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A Multivalued Nonlinear System with the Vector p-Laplacian on the Semi-Infinity Interval

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Abstract. We study a second order nonlinear system driven by the vector p-Laplacian, with a multivalued nonlinearity and defined on the positive time semi-axis $\mathbb{R}_+$. Using degree theoretic techniques we solve an auxiliary mixed boundary value problem defined on the finite interval $[0, n]$ and then via a diagonalization method we produce a solution for the original infinite time horizon system.

1 Introduction

The purpose of this paper is to study the existence of a solution for a second order nonlinear system driven by the ordinary vector p-Laplacian differential operator, with a multivalued right-hand side and defined on the semi-infinite time interval $\mathbb{R}_+$. So the problem under consideration is the following:

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' \in F(t, x(t), x'(t)) \text{ a.e. on } \mathbb{R}_+, \\ &x(0) = 0,\ x \text{ is bounded on } \mathbb{R}_+,\ 1 < p < +\infty. \end{aligned} \tag{1.1}$$

Problems of this type have been studied by Agarwal and O'Regan [1], Constantin [3], Granas, Guenther, Lee, and O'Regan [7] and Ma [10]. In all these works $p = 2$ (semilinear problems), the right-hand side nonlinearity F is single-valued and the problem is scalar. Prolems with multivalued nonlinearities defined on a finite interval can be found in [6].

Our method of proof uses a degree theoretic technique based on the fixed point index of Bader [2] to produce solutions valid on the finite intervals $[0, n]$, $n \geq 1$ and finally through a diagonalization argument we obtain a solution for the infinite time horizon problem.

2 Mathematical Background

In addition to degree theory, our approach uses notions and results from the theory of nonlinear operators of monotone type and from multivalued analysis. For easy reference we recall them in this section. Details can be found in [5, 9].

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Let (Ω, Σ, μ) be a complete σ-finite measure space and X a separable Banach space. In what follows we will use the following notations:

$$P_{f(c)}(X) = \{C \subseteq X : \text{nonempty, closed, (convex)}\},$$

$$P_{(w)k(c)}(X) = \{C \subseteq X : \text{nonempty, (weakly-) compact, (convex)}\}.$$

A multifunction $G\colon \Omega \to 2^X \setminus \{\varnothing\}$ is said to be *measurable*, if for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \to d(x, G(\omega)) = \inf\{\|x - u\| : u \in G(\omega)\}$ is Σ-measurable. We say that G is *graph measurable*, if

$$GrG = \{(\omega, x) \in \Omega \times X : x \in G(\omega)\} \in \Sigma \times B(X),$$

with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions (*i.e.,* multifunctions with nonempty, closed values), measurability and graph measurability are equivalent notions. Moreover, by virtue of the Yankov–von Neumann–Aumann selection theorem, a graph measurable multifunction $G\colon \Omega \to 2^X \setminus \{\varnothing\}$ admits a Σ-measurable selection, *i.e.,* there exists a Σ-measurable function $g\colon \Omega \to X$ such that $g(\omega) \in G(\omega)$ for all $\omega \in \Omega$. Given $1 \leq p \leq \infty$ and a graph measurable multifunction $G\colon \Omega \to 2^X \setminus \{\varnothing\}$, we set $S_G^p = \{g \in L^p(\Omega, X) : g(\omega) \in G(\omega)\mu - \text{a.e.}\}$ (the set of L^p-selectors of G). This set may be empty. A straightforward argument involving the Yankov–von Neumann–Aumann selection theorem, reveals that if $\inf\{\|u\| : u \in G(\omega)\} \in L^p(\Omega)_+$, then $S_G^p \neq \varnothing$.

Let Y, Z be Hausdorff topological spaces and $H\colon Y \to 2^Z \setminus \{\varnothing\}$ a multifunction. We say that H is upper semicontinuous (usc for short), if for all $U \subseteq Z$ open, the set $H^+(U) = \{y \in Y : H(y) \subseteq U\}$ is open in Y. We say that H has a closed graph, if $GrH = \{(y, z) \in Y \times Z : z \in H(y)\}$ is closed in $Y \times Z$. If Z is regular and H has closed values, then upper semicontinuity of H implies that H has a closed graph. The converse is true if H has closed values and is locally compact, *i.e.,* for every $y \in Y$ we can find a neighborhood U of y such that $\overline{H(U)}$ is compact in Z.

Suppose that V, W are Banach spaces and $K\colon D \subseteq V \to W$. We say that K is completely continuous, if for every sequence $\{v_n\}_{n \geq 1} \subseteq D$ such that $v_n \xrightarrow{w} v \in D$, we have $K(v_n) \to K(v)$ in W. If V is reflexive and $D \subseteq V$ is nonempty, closed and convex, then complete continuity of K implies that K is compact, *i.e.,* K is continuous and maps bounded sets onto relatively compact sets.

