FINITE 3-GROUPS ACTING ON BORDERED SURFACES†

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1. Introduction. Let G be a finite group. The *real genus* $\rho(G)$ [**8**] is the minimum algebraic genus of any compact bordered Klein surface on which G acts. There are now several results about the real genus parameter. The groups with real genus $\rho \leq 5$ have been classified [**8**,**9**,**12**], and genus formulas have been obtained for several classes of groups [**8**,**9**,**10**,**11**,**12**]. Most notably, McCullough calculated the real genus of each finite abelian group [**13**]. In addition, there is a good general lower bound for the real genus of a finite group [**11**].

Here we consider finite 3-groups acting on bordered Klein surfaces. We begin by specializing the approach in [**11**] to obtain a lower bound for the real genus of a 3-group. Then we determine the real genus of several infinite families of 3-groups. The lower bound is attained for most of these families. We also develop some general ideas about 3-groups acting on bordered surfaces. Finally, we determine the real genus of all groups with order 81.

We use the standard representation of a group G as a quotient of a non-euclidean crystallographic group Γ by a bordered surface group K; then G acts on the Klein surface U/K, where U is the open upper half-plane.

2. Preliminaries. We shall assume that all surfaces are compact. A bordered surface X can carry a dianalytic structure [**1**, p. 46] and be considered a Klein surface or a non-singular real algebraic curve. Thus the surface X has an algebraic genus g. The algebraic genus appears naturally in bounds for the order of the automorphism group of a Klein surface, and the real genus of a group is defined in terms of the algebraic genus.

There is a useful upper bound for the real genus of a finite group in terms of the orders of the elements in a generating set [**8**, p. 712].

THEOREM A [**8**]. *Let G be a finite group with generators $z_1, \ldots, z_c$, where $o(z_i) = m_i$. Then*

$$\rho(G) \leq 1 + o(G)\left[c - 1 - \sum_{i=1}^{c} \frac{1}{m_i}\right]. \tag{2.1}$$

Group actions on Klein surfaces have often been investigated using non-euclidean crystallographic (NEC) groups; here see the monograph [**2**], an excellent reference for the work on Klein surfaces. Let $\mathscr{L}$ denote the group of all dianalytic automorphisms of the open upper half-plane U. An NEC group is a discrete subgroup Γ of $\mathscr{L}$ (with the quotient space U/Γ compact). Associated with the NEC group Γ is its *signature*, which has the form

$$\left(p; \pm; [\lambda_1, \ldots, \lambda_r]; \left\{\left(\nu_{11}, \ldots, \nu_{1s_1}\right), \ldots, \left(\nu_{k1}, \ldots, \nu_{ks_k}\right)\right\}\right). \tag{2.2}$$

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The quotient space $X = U/\Gamma$ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and non-orientable otherwise. The *ordinary periods* $\lambda_1, \ldots, \lambda_r$ are the ramification indices of the natural quotient mapping from U to X in fibers above interior points of X. The *link periods* $\nu_{i1}, \ldots, \nu_{is_i}$, are the ramification indices in fibers above points on the ith boundary component of X. Associated with the signature (2.2) is a presentation for the NEC group Γ. For more information about signatures, see [**14**] and [**2**].

Let Γ be an NEC group with signature (2.2) and assume $k \geq 1$ so that the quotient space U/Γ is a bordered surface. Then the non-euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature [**14**, p. 235]:

$$\mu(\Gamma)/2\pi = \gamma - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{\lambda_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{\nu_{ij}}\right), \tag{2.3}$$

where γ is the algebraic genus of the quotient space U/Γ. If Λ is a subgroup of finite index in Γ, then

$$[\Gamma : \Lambda] = \mu(\Lambda)/\mu(\Gamma). \tag{2.4}$$

An NEC group K is called a *surface group* if the quotient map from U to U/K is unramified. If the quotient space U/K has a non-empty boundary, then K is called a *bordered surface group*. Bordered surface groups contain reflections but no other elements of finite order.

Let X be a bordered Klein surface of algebraic genus $g \geq 2$, and let G be a group of dianalytic automorphisms of X. Then X can be represented as U/K, where K is a bordered surface group with $\mu(K) = 2\pi(g-1)$. Further, there exist an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$ [**7**]. The group $G \cong \Gamma/K$, so that from (2.4) the algebraic genus g of the bordered surface X on which G acts is given by

$$g = 1 + o(G) \cdot \mu(\Gamma)/2\pi. \tag{2.5}$$

Thus (2.5) and (2.3) give the relationship between the algebraic genera g and γ of X and U/Γ, respectively. This relationship is sometimes given as the Riemann-Hurwitz formula for the quotient mapping $X \to X/G = U/\Gamma$; see [**6**], for example. In the following we will consistently use g and γ for the algebraic genera of the surfaces X and U/Γ, respectively.

