Remark 1. In an earlier version of this paper, the first author showed that

$$(4.5) \qquad |H_{\bar{m}}(D)^{-1}P(z)| \leqq C(\bar{m})n^{-\Sigma m_i} \log n \max_{|w| \leqq 1} |P(w)|$$

for any polynomial $P(z)$, of degree n. This introduced an extra $\log n$ in Theorem 3.4 and (4.4), necessitating $\log^2 1/\delta$ in (4.1). For $\bar{m} = (0, m_1)$

and for special cases of $(0, m_1, m_2)$, the log n term in (4.5) is not needed. We conjecture that (4.5) is valid without the log n term in any case.

Remark 2. If $|z| \leqq \rho < 1$, then the quantity on the left in inequality (3.8) is bounded by a constant depending only on ρ. Thus, in (3.7) and (3.9) it is possible to delete the log n from the right hand side if we are restricted to $|z| \leqq \rho < 1$. This allows one to prove a stronger form of Theorem 4.1 for convergence on $|z| < 1$: If $f(z)$ is analytic for $|z| < 1$ and continuous on $|z| = 1$, then the operator $\tilde{Q}_{\bar{m},n}(f, z)$ converges uniformly to $f(z)$ on compact subsets of $|z| < 1$. Here the requirement (4.4) could be weakened by omitting the log n term on the right side of the expression.

5. An example. It is natural to ask whether the Dini–Lipschitz condition is necessary in Theorem 4.1. We thank the referee for conveying to us an example suggested by Professor P. Nevai which we modify slightly for the following.

This example is a slight modification of a construction due to Fejer which can be found in [**11**]. We shall give here only the major points of the construction; the reader may consult [**11**] for details. Let

$$(5.1) \qquad P_{2n}(z) = \sum_{\nu=0}^{n-1} \frac{(-1)^\nu z^\nu}{n-\nu} - z^n \sum_{\nu=1}^{n} \frac{(-1)^{n+\nu} z^\nu}{\nu}.$$

Then

$$(5.2) \qquad |P_{2n}(z)| \leqq 4\sqrt{\pi}, \quad |z| = 1$$

and

$$(5.3) \qquad P_{2n}(-1) = 0.$$

Let $L_n(f, z)$ be the Lagrange interpolating polynomial of degree n at the roots of unity. Then for any odd integers n and N, $N \geqq 1$, it can be shown that

$$(5.4) \qquad L_n(P_{2nN}, -1) = 2 \sum_{\nu=1}^{n-1} \sum_{j=0}^{n-1} \frac{(-1)^j}{\nu + jn} > \log n.$$

Furthermore, for $2N < n$

$$(5.5) \qquad L_n(P_{2N}, z) = P_{2N}(z), \quad L_n(P_{2N}, -1) = 0.$$

For a positive odd integer N, we consider the function

$$(5.6) \qquad f_N(z) = \sum_{k=1}^{\infty} \frac{P_{2\cdot N^k}(z)}{k^2}.$$

Since the series converges uniformly in $|z| \leqq 1$, the functions f_N are analytic in $|z| < 1$ and continuous for $|z| = 1$. We next show that

$$(5.7) \qquad \omega(f_N, \delta) \leqq C(\log 1/\delta)^{-1},$$

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where the constant depends only on N. Now the error of approximation of the 2π-periodic function $f_N(e^{i\theta})$ by trigonometric polynomials can be estimated by using the partial sum of the series (5.6) to obtain

$$E_n^*(f) \leqq \left\| \sum_{2\cdot N^k > n} \frac{P_{2\cdot N^k}(e^{i\theta})}{k^2} \right\| \leqq C'/\log n$$

where C' depends on N. Inequality (5.7) follows by means of the well-known estimate [**3**, p. 59]

$$\omega(f, h) \leqq Mh \sum_{0 \leqq n \leqq h^{-1}} E_n^*(f).$$

Finally, we show that the Lagrange interpolation operator cannot converge to $f_N(z)$. Indeed,

$$L_{N^n}(f_N, -1) = \sum_{k=1}^{\infty} \frac{L_{N^n}(P_{2\cdot N^k}, -1)}{k^2} \geqq \sum_{k \geqq n} \frac{L_{N^n}(P_{2\cdot N^k}, -1)}{k^2}$$

$$\geqq \sum_{k \geqq n} \frac{\log N^n}{k^2} \geqq \log N.$$

Hence

$$\overline{\lim}_{n\to\infty} L_n(f_N, -1) \geqq \log N \neq 0 = f_N(-1). \tag{5.8}$$

An idea of Lozinski [**6**] extends this example to operators of Hermitian type. Let $\bar{m} = (0, 1, \ldots, q)$ and let $H_{q,n}(f, z)$ be the operator of (4.3) corresponding to this $\bar{m}$ where all coefficients $\beta_{k,j}(n)$ are set equal to zero. Lozinski showed that

$$H_{q,n}(f, z) = (1 - z^n)^{q-1} L_n(f, z) + O(1), \quad |z| = 1, \tag{5.9}$$

where the $O(1)$ term depends only on q and $|f|$. Since the $|f_N|$ are uniformly bounded on $|z| = 1$, from (5.8) and (5.9) we see that for suitable N

$$\overline{\lim}_{n\to\infty} H_{q,n}(f_N, -1) > 0 = f_N(-1).$$

This example shows that Theorem 4.1 is in some sense best possible. Professor P. O. H. Vertesi has kindly communicated to us a more general result which may give examples for lacunary cases.

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University of Alberta,
Edmonton, Alberta