Suppose that $C \subseteq V$ and $D \subseteq W$ are nonempty closed and convex sets and $G\colon C \to 2^D \setminus \{\varnothing\}$ a multifunction with weakly compact convex values which is usc from C with the relative norm topology into D with the relative weak topology. Also let $K\colon D \to C$ be completely continuous and set $S = K \circ G$. We assume that S is compact, *i.e.,* it maps bounded sets into relatively compact sets (this is the case if, for example, G maps bounded sets to norm bounded sets and X is reflexive; also we should mention that the compactness assumption implies that S is usc). The interesting feature of S is that it need not have convex values. Finally let U be a bounded and relatively open subset of C such that $\text{Fix}(S) \cap \partial U = \varnothing$, where $\text{Fix}(S) = \{v \in C : v \in S(v)\}$ (the set of fixed points of the composite multifunction S). For such triples (S, U, C) Bader [2] defined a fixed point index $i_C(S, U)$ which exhibits all the usual properties. In particular, if $S_0 = K_0 \circ G_0$ and

$S_1 = K_1 \circ G_1$, then we say that S_0 and S_1 are homotopic if there is a usc multifunction $F\colon [0,1] \times C \to 2^D \setminus \{\varnothing\}$ (D equipped with the relative weak topology) which has weakly compact convex values such that $F(0,\cdot) = G_0$ and $F(1,\cdot) = G_1$ and a sequentially continuous map $N\colon [0,1] \times D \to C$ (D always with the relative weak topology) such that $N(0,\cdot) = K_0$ and $N(1,\cdot) = K_1$. We set $H(\lambda, x) = (N \circ F)(\lambda, x)$ and in the homotopy invariance property of the index we require that $x \notin H(\lambda, x)$ for all $(\lambda, x) \in [0,1] \times \partial U$ and H is compact. Finally we should mention that O'Regan [12] recently extended the Leray–Schauder alternatives obtained by Bader [2].

If X is a reflexive Banach space, a nonlinear operator $A\colon D(A) \subseteq X \to X^*$ is said to be generalized pseudomonotone, if for any sequence $\{x_n\}_{n\geq 1} \subseteq D(A)$ with $x_n \xrightarrow{w} x$ in X, $A(x_n) \xrightarrow{w} u^*$ in X^* and $\limsup \langle A(x_n), x_n - x\rangle \leq 0$, we have $u^* = A(x)$ and $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$. A maximal operator is generalized pseudomonotone. A generalized pseudomonotone operator A such that $D(A) = X$ and which is bounded (maps bounded sets to bounded sets), it is pseudomonotone. Recall that a pseudomonotone coercive operator is surjective.

3 Auxiliary results

Our hypotheses on the multivalued nonlinearity $F(t, x, y)$ are the following:

(H(F)) $$F\colon \mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to P_{kc}(\mathbb{R}^{\mathbb{N}})$$

is a multifunction such that

(i) for all $(x, y) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $t \to F(t, x, y)$ is graph measurable;

(ii) for almost all $t \in T$, $(x, y) \to F(t, x, y)$ has a closed graph;

(iii) for every $r > 0$, there exists $\alpha_r \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that for almost all $t \in \mathbb{R}_+$, all $\|x\|, \|y\| \leq r$ and all $u \in F(t, x, y)$, we have $\|u\| \leq \alpha_r(t)$;

(iv) there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{\mathbb{R}^{\mathbb{N}}} = 0$, we can find $\delta, \gamma > 0$ such that for almost all $t \in \mathbb{R}_+$, we have

$$\inf\bigl\{ (-u, x)_{\mathbb{R}^{\mathbb{N}}} + \|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta,\ u \in F(t, x, y)\bigr\} \geq \gamma > 0;$$

(v) there exist $\eta \in L^1(\mathbb{R}_+)$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ a nondecreasing function such that $\frac{1}{\psi(\cdot)}$ is locally integrable, $\int_0^\infty \frac{ds}{\psi(s)} \geq \|\eta\|_1$ and for almost all $t \in \mathbb{R}_+$, all $\|x\| \leq M$, all $y \in \mathbb{R}^{\mathbb{N}}$ and all $u \in F(t, x, y)$, we have $\|u\| \leq \eta(t)\psi(\|y\|^{p-1})$.

Remark 3.1 Hypothesis H(F)(iv) is a version of the Hartman condition [8, p. 433], which is adapted to the present multivalued, Caratheodory setting. Hypothesis H(F)(v) is a version of the well-known Bernstein–Nagumo–Wintner growth condition (see [11, 13]) and it is satisfied if for almost all $t \in T$, all $\|x\| \leq M$, all $y \in \mathbb{R}^{\mathbb{N}}$ and all $u \in F(t, x, y)$, we have $\|u\| \leq \alpha(t) + c\|y\|^p$ with $\alpha \in L^1(T)_+$, $c > 0$ (Bernstein's choice). Hypothesis H(F)(iv) leads to *a priori* bounds for $\|x(t)\|$ and hypothesis H(F)(v) produces *a priori* bounds for $\|x'(t)\|$.

First we will consider a sequence of boundary value problems with a finite time horizon. For this purpose, for every $n \geq 1$, we consider the following function space:

$$\widehat{W}_n = \{x \in W^{1,p}((0, n), \mathbb{R}^{\mathbb{N}}) : x(0) = 0\}.$$

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The continuous (in fact compact) embedding of $W^{1,p}((0,n),\mathbb{R}^{\mathbb{N}})$ into $C([0,n],\mathbb{R}^{\mathbb{N}})$ justifies the evaluation at $t=0$. Also using this embedding we can check that $\widehat{W}_n$ furnished with the Sobolev norm, becomes a closed subspace of the Sobolev space $W^{1,p}((0,n),\mathbb{R}^{\mathbb{N}})$. In particular, then $\widehat{W}_n$ is a separable reflexive Banach space. Evidently the only constant function belonging in $\widehat{W}_n$ is the zero function. This implies that the Poincaré inequality is valid on $\widehat{W}_n$. For the benefit of the reader, in the next lemma we prove this fact in full generality.