Minimizing the algebraic genus g for a particular group G is therefore equivalent to minimizing $\mu(\Gamma)$. Among the NEC groups Γ for which G is a quotient of Γ by a bordered surface group, then, we want to identify one for which $\mu(\Gamma)$ is as small as possible.

3. A lower bound. Here we establish a useful lower bound for the real genus of a finite 3-group. We specialize the approach in [**11**, §3] to obtain a bound that is much better for 3-groups. The approach in [**11**] considered generators as having order 2, order 3 or "high" order, and the ones of high order were all treated as having order 4 to produce the lower bound. In a 3-group there are no generators of order 2, of course, and high order generators must have order at least 9.

Let G be a finitely presented 3-group and S a generating set for G. Let $t_3(S)$ denote the number of generators in S of order 3, and let $t_h(S)$ be the number of generators of order larger than 3 (and thus at least 9). We will write simply t_3 and t_h if the generating set is obvious. Then $|S| = t_3 + t_h$. We define

$$\theta(G) = \text{minimum}\{8t_h(S) + 6t_3(S) \mid S \text{ a generating set for } G\}.$$

A generating set for which $\theta(G)$ is attained is said to be *θ-minimal.* The quantity $\theta(G)/9$ arises naturally when obtaining a lower bound for (2.3) by treating all generators of high order as having order 9; see the proof of Theorem 1. The following result is basic.

LEMMA 1. *Let G' be a quotient group of the finitely presented group G. Then*

$$\theta(G) \geq \theta(G').$$

Proof. Let $\pi : G \to G'$ denote the quotient map. Then let S be a θ-minimal generating set for G, and let S' be the induced generating set for G'. Write $t_3 = t_3(S)$ and $t_3' = t_3(S')$ and so forth. Clearly $|S| \geq |S'|$ so that

$$t_h + t_3 \geq t_h' + t_3'.$$

If $y \in S'$ with $o(y) > 3$, then there is at least one generator x in S such that $\pi(x) = y$. But $o(x) > 3$ also. Hence we have

$$t_h \geq t_h'.$$

Now since S is θ-minimal,

$$\theta(G) = 8t_h + 6t_3 = 6(t_h + t_3) + 2t_h \geq 6\left(t_h' + t_3'\right) + 2t_h' \geq \theta(G'),$$

whether or not the generating set S' is θ-minimal.

Let G be a finite 3-group, and suppose there exist an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $K = \text{kernel}\,\phi$ is a bordered surface group. Since the surface group K contains no analytic elements of finite order, each ordinary period of Γ must be a power of 3. We need an upper bound for $\theta(G)$.

LEMMA 2. *Let G be a finite 3-group. Suppose there exist an NEC group Γ with signature* (2.2) *and a homomorphism $\phi : \Gamma \to G$ onto G such that K =kernel ϕ is a bordered surface group. Let r_3 denote the number of ordinary periods of Γ equal to* 3 *and r_h the number greater than* 3. *Then*

$$\theta(G) \leq 8(\gamma + r_h) + 6r_3,$$

where γ is the algebraic genus of the quotient space U/Γ.

Proof. Simplify the canonical presentation for Γ as in [**11**, §2]; the simplified presentation has $\gamma + r$ generators (that are elliptic, hyperbolic or glide reflections) plus some additional reflections. Let S be the induced generating set for G. Since the order of G is odd, all reflections in Γ are in the bordered surface group $K = \text{kernel}\,\phi$. Therefore the generating set S has at most $\gamma + r$ elements. Of the elements in S, clearly at most $\gamma + r_h$ can have order larger than three. Now apply the definition of $\theta(G)$.

Now we establish our general lower bound.