Lemma 3.2 *If $Z\subseteq\mathbb{R}^{\mathbb{N}}$ is a bounded domain with a Lipschitz boundary ∂Z and V is a closed vector subspace of $W^{1,p}(Z)$ such that the only constant function in V is the zero function, then there exists $\beta>0$ such that $\|x\|_p\le\beta\|Dx\|_p$ for all $x\in V$.*

Proof We argue indirectly. Suppose that the lemma is not true. Then we can find $\{x_n\}_{n\ge1}\subseteq V$ such that $\|x_n\|_p>n\|Dx_n\|_p$ for all $n\ge1$. Set $y_n=\frac{x_n}{\|x_n\|_p}$, $n\ge1$. We have $1>n\|Dy_n\|_p$ for all $n\ge1$, hence $Dy_n\to0$ in $L^p(Z,\mathbb{R}^{\mathbb{N}})$. In particular then, $\{y_n\}_{n\ge1}\subseteq W^{1,p}(Z)$ is bounded and so by passing to a suitable subsequence if necessary, we may assume that $y_n\xrightarrow{w}y$ in $W_0^{1,p}(Z)$. Since $W^{1,p}(Z)$ is embedded compactly in $L^p(Z)$, we have that $y_n\to y$ in $L^p(Z)$ and so $\|y\|_p=1$, *i.e.*, $y\ne0$. Also note that $y_n\to y$ in $W^{1,p}(Z)$ and so $\|Dy\|_p=0$. Therefore $y\equiv\xi\in\mathbb{R}$, hence $y=0$ (since $y\in V$), a contradiction. ■

Remark 3.3 If $Z=(0,n)$ and we apply the lemma on each component, we see that the Poincaré inequality is valid on $\widehat{W}_n$.

For $h\in L^1([0,n],\mathbb{R}^{\mathbb{N}})$ we consider the following mixed boundary value problem:

$$\begin{aligned}&-(\|x'(t)\|^{p-2}x'(t))'=h(t)\text{ a.e. on }[0,n],\\&x(0)=0,\ x'(n)=0,\ 1<p<\infty.\end{aligned}\tag{3.1}$$

Proposition 3.4 *Problem (3.1) has a unique solution $V_n(h)\in C^1([0,n],\mathbb{R}^{\mathbb{N}})$ and the solution map $V_n\colon L^1([0,n],\mathbb{R}^{\mathbb{N}})\to C^1([0,n],\mathbb{R}^{\mathbb{N}})$ is completely continuous.*

Proof Consider the nonlinear operator $A\colon\widehat{W}_n\to\widehat{W}_n^*$ defined by

$$\langle A(x),y\rangle_n=\int_0^n\|x'(t)\|^{p-2}(x'(t),y'(t))_{\mathbb{R}^{\mathbb{N}}}\,dt\quad\text{for all }x,y\in\widehat{W}_n.$$

Hereafter by $\langle\cdot,\cdot\rangle_n$ we denote the duality brackets for the pair $(\widehat{W}_n,\widehat{W}_n^*)$. It is easy to see that A is monotone, demicontinuous, hence it is maximal monotone. Also $\langle A(x),x\rangle_n=\|x'\|_p^p$. By virtue of Lemma 3.2 this means that the operator A is coercive. But recall that a maximal monotone, coercive operator is surjective (see [5, p. 49]. Since $L^1([0,n],\mathbb{R}^{\mathbb{N}})\subseteq\widehat{W}_n^*$, we can find $x=V_n(h)\in\widehat{W}_n$ such that $A(x)=h$. Let $\varphi\in C_c^\infty((0,b),\mathbb{R}^{\mathbb{N}})$ (*i.e.*, $\varphi\colon(0,b)\to\mathbb{R}^{\mathbb{N}}$ is C^∞ and has compact support in $(0,b)$). We have

$$\langle A(x),\varphi\rangle_n=\int_0^n(h(t),\varphi(t))_{\mathbb{R}^{\mathbb{N}}}\,dt,$$

therefore, $\int_0^n \|x'(t)\|^{p-2}(x'(t),\varphi'(t))_{\mathbb{R}^{\mathbb{N}}}\,dt = \int_0^n (h(t),\varphi(t))_{\mathbb{R}^{\mathbb{N}}}\,dt$. From the representation theorem for the elements of $W^{-1,p'}((0,n),\mathbb{R}^{\mathbb{N}}) = W_0^{1,p}((0,b),\mathbb{R}^{\mathbb{N}})^*$, with $\frac{1}{p}+\frac{1}{p'}=1$ (see for example [4, p. 362]), we know that

$$\left(\|x'\|^{p-2}x'\right)' \in W^{-1,p'}((0,n),\mathbb{R}^{\mathbb{N}}).$$

By $\langle\cdot,\cdot\rangle_{0,n}$ we denote the duality brackets for the pair

$$(W_0^{1,p}((0,n),\mathbb{R}^{\mathbb{N}}), \quad W^{-1,p'}((0,b),\mathbb{R}^{\mathbb{N}})).$$

Then, through an integration by parts, we have

$$\begin{aligned}\langle -(\|x'\|^{p-2}x')',\varphi\rangle_{0,n} &= \int_0^n (h(t),\varphi(t))_{\mathbb{R}^{\mathbb{N}}}\,dt\\ &= \langle h,\varphi\rangle_{0,n} \quad \text{for all } \varphi\in C_c^\infty((0,b),\mathbb{R}^{\mathbb{N}}).\end{aligned}$$