THEOREM 1. *Let G be a finite 3-group. Then*

$$\rho(G) \geq 1 + o(G)[\theta(G) - 9]/9. \tag{3.1}$$

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Proof. The only 3-groups with $\rho \leq 1$ are cyclic [**8**]. For a cyclic group G, $\rho(G) = 0$, and it is a simple matter to check that the inequality holds. Assume then that $\rho(G) \geq 2$, and let G act on the bordered surface X of algebraic genus $g \geq 2$. Then represent X as U/K where K is a bordered surface group, and obtain an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$. We use the notation of Lemma 2. In particular, γ denotes the algebraic genus of the quotient space U/Γ. Each ordinary period of Γ must be a power of 3. Using (2.3) we obtain

$$\mu(\Gamma)/2\pi \geq \gamma - 1 + r_3 \cdot 2/3 + r_h \cdot 8/9.$$

Therefore

$$9[\mu(\Gamma)/2\pi] \geq 9_\gamma + 8r_h + 6r_3 - 9.$$

Since $\gamma \geq 0$, applying Lemma 2 yields

$$9[\mu(\Gamma)/2\pi] \geq \theta(G) - 9.$$

Now from (2.5) we have $g \geq 1 + o(G)[\theta(G) - 9]/9$. Thus $\rho(G) \geq 1 + o(G)[\theta(G) - 9]/9$.

We believe the lower bound (3.1) is quite useful, in general. We shall see examples of infinite families of 3-groups for which the lower bound gives the real genus. Indeed, the lower bound of Theorem 1 always gives the genus of a 3-group G if G has a θ-minimal generating set that only contains elements of orders 3 and 9.

THEOREM 2. *Let G be a finite 3-group. If a θ-minimal generating set for G contains t elements of order* 3, *n elements of order* 9, *and no other elements, then*

$$\rho(G) = 1 + o(G)(6t + 8n - 9)/9. \tag{3.2}$$

Proof. We have $\theta(G) = 6t + 8n$ and (3.1) holds. But from the general upper bound (2.1), we also obtain

$$\rho(G) \leq 1 + o(G)(n + t - 1 - t/3 - n/9) = 1 + o(G)(6t + 8n - 9)/9.$$

4. General results. Next we use our lower bound to obtain some general results about 3-groups acting on bordered surfaces.

Let X be a bordered Klein surface of algebraic genus $g \geq 2$. Then the automorphism group G of X has order at most $12(g - 1)$[**6**]. This general upper bound can be improved, of course, in special cases. (See [**2**] for a survey of these results.) For example, there is a basic upper bound for p-groups [**3**], where p is an odd prime. In particular, if the automorphism group G is a 3-group, then $o(G) \leq 3(g - 1)$. An immediate consequence of this result is a lower bound for the real genus of a 3-group. Here we obtain this lower bound as a simple consequence of Theorem 1.

THEOREM 3. *Let G be a finite 3-group that is not cyclic. Then*

$$\rho(G) \geq 1 + o(G)/3.$$

Proof. Any generating set for G must have at least two elements, and obviously $\theta(G) \geq 2 \cdot 6$. Now (3.1) gives the result.

There are infinite families of 3-groups for which these bounds are attained. See [**3**, §5], [**2**, pp. 130, 131], and [**11**, §5]. The bound of Theorem 3 can be improved in special cases, of course.

THEOREM 4. *Let G be a finite 3-group with $\rho(G) \geq 2$. If G is not generated by elements of order* 3, *then*

$$\rho(G) \geq 1 + 5o(G)/9.$$

Proof. Since $\rho(G) \neq 0$, the rank of G is at least two, and a generating set for G must contain at least one element of high order. Hence $\theta(G) \geq 8 + 6$, and (3.1) gives the lower bound.

Let G be a finite 3-group. It may well be that in any presentation for G, there are at least two generators of high order. Let Ω be the subgroup of G generated by the elements of order 3. Then Ω is a characteristic subgroup of G.

THEOREM 5. *Let G be a finite 3-group. If G/Ω is not cyclic, then*

$$\rho(G) \geq 1 + 7o(G)/9.$$

Proof. Since G/Ω is not cyclic, any generating set for G must have at least two elements of order larger than 3. Hence $\theta(G) \geq 2{\cdot}8$, and the result follows from (3.1).

The lower bound for the real genus is even better for 3-groups that have rank three or more.

THEOREM 6. *Let G be a finite 3-group. If* $\mathrm{rank}(G) > 2$, *then*

$$\rho(G) \geq o(G) + 1.$$

Proof. Since $\mathrm{rank}(G) \geq 3$, obviously $\theta(G) \geq 3{\cdot}6$.

This series of results about a 3-group G can also be obtained by considering the possible signatures for an NEC group Γ such that G is a quotient of Γ by a bordered surface group and the non-euclidean area $\mu(\Gamma)$ is small. For instance, if G is a 3-group of the maximum possible order for the value of the genus, then G must be a quotient of an NEC group with signature $\big(0; +; [3, 3]; \{(\,)\}\big)$. For examples of this approach, see [**9**, §3] and [**12**, §3].

5. Genus formulas for particular families. We begin with an easy application of (3.2) to abelian groups that only have factors of Z_3 and Z_9.

THEOREM 7.

$$\rho\big((Z_3)^t \times (Z_9)^n\big) = 1 + 3^{t+2n-2}(6t + 8n - 9).$$

Proof. Let $G = (Z_3)^t \times (Z_9)^n$. Clearly $\mathrm{rank}(G) = t + n$, with a θ-minimal generating set for G containing t elements of order 3 and n elements of order 9. Now (3.2) gives

$$\rho(G) = 1 + 3^t 9^n (6t + 8n - 9)/9.$$

The formula of Theorem 7 can also be obtained from the general results in [**13**], although it does not appear there explicitly. The approach in [**13**] utilizes graphs of groups and is quite different, however. In addition, see [**11**, p. 1284], where elementary abelian 3-groups are considered.

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Next let K be the nonabelian group of order 27 with no element of order 9. The group K has presentation [**11**, p. 1282]

$$R^3 = S^3 = (RS)^3 = \left(R^{-1}S\right)^3 = 1.$$

The group K is a semi-direct product $(Z_3)^2 \times_\phi Z_3$.

THEOREM 8. $\rho(K^n) = 1 + 3^{3n-1}(4n-3)$.

Proof. The group K is generated by two elements of order 3, and it is clear that $\theta(K) = 12$ and $\theta(K^n) = 12n$, with a θ-minimal generating set for K^n containing $2n$ elements of order 3. Now (3.2) yields $\rho(K^n) = 1 + 3^{3n}(6\cdot 2n - 9)/9$.

In particular, $\rho(K) = 10$ [**11**, p. 1282].

For $m \geq 3$, let M_m be the group with generators X, Y and defining relations

$$X^{3^{m-1}} = Y^3 = 1,\ Y^{-1}XY = X^{1+3^{m-2}}. \tag{5.1}$$

The group M_m is a nonabelian group of order 3^m [**5**, p. 190]. The properties of these groups are well-known, of course [**5**, pp. 190–194]. Each possesses a maximal cyclic subgroup of order 3^{m-1}. In fact, these groups are characterized among all nonabelian 3-groups by this property [**5**, p. 193].

The group M_m is not generated by elements of order 3 and 9 (if $m > 3$), and the lower bounds of Theorems 1 and 4 are not attained. To obtain the lower bound for the real genus of M_m, we modify the proof of Proposition 1 of [**12**].

THEOREM 9. $\rho(M_m) = 2\left(3^{m-1} - 1\right)$ *for* $m \geq 3$.

Proof. Write $G = M_m$. We know $\rho(G) \geq 2$. Let G act on a bordered Klein surface X of algebraic genus $g \geq 2$. Then represent X as U/K, where K is a bordered surface group, and obtain an NEC group Γ with signature (2.2) and a homomorphism $\alpha : \Gamma \to G$ onto G such that kernel $\alpha = K$. Let γ be the algebraic genus of the quotient space U/Γ. Since the order of G is odd, it is basic that all period cycles of Γ are empty (Each reflection is in the kernel K, but the surface group K contains no analytic elements of finite order).

Simplify the canonical presentation for Γ as in [**11**, §2]. In this simplified presentation, there must be at least two elements with order 3 or more, since $\Gamma/K \cong G$. The number of generators of Γ with order larger than two is at most $\gamma + r$, where r is the number of ordinary periods. Therefore $\gamma + r \geq 2$. Let $A = \mu(\Gamma)/2\pi$, which is given by (2.3). We obtain a lower bound for A. Again, each ordinary period of Γ must be a power of 3. If $\gamma \geq 2$, then obviously $A \geq 1$. If $\gamma = 1$, then $r \geq 1$ and $A \geq 2/3$.