Because $C_c^\infty((0,b),\mathbb{R}^{\mathbb{N}})$ is dense in $W_0^{1,p}((0,n),\mathbb{R}^{\mathbb{N}})$, it follows that

$$-(\|x'\|^{p-2}x'(t))' = h(t) \quad \text{a.e. on } [0,n],\ x(0)=0. \tag{3.2}$$

From (3.2), we obtain that $\|x'\|^{p-2}x' \in W^{1,1}((0,b),\mathbb{R}^{\mathbb{N}}) \subseteq C([0,n],\mathbb{R}^{\mathbb{N}})$. Since $\theta_p\colon \mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}$ defined by

$$\theta_p(v) = \begin{cases}\|v\|^{p-2}v & \text{if } v\neq 0,\\ 0 & \text{if } v=0,\end{cases}$$

is a homeomorphism, it follows that $x'\in C([0,n],\mathbb{R}^{\mathbb{N}})$, hence $x\in C^1([0,n],\mathbb{R}^{\mathbb{N}})$.

Now let $v\in\widehat{W}_n$. We have

$$\langle A(x),v\rangle_n = \int_0^n (h(t),v(t))_{\mathbb{R}^{\mathbb{N}}}\,dt.$$

Therefore,

$$\int_0^n \|x'(t)\|^{p-2}(x'(t),v'(t))_{\mathbb{R}^{\mathbb{N}}}\,dt = \int_0^n (h(t),v(t))_{\mathbb{R}^{\mathbb{N}}}\,dt. \tag{3.3}$$

Performing an integration by parts on the integral in the left-hand side of (3.3), we obtain

$$\|x'(n)\|^{p-2}(x'(n),v(n))_{\mathbb{R}^{\mathbb{N}}} - \int_0^n ((\|x'(t)\|^{p-2}x'(t))',v(t))_{\mathbb{R}^{\mathbb{N}}}\,dt = \int_0^n (h(t),v(t))_{\mathbb{R}^{\mathbb{N}}}\,dt,$$

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Therefore, $\|x'(n)\|^{p-2}(x'(n), v(n))_{\mathbb{R}^N} = 0$ for all $v \in \widehat{W}_n$ (see (3.3)), hence $x'(n) = 0$. Therefore $x \in C^1([0,n],\mathbb{R}^N)$ is a solution of problem (3.1). Moreover, due to the strict monotonicity of the operator A, the solution $x = V_n(h) \in C^1([0,n],\mathbb{R}^N)$ is unique.

We consider the solution map $V_n\colon L^1([0,n],\mathbb{R}^N) \to C^1([0,n],\mathbb{R}^N)$. We will show that V_n is completely continuous. To this end suppose that $h_m \xrightarrow{w} h$ in $L^1([0,n],\mathbb{R}^N)$ and set $x_m = V_n(h_m) \in C^1([0,n],\mathbb{R}^N)$. We have $A(x_m) = h_m$ in $\widehat{W}_n^*$ for all $n \geq 1$. Using as a test function $x_m \in \widehat{W}_n$, we obtain

$$\begin{aligned} \|x_m'\|_p^p &= \int_0^n (h_m(t), x_m(t))_{\mathbb{R}^N}\, dt \\ &\leq \|h_m\|_1 \|x_m\|_\infty \quad \text{(by Hölder's inequality)} \\ &\leq c_1 \|h_m\|_1 \|x_m'\|_p \quad \text{for some } c_1 > 0, \text{ all } m \geq 1. \end{aligned} \tag{3.4}$$

Here in the last inequality we have used Lemma 3.2 and the fact that $\widehat{W}_n$ is embedded continuously (in fact compactly) into $C([0,n],\mathbb{R}^N)$. So using once more Lemma 3.2, from (3.4) we infer that $\{x_m\}_{m\geq 1} \subseteq \widehat{W}_n$ is bounded. By passing to a suitable subsequence if necessary, we may assume that

$$x_m \xrightarrow{w} x \text{ in } \widehat{W}_n \quad \text{and} \quad x_m \to x \text{ in } C([0,n],\mathbb{R}^N) \text{ as } m \to \infty.$$

We have

$$\langle A(x_m), x_m - x\rangle_n = \int_0^n (h_m(t), x_m(t) - x(t))_{\mathbb{R}^N}\, dt \to 0 \text{ as } m \to \infty. \tag{3.5}$$

Recall that the nonlinear operator A is maximal monotone, hence it is generalized pseudomonotone (see [5, p. 58]). So from (3.5) it follows that $\langle A(x_m), x_m\rangle_n \to \langle A(x), x\rangle_n$, hence $\|Dx_m\|_{L^p([0,n],\mathbb{R}^N)} \to \|Dx\|_{L^p([0,n],\mathbb{R}^N)}$. We know that $Dx_m \xrightarrow{w} Dx$ in $L^p([0,n],\mathbb{R}^N)$. Also the space $L^p([0,n],\mathbb{R}^N)$ is uniformly convex. So by the Kadec–Klee property we have $Dx_m \to Dx$ in $L^p([0,n],\mathbb{R}^N)$, therefore $x_m \to x$ in $\widehat{W}_n$ as $m \to \infty$. Passing to the limit as $m \to \infty$, we obtain $A(x) = h$, and thus $V_n(h) = x$. Because $-(\|x_m'(t)\|^{p-2}x_m'(t))' = h_m(t)$ a.e. on $[0,n]$ for all $m \geq 1$, it follows that