Suppose $\gamma = 0$ so that $r \geq 2$. If $r \geq 3$, then $A \geq -1 + 3 \cdot 2/3 = 1$. Assume $r = 2$. Then the group Γ has signature $\left(0; +; [\lambda_1, \lambda_2]; \{(\,)\}\right)$, where we may take $\lambda_1 \leq \lambda_2$. From (2.3)

$$A = 1 - \frac{1}{\lambda_1} - \frac{1}{\lambda_2}.$$

If $\lambda_2 \geq \lambda_1 \geq 9$, then $A \geq 1 - 2 \cdot (1/9) = 7/9 > 2/3$.

Assume, then, that $\gamma = 0$, $r = 2$, and $\lambda_1 = 3$. The group Γ has presentation

$$x^3 = y^{\lambda_2} = c^2 = [c, e] = xye = 1.$$

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But the only generating reflection c must be in the bordered surface group K, and e is redundant. Thus the quotient group $G \cong \Gamma/K$ is generated by the two elements $\alpha(x)$ and $\alpha(y)$. In any presentation for G, there must be at least one element of order 3^{m-1}, since the subgroup generated by elements of order dividing 3^{m-2} has index 3 in G [**5**, Th. 4.3(i)(c), p. 191]. Therefore $\lambda_2 = 3^{m-1}$, and

$$A = 1 - \frac{1}{3} - \frac{1}{3^{m-1}}.$$

In this case, then, $A < 2/3$ and from (2.5) $g = 1 + 3^m \cdot A = 1 + 3^m(2/3 - 1/3^{m-1}) = 2(3^{m-1} - 1)$. In all other cases, $A \geq 2/3$. Thus $\rho(G) \geq 2(3^{m-1} - 1)$.

The upper bound for $\rho(G)$ is provided by (2.1) applied to the defining presentation (5.1).

In particular, the group M_3 of order 27 has real genus 16 [**12**, p. 405]. Write $M = M_3$; M is a semi-direct product $Z_9 \times_\theta Z_3$.

THEOREM 10. $\rho(M^n) = 1 + 3^{3n-2}(14n - 9)$.

Proof. The group M is generated by an element of order 9 and one of order 3, of course. Clearly $\theta(M) = 14$ and $\theta(M^n) = 14n$, with a θ-minimal generating set for M^n containing n elements of order 3 and n of order 9. Now (3.2) gives $\rho(M^n) = 1 + 3^{3n}(8n + 6n - 9)/9$.

We also briefly consider direct products of elementary abelian 3-groups and the groups K and M.

THEOREM 11. $\rho((Z_3)^n \times K) = 1 + 3^{n+2}(2n + 1)$.

Proof. A θ-minimal generating set for $(Z_3)^n \times K$ contains $n + 2$ elements of order 3.

THEOREM 12. $\rho((Z_3)^n \times M) = 1 + 3^{n+1}(6n + 5)$.

Proof. A θ-minimal generating set for $(Z_3)^n \times M$ contains $n + 1$ elements of order 3 and one of order 9.

The general lower bound of Theorem 1 is attained for the groups of Theorems 7, 8, 10, 11 and 12.

6. The groups of order 81. The real genus of each group with order less than 32 has been determined [**12**]. For 3-groups the next order of interest is 81. There are 15 groups of order 81; five of these are abelian. These groups are listed in Burnside's classic book [**4**, pp. 145, 146], and our notation G_n refers to the nth group in Burnside's list. The real genus of each abelian group has been determined [**13**]. Interestingly, all the nonabelian groups of order 81 yield to the methods of §§3–5. The group G_9 is the direct product $Z_3 \times M$, and G_{14} is the direct product $Z_3 \times K$. Also G_6 is the group M_4. We consider each remaining group as an extension of a large normal subgroup.

The groups G_7, G_{10}, and G_{15} are extensions of $Z_3 \times Z_9$. The abelian group $A = Z_3 \times Z_9$ has presentation

$$P^9 = Q^3 = 1, PQ = QP. \tag{6.1}$$

To obtain the group G_{15}, adjoin to A an element R of order 3 that transforms the elements of A according to the automorphism $P \to PQ, Q \to P^{-3}Q$. Then the group G_{15} [**4**, p. 146] has generators P, Q, R and defining relations (6.1) together with

$$R^3 = 1, R^{-1}PR = PQ, R^{-1}QR = P^{-3}Q.$$

Then the element $R^{-1}P$ has order 3, and clearly $G_{15} = \langle R, R^{-1}P\rangle$. Thus G_{15} is generated by two elements of order 3. Now $\theta(G_{15}) = 12$ and $\rho(G_{15}) = 28$ by (3.2).