$$\{(\|x_m'\|^{p-2}x_m')'\}_{m\geq 1} \subseteq L^1([0,n],\mathbb{R}^N)$$

is uniformly integrable and so we deduce that $\{\|x_m'\|^{p-2}x_m'\}_{m\geq 1} \subseteq C([0,n],\mathbb{R}^N)$ is equicontinuous. Moreover, since $x_m'(n) = 0$, we have that $\{\|x_m'\|^{p-2}x_m'\}_{m\geq 1} \subseteq C([0,n],\mathbb{R}^N)$ is bounded. Invoking the Arzela–Ascoli theorem, we obtain that

$$\{\|x_m'\|^{p-2}x_m'\}_{m\geq 1} \subseteq C([0,n],\mathbb{R}^N)$$

is relatively compact. Recall that the map $\theta_p\colon \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\theta_p(v) = \begin{cases} \|v\|^{p-2}v & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases}$$

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is homeomorphism.

So it follows that $\{x'_m\}_{m\geq 1} \subseteq C([0,n],\mathbb{R}^{\mathbb{N}})$ is relatively compact. Therefore, at least for a subsequence, we can say that $x'_m \to x'$ in $C([0,n],\mathbb{R}^{\mathbb{N}})$, hence $x_m \to x$ in $C^1([0,n],\mathbb{R}^{\mathbb{N}})$. Finally, Urysohn's criterion for convergent sequences implies that for the original sequence we have $V_n(h_m) = x_m \to x = V_n(h)$ in $C^1([0,n],\mathbb{R}^{\mathbb{N}})$. This proves the desired complete continuity of the solution map $V_n(\,\cdot\,)$. ∎

Next we consider the Nemytskii (superposition) operator corresponding to the multifunction F. Namely let $N_F\colon C^1([0,n],\mathbb{R}^{\mathbb{N}}) \to 2^{L^1([0,n],\mathbb{R}^{\mathbb{N}})}$ be defined by

$$N_F(x) = S^1_{F(\,\cdot\,,x(\,\cdot\,),x'(\,\cdot\,))}.$$

Proposition 3.5 *If hypotheses* H(F)(i–iii) *hold, then N_F has values in*

$$P_{wkc}(L^1([0,n],\mathbb{R}^{\mathbb{N}}))$$

and is usc from $C^1([0,n],\mathbb{R}^{\mathbb{N}})$ with the norm topology into $L^1([0,n],\mathbb{R}^{\mathbb{N}})$ furnished with the weak topology (denoted hereafter by $L^1([0,n],\mathbb{R}^{\mathbb{N}})_w$).

Proof First we show that N_F has nonempty values. Note that hypotheses H(F)(i,ii) do not in general imply the measurability of $(t,x,y) \to F(t,x,y)$ (see [9, p. 227]. So the nonemptiness of the values of N_F it is not immediately clear. Let

$$x \in C^1([0,n],\mathbb{R}^{\mathbb{N}}).$$

Then we can find step function $\{s_m\}_{m\geq 1}$, $\{r_m\}_{m\geq 1}$ such that $\|s_m(t)\| \leq \|x(t)\|$, $\|r_m(t)\| \leq \|x'(t)\|$ and $s_m(t) \to x(t)$, and $r_m(t) \to x'(t)$ a.e. on $[0,n]$ as $m \to \infty$. Then by virtue of hypothesis H(F)(i), for every $m \geq 1$, $t \to F(t,s_m(t),r_m(t))$ is Lebesgue measurable (see Section 2) and so, applying the Yankov–von Neumann–Aumann selection theorem, we can find a Lebesgue measurable function

$$u_m\colon [0,n] \to \mathbb{R}^{\mathbb{N}}$$

such that $u_m(t) \in F(t,s_m(t),r_m(t))$ a.e. on $[0,n]$, for all $m \geq 1$. Because of hypothesis H(F)(iii), we have $\|u_m(t)\| \leq \alpha_r(t)$ a.e. on T with $r = \|x\|_{C^1([0,n],\mathbb{R}^{\mathbb{N}})}$ for all $m \geq 1$. So $\{u_m\}_{m\geq 1} \subseteq L^1([0,n],\mathbb{R}^{\mathbb{N}})$ is uniformly integrable, and by the Dunford–Pettis theorem, we may assume that $u_m \xrightarrow{w} u$ in $L^1([0,n],\mathbb{R}^{\mathbb{N}})$. Then from [4, p. 484], we have that

$$u(t) \in \overline{\operatorname{conv}} \limsup_{m\to\infty} F(t,s_m(t),r_m(t)) \subseteq F(t,x(t),x'(t)) \ \text{ a.e. on } \ [0,n],$$

where the last inclusion is a consequence of hypothesis H(F)(ii) and the fact that F has values in $P_{kc}(\mathbb{R}^{\mathbb{N}})$. Therefore $u \in S^1_{F(\,\cdot\,,x(\,\cdot\,),x'(\,\cdot\,))} = N_F(x)$ and so $N_F(x) \neq \varnothing$. Clearly the values of N_F are closed and convex. Moreover, hypothesis H(F)(iii) and the Dunford–Pettis theorem imply that for all $x \in C^1([0,n],\mathbb{R}^{\mathbb{N}})$, $N_F(x)$ belongs in $P_{wkc}(L^1([0,n],\mathbb{R}^{\mathbb{N}}))$.

The above argument also shows that the multifunction N_F has a graph which is sequentially closed in $C([0,n],\mathbb{R}^{\mathbb{N}}) \times L^1([0,n],\mathbb{R}^{\mathbb{N}})_w$. Also hypothesis H($F$)(iii) and the Dunford–Pettis theorem imply that N_F is locally compact from $C^1([0,n],\mathbb{R}^{\mathbb{N}})$ into $L^1([0,n],\mathbb{R}^{\mathbb{N}})_w$. Since weakly compact sets in $L^1([0,n],\mathbb{R}^{\mathbb{N}})$ furnished with the relative weak topology are metrizable, we conclude that N_F is usc from $C^1([0,n],\mathbb{R}^{\mathbb{N}})$ with the norm topology into $L^1([0,n],\mathbb{R}^{\mathbb{N}})$ with the weak topology. ■

We consider the following finite time horizon approximation of problem (1.1):

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' \in F(t,x(t),x'(t)) \text{ a.e. on } [0,n],\\ &x(0)=0,\ x'(n)=0,\ n\geq 1. \end{aligned} \tag{3.6}$$

In the next proposition, we establish the solvability of (3.6).

Proposition 3.6 *If hypotheses* H(F) *hold, then problem* (3.6) *has a solution* $x_n \in C^1([0,n],\mathbb{R}^{\mathbb{N}})$ *such that for some* $M_1 \geq M$ *we have*

$$\|x_n(t)\|,\ \|x_n'(t)\| \leq M_1 \quad \text{for all } t\in[0,n] \text{ and } n\geq 1.$$

Proof Consider the multivalued homotopy

$$H_n\colon [0,1]\times C^1([0,n],\mathbb{R}^{\mathbb{N}}) \to P_k(C^1([0,n],\mathbb{R}^{\mathbb{N}}))$$

defined by $H_n(\lambda,x) = (V_n\circ\lambda N_F)(x)$ for all $(\lambda,x)\in[0,1]\times C^1([0,n],\mathbb{R}^{\mathbb{N}})$. We will show that there exists $R>0$ such that $x\notin H(\lambda,x)$ for all $\lambda\in[0,1]$ and all $x\in\partial B_R(0)$. Since $V_n(0)=0$, we may assume $0<\lambda\leq 1$. If $x\in(V\circ\lambda N_F)(x)$ with $\lambda\in(0,1]$, we have

$$\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' = \lambda u(t) \text{ a.e. on } [0,n]\\ &x(0)=0,\ x'(n)=0 \text{ and } u\in N_F(x). \end{aligned} \tag{3.7}$$

First we show that $\|x(t)\|\leq M$ for all $t\in[0,n]$, with $M>0$ as in hypothesis H(F)(iv) (the Hartman condition). For this purpose let $\xi(t) = \|x(t)\|^p$ and let $t_0 = [0,n]$ be the point where ξ attains its maximum on $[0,n]$. Suppose that $M^p < \xi(t_0)$. First assume that $t_0\in(0,n)$. Then $0=\xi'(t_0) = p\|x(t_0)\|^{p-2}(x'(t_0),x(t_0))_{\mathbb{R}^{\mathbb{N}}}$, hence $(x'(t_0),x(t_0))_{\mathbb{R}^{\mathbb{N}}}=0$. By virtue of hypothesis H(F)(iv), there exist $\delta>0$ and $\gamma>0$ such that

$$\inf\bigl\{(-u,x)_{\mathbb{R}^{\mathbb{N}}}+\|y\|^p : \|x-x(t_0)\|+\|y-x'(t_0)\|<\delta,\ u\in F(t,x,y)\bigr\}\geq\gamma>0. \tag{3.8}$$

Since $x\in C^1([0,n],\mathbb{R}^{\mathbb{N}})$, given $\delta>0$ as in (3.8), we can find $\delta_1>0$ such that if $t\in(t_0,t_0+\delta_1]\subseteq[0,n]$, we have $\|x(t)-x(t_0)\|+\|x'(t)-x'(t_0)\|<\delta$. Since $u(t)\in F(t,x(t),x'(t))$ a.e on $[0,n]$, we have

$$(-u(t),x(t))_{\mathbb{R}^{\mathbb{N}}} + \|x'(t)\|^p \geq \gamma > 0 \quad \text{for almost all } t\in(t_0,t_0+\delta_1],$$

therefore, $((\|x'(t)\|^{p-2}x'(t))',x(t))_{\mathbb{R}^{\mathbb{N}}} + \lambda\|x'(t)\|^p \geq \lambda\gamma > 0$ for almost all $t\in(t_0,t_0+\delta_1]$ (see (3.7) and recall $\lambda>0$), hence

$$\int_{t_0}^t ((\|x'(s)\|^{p-2}x'(s))',x(s))_{\mathbb{R}^{\mathbb{N}}} + \lambda\int_{t_0}^t \|x'(s)\|^p ds \geq \lambda\gamma(t-t_0)$$

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for all $t \in (t_0, t_0 + \delta_1]$. Performing an integration by parts on the first integral of the left-hand side of the above inequality and recalling that $(x'(t_0), x(t_0))_{\mathbb{R}^N} = 0$, we obtain

$$\|x'(t)\|^{p-2}(x'(t), x(t))_{\mathbb{R}^N} + (\lambda - 1)\int_{t_0}^{t} \|x'(s)\|^p ds \geq \lambda\gamma(t - t_0) > 0,$$

therefore $\|x'(t)\|^{p-2}(x'(t), x(t))_{\mathbb{R}^N} > 0$ for all $t \in (t_0, t_0 + \delta_1]$ since $0 < \lambda \leq 1$), hence $\xi'(t) > 0$ for all $t \in (t_0, t_0 + \delta_1]$, which contradicts the choice of $t_0 \in [0, n]$. Therefore we have $\|x(t)\| \leq M$ for all $t \in [0, n]$.