To obtain the group G_{10}, adjoin to $A = Z_3 \times Z_9$ an element S of order 3 that transforms the elements of A according to the automorphism $P \to PQ, Q \to Q$. Then the group G_{10} [**4**, p. 145] is generated by P, Q, S with defining relations (6.1) and

$$S^3 = 1, S^{-1}PS = PQ, QS = SQ.$$

Then the subgroup $\Omega(G_{10}) \cong (Z_3)^3$, so that G_{10} is not generated by elements of order 3. But obviously $G_{10} = \langle S, P\rangle$, and $\theta(G_{10}) = 14$. Hence $\rho(G_{10}) = 46$.

The group G_7 is a third extension of $Z_3 \times Z_9$. The group G_7 has presentation [**4**, p. 145]

$$P^9 = Q^3 = R^3 = 1, PQ = QP, PR = RP, R^{-1}QR = QP^3.$$

The Frattini subgroup $\Phi(G_7) = \langle P^3\rangle \cong Z_3$, so that $G_7/\Phi \cong (Z_3)^3$ and G_7 has rank 3. Also $\Omega(G_7)$ is a nonabelian subgroup of order 27, so that G_7 is not generated by elements of order 3. But $G_7 = \langle P, Q, R\rangle$, of course, and a θ-minimal generating set for G_7 has two elements of order 3 and one of order 9. Thus $\rho(G_7) = 100$, using (3.2).

The groups G_{11}, G_{12}, and G_{13} are extensions of the nonabelian group $M = M_3$ with presentation (from (5.1))

$$X^9 = Y^3 = 1, Y^{-1}XY = X^4. \tag{6.2}$$

First, to obtain the group G_{11}, adjoin to M an element W of order 3 that transforms the elements of M according to the automorphism $X \to XY, Y \to Y$. The group G_{11} [**4**, p. 145] has generators X, Y, W and defining relations (6.2) plus

$$W^3 = 1, W^{-1}XW = XY, YW = WY.$$

Then the element $W^{-1}X$ has order 3, and $G_{11} = \langle W, W^{-1}X\rangle$. Thus G_{11} is generated by two elements of order 3 and $\theta(G_{11}) = 12$. Hence $\rho(G_{11}) = 28$.

The group G_{12} [**4**, p. 145] has generators X, Y, W and defining relations (6.2) and

$$W^{-1}XW = XY, YW = WY, W^3 = X^3.$$

Then $\Omega(G_{12})$ is a nonabelian subgroup of order 27, so that G_{12} is not generated by elements of order 3. But the element WX has order 3 and $G_{12} = \langle X, WX\rangle$. Now $\theta(G_{12}) = 14$ and $\rho(G_{12}) = 46$.

The group G_{13} is a third extension of M. The group G_{13} [**4**, p. 145] has presentation (6.2) together with

$$W^{-1}XW = XY, YW = WY, W^3 = X^6.$$

The element W has order 9. Here we have $\Phi(G_{13}) = \Omega(G_{13}) = \langle Q, P^3 \rangle \cong (Z_3)^2$ and $G_{13}/\Phi \cong (Z_3)^2$. Thus any generating set for G_{13} must have at least two elements of order larger than 3. Since $G_{13} = \langle X, W \rangle$, we have $\theta(G_{13}) = 16$ and $\rho(G_{13}) = 64$ by (3.2).

Finally consider G_8, the group with presentation [**4**, p. 145]

$$P^9 = Q^9 = 1, Q^{-1}PQ = P^4.$$

This group is a semi-direct product $Z_9 \times_\phi Z_9$. For this group $\Phi(G_8) = \Omega(G_8) = \langle Q^3, P^3 \rangle \cong (Z_3)^2$, and $G_8/\Phi \cong (Z_3)^2$. Thus the two elements P and Q of order 9 form a θ-minimal generating set for G. Hence $\theta(G_8) = 16$ and $\rho(G_8) = 64$.

The following table gives $\rho(G)$ for each nonabelian group G of order 81.

Group	ρ	Group	ρ
M_4	52	G_{11}	28
G_7	100	G_{12}	46
G_8	64	G_{13}	64
$Z_3 \times M$	100	$Z_3 \times K$	82
G_{10}	46	G_{15}	28

The nonabelian groups of order 81 provide examples of groups for which the bounds of §4 are attained. For instance, each of the groups with $\rho = 46$ is generated by an element of order 3 and one of order 9, and the bound of Theorem 4 is realized.

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