If $t_0 = 0$, then $x \equiv 0$ (since $x(0) = 0$). Finally if $t_0 = n$, then since $x'(n) = 0$ (see (3.6)) we have that $\xi'(t_0) = 0$ and so the above argument applies. Therefore in all three cases, we have $\|x(t)\| \leq M$ for all $t \in [0, n]$ and all $n \geq 1$. Next because of (3.7) and hypothesis H(F)(v) (the Bernstein–Nagumo–Wintner growth condition), we have

$$\begin{aligned} \frac{d}{dt}(\|x'(t)\|^{p-1}) &= \frac{d}{dt}\big(\big\| \|x'(t)\|^{p-2}x'(t)\big\|\big) \\ &\leq \big\| \big(\|x'(t)\|^{p-2}x'(t)\big)'\big\| \\ &= \lambda\|u(t)\| \qquad \text{(see (3.7))} \\ &\leq \|u(t)\| \qquad (\text{since } 0 < \lambda \leq 1) \\ &\leq \eta(t)\psi(\|x'(t)\|^{p-1}) \qquad \text{a.e. on } [0, n]. \end{aligned} \tag{3.9}$$

Set $z(t) = \int_0^t \eta(s)\psi\left(\|x'(s)\|^{p-1}\right) ds + \|x'(0)\|^{p-1}$, $t \in [0, n]$. Evidently $z\colon [0, n] \to \mathbb{R}^N$ is absolutely continuous and we have

$$z'(t) = \eta(t)\psi\left(\|x'(t)\|^{p-1}\right) \quad \text{a.e. on } [0, n]. \tag{3.10}$$

Moreover, by virtue of (3.9), we have

$$\|x'(t)\|^{p-1} \leq z(t) \quad \text{for almost all } t \in [0, n]. \tag{3.11}$$

Since by hypothesis H(F)(v), ψ is nondecreasing, from (3.10) and (3.11) we have $z'(t) \leq \eta(t)\psi(z(t))$ a.e. on $[0, n]$. Therefore, $\frac{z'(t)}{\psi(z(t))} \leq \eta(t)$ a.e. on $[0, n]$. Hence,

$$\int_0^t \frac{z'(t)}{\psi(z(s))}\, ds \leq \|\eta\|_{L^1(\mathbb{R}_+)} \quad \text{for all } t \in [0, n], n \geq 1.$$

Then by a change of variables, we have

$$\int_0^{z(t)} \frac{ds}{\psi(s)} \leq \|\eta\|_{L^1(\mathbb{R}_+)} \quad \text{for all } t \in [0, n], n \geq 1.$$

By virtue of hypothesis H(F)(v), the above inequality implies that there is an $M_1 \geq M$ such that $\|x'(t)\| \leq M_1$ for all $t \in [0, n]$, all $n \geq 1$. Therefore if $R > M_1$, then we have that $x \notin H(\lambda, x)$ for all $\lambda \in [0, 1]$ and all

$$x \in \partial B_R(0) = \{x \in C^1([0, n], \mathbb{R}^N) : \|x\|_{C^1([0,n],\mathbb{R}^N)} = R\}$$

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for all $n \geq 1$. From the homotopy invariance and the normalization properties of the fixed point index of Bader [2], we have

$$i_{C^1([0,n],\mathbb{R}^N)}(V_n \circ N_F, B_R(0)) = 1 \quad \text{for all} \quad n \geq 1. \tag{3.12}$$

Here $B_R(0) = \{x \in C^1([0,n],\mathbb{R}^N) : \|x\|_{C^1([0,n],\mathbb{R}^N)} < R\}$. From (3.12) and the existence property of the fixed point index, we see that we can find $x_n \in B_{\mathbb{R}}$ such that $x_n \in (V_n \circ N_F)(x_n)$. Then $A(x_n) = u$ with $u \in N_F(x_n)$, and $x_n \in C^1([0,n],\mathbb{R}^N)$ is a solution of problem (3.6). ■

Now using a diagonal argument, we will establish the existence of a solution for the original infinite time horizon problem (1.1)

Theorem 3.7 *If hypotheses* H(F) *hold, then* (1.1) *has a solution* $x \in C^1(\mathbb{R}_+,\mathbb{R}^N)$.

Proof From Proposition 3.6, we know that for every $n \geq 1$, problem (3.6) has a solution $x_n \in C^1([0,n],\mathbb{R}^N)$. We set

$$\widehat{x}_n(t) = \begin{cases} x_n(t) & \text{if } t \in [0,n], \\ x_n(n) & \text{if } t \in [n,+\infty), \end{cases} \quad n \geq 1.$$

Since $x_n'(n) = 0$, we see that $x_n \in C^1(\mathbb{R}_+,\mathbb{R}^N)$. Moreover, from Proposition 3.6 we know that $\|\widehat{x}_n(t)\|$, $\|\widehat{x}_n'(t)\| \leq M_1$ for all $t \in \mathbb{R}_+$ and all $n \geq 1$. Then by virtue of hypothesis H(F)(iii), we have

$$\left\| \left(\|\widehat{x}_n'(t)^{p-1}\widehat{x}_n'(t) \right)' \right\| \leq \alpha_{M_1}(t) \quad \text{a.e. on } [0,b] \text{ for all } b > 0,$$

Therefore,

$$\int_s^t \left\| \left(\|\widehat{x}_n'(\tau)\|^{p-2}\widehat{x}_n'(\tau) \right) \right\| d\tau \leq \int_s^t \alpha_{M_1}(\tau)\, d\tau \quad \text{for all } t,s \in [0,b], s \leq t, b > 0;$$

$$\left\| \int_s^t (\|\widehat{x}_n'(\tau)\|^{p-2}\widehat{x}_n'(\tau))'\, d\tau \right\| \leq \int_s^t \alpha_{M_1}(\tau)\, d\tau \quad \text{for all } t,s \in [0,b], s \leq t, b > 0;$$

$$\left\| \|\widehat{x}_n'(t)\|^{p-2}\widehat{x}_n'(t) - \|\widehat{x}_n'(s)\|^{p-2}\widehat{x}_n'(s) \right\| \leq \int_s^t \alpha_{M_1}(\tau)\, d\tau$$
$$\text{for all } t,s \in [0,b], s \leq t, b > 0;$$

$$\{\|\widehat{x}_n'(\cdot)\|^{p-2}\widehat{x}_n'(\cdot)\}_{n\geq 1} \subseteq C([0,b],\mathbb{R}^N)$$
$$\text{is equicontinuous and bounded for all } b > 0;$$

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$\{\|\widehat{x}_n'(\cdot)\|^{p-2}\widehat{x}_n'(\cdot)\}_{n\geq 1} \subseteq C([0,b],\mathbb{R}^{\mathbb{N}})$

is relatively compact for all $b > 0$ (Arzela–Ascoli theorem);

$$\{\widehat{x}_n'\}_{n\geq 1} \subseteq C([0,b],\mathbb{R}^{\mathbb{N}}) \quad \text{is relatively compact for all } b > 0.$$

Therefore we can find S_1 a subsequence of $\mathbb{N}$ and $w_1 \in C^1([0,b],\mathbb{R}^{\mathbb{N}})$ such that

$$\widehat{x}_n \to w_1 \quad \text{in } C^1([0,1],\mathbb{R}^{\mathbb{N}}),\, n \in S_1.$$

Let $\widehat{S}_1 = S_1\backslash\{1\}$. Then we can find a subsequence S_2 of $\widehat{S}_1$ and $w_2 \in C^1([0,2],\mathbb{R}^{\mathbb{N}})$ such that $\widehat{x}_n \to w_2$ in $C^1([0,2],\mathbb{R}^{\mathbb{N}})$, $n \in S_2$. Evidently $w_2|_{[0,1]} = w_1$. Inductively we generate a subsequence S_{k+1} of $\widehat{S}_k = S_k\backslash\{k\}$ and a $w_{k+1} \in C^1([0,k],\mathbb{R}^{\mathbb{N}})$ such that $\widehat{x}_n \to w_{k+1}$ in $C^1([0,k],\mathbb{R}^{\mathbb{N}})$, $n \in S_{k+1}$ and $w_{k+1}|_{[0,k]} = w_k$ for all $k \geq 1$. Now let $\widehat{x}(t) = w_k(t)$ for all $t \in [0,k]$. Clearly this function is well defined and belongs in $C^1(\mathbb{R}_+,\mathbb{R}^{\mathbb{N}})$. Also if $k \geq 1$, then for each $n \in \widehat{S}_k = S_k\backslash\{k\}$, we have

$$\begin{aligned} &-\left(\|\widehat{x}_n'(t)\|^{p-2}\widehat{x}_n'(t)\right)' = u_n(t) \text{ a.e. on } [0,k],\\ &\text{with } u_n \in N_F(\widehat{x}_n) \text{ on } [0,k],\ \widehat{x}_n(0) = 0,\ \widehat{x}_n'(k) = 0. \end{aligned} \tag{3.13}$$

Because of hypothesis H(F)(iii) and the Dunford–Pettis theorem, we may assume that $u_n \xrightarrow{w} v_k$ in $L^1([0,k],\mathbb{R}^{\mathbb{N}})$ as $n \to \infty$ for all $k \geq 1$. Because of Proposition 3.5, we have that $v_k \in N_F(w_k)$ for all $k \geq 1$. So from (3.13) in the limit as $n \to \infty$, we obtain

$$-\left(\|w_k'(t)\|^{p-2}w_k(t)\right)' = v_k(t) \text{ a.e. on } [0,k],$$

$$v_k \in N_F(w_k),\ v_k(0) = 0,\ w_k'(k) = 0, \quad k \geq 1.$$

If we set $\widehat{v}(t) = v_k(t)$ for all $t \in [0,k]$, then $\widehat{v}$ is well-defined, $\widehat{v} \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^N)$, $\widehat{v}(t) \in F\left(t,\widehat{x}(t),\widehat{x}'(t)\right)$ a.e. on $\mathbb{R}_+$ and

$$\begin{aligned} &-(\|\widehat{x}'(t)\|^{p-2}\widehat{x}'(t))' = \widehat{v}(t) \text{ a.e. on } \mathbb{R}_+\\ &\widehat{x}(0) = 0,\ \widehat{x} \text{ is bounded on } \mathbb{R}_+, \end{aligned} \quad .$$

Therefore $\widehat{x} \in C^1(\mathbb{R}_+,\mathbb{R}^{\mathbb{N}})$ is a solution of (1.1). ∎

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