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SOME TRIANGLE INEQUALITIES AND GENERALIZATIONS

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ABSTRACT. Let $D_{n,s}(x) = \Pi(-sa_1^x + a_2^x + \cdots + a_n^x)$, where a_i, s, and x are real, and Π denotes the product over cyclic rearrangements of the subscripts. We show that, in five special cases, $D_{n,s}(x)D_{n,s}(y)$ is greater than a fixed multiple of $D_{n,s}(x+y)$.

Introduction and results. A plane triangle ABC has an incircle of radius r and a circumcircle of radius R. By a known trigonometric formula [5, p. 200],

$$2r^2 - 4R^2 \cos A \cos B \cos C = IP^2 \geq 0.$$

In terms of the sides of the triangle, we obtain

$$(1)\quad (-a+b+c)^2(a-b+c)^2(a+b-c)^2 \geq (-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2).$$

The functions appearing here are special cases of the general function

$$D_{n,s}(x) = \Pi(-sa_1^x + a_2^x + a_3^x + \cdots + a_n^x),$$

where s and x are real and Π denotes the product over the n cyclic rearrangements of the subscripts. Although (1) holds for all real a, b, and c, our theorems 1, 2, 3, and 5 are valid only for positive values of $a_1, a_2, \ldots, a_n$. The inequalities connecting $D_{n,s}(x)D_{n,s}(y)$ and $D_{n,s}(x+y)$ seem not to appear in standard treatises on inequalities [1, 4, 7], but we shall mention a slight connection between the work of Gårding [3] and the proof of

THEOREM 1. *If $n > 1$ and $s \leq 0$, then $D_{n,s}(x)D_{n,s}(y) > (-s)D_{n,s}(x+y)$.*

The inequalities for positive values of s are more interesting and more difficult.

THEOREM 2.

$$(2)\qquad [D_{4,1}(1)]^2 > 3D_{4,1}(2).$$

We remark that (2) *becomes an equality if we set $a_1 = a_2 = a_3$ and take the limit as $a_4 \to 0$. If we let $a_4 \to 0$ and then use the elementary inequality $3(a^2+b^2+c^2) \geq (a+b+c)^2$, we recover* (1).

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Jensen's inequality [6] suggests the following generalizations:

THEOREM 3. *If x and y are positive, then*

$$D_{3,1}(x)D_{3,1}(y) \geq D_{3,1}(x+y), \tag{3}$$

with equality iff $a_1 = a_2 = a_3$.

THEOREM 4. *If* a_1, a_2, s, x, and y are positive, or if $s>0$ and x and y are positive integers,

$$D_{2,s}(x)D_{2,s}(y) \geq (1-s)^2 D_{2,s}(x+y), \tag{4}$$

with equality only if $|a_1| = |a_2|$.

Finally, we shall prove

THEOREM 5. *Suppose* a_1^{x+y}, a_2^{x+y}, a_3^{x+y} *are the sides of a triangle. If x and y are positive and* $s \leq 1$, *then*

$$D_{3,s}(x)D_{3,s}(y) > 2(1-s)^2 D_{3,s}(x+y). \tag{5}$$

Proof of Theorem 1. The proof is trivial if $s=0$. If $n>1$ and $s<0$, we have $D_{n,s}(x)D_{n,s}(y) > \Pi(s^2 a_1^{x+y} + a_2^{x+y} + \cdots + a_n^{x+y})$. The quantity on the right is greater than $(-s)D_{n,s}(x+y)$, except that it is equal to $D_{n,s}(x+y)$ when $s=-1$. To prove this, we need

LEMMA 1. *If* $A_1, A_2, \ldots, A_n$ *are positive,* $n>1$, $s<0$, *and* $s \neq -1$, *then*

$$\Pi(s^2 A_1 + A_2 + \cdots + A_n) + s\Pi(-sA_1 + A_2 + \cdots + A_n) \tag{6}$$

is positive.

The proof can be done by induction on n or by use of Cauchy's inequality followed by Hölder's inequality. We plan to give full details and extensions of Lemma 1 in a separate publication. We conjecture that the polynomial (6) is hyperbolic in the sense of Gårding [3].

Proof of Theorem 2. If the numbers $a_1 \cdots a_4$ are not all distinct, the inequality is easily proved. Let $a_1 = a_2$. Then

$$D_{4,1}(1) = 4a_2^2(a_3+a_4)^2 - (a_3^2 - a_4^2)^2$$

and

$$D_{4,1}(2) = 4a_2^4(a_3^2+a_4^2)^2 - (a_3^2-a_4^2)^2(a_3^2+a_4^2)^2.$$

Hence,

$$12a_2^4(a_3+a_4)^4 - 3(a_3^2-a_4^2)^4 > 3D_{4,1}(2).$$

Finally,

$$[D_{4,1}(1)]^2 - 12a_2^4(a_3+a_4)^4 + 3(a_3^2-a_4^2)^4 = [2a_2^2(a_3+a_4)^2 - 2(a_3^2-a_4^2)^2]^2 \geq 0.$$

In the general case, we can write $D_{4,1}(1)$ and $D_{4,1}(2)$ in terms of the elementary symmetric functions $\sigma_1 \cdots \sigma_4$. They are defined by

$$\Pi(x-a_i) \equiv x^4 - \sigma_1 x^3 + \sigma_2 x^2 - \sigma_3 x + \sigma_4. \tag{7}$$

Some computations give

$$D_{4,1}(1) = -\sigma_1^4 + 4\sigma_1^2\sigma_2 - 8\sigma_1\sigma_3 + 16\sigma_4,$$
$$D_{4,1}(2) = -(\sigma_1^2 - 2\sigma_2)^4 + 4(\sigma_1^2 - 2\sigma_2)^2(\sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4)$$
$$-8(\sigma_1^2 - 2\sigma_2)(\sigma_3^2 - 2\sigma_2\sigma_4) + 16\sigma_4^2,$$

and

$$[D_{4,1}(1)]^2 - 3D_{4,1}(2) = P(\sigma_1, \sigma_2, \sigma_3) - 8(7\sigma_1^4 - 22\sigma_1^2\sigma_2 + 32\sigma_1\sigma_3 - 26\sigma_4)\sigma_4, \tag{8}$$

where $P(\sigma_1, \sigma_2, \sigma_3)$ is a polynomial. We shall show that (8) is a decreasing function of σ_4, when $\sigma_1, \sigma_2, \sigma_3$ are fixed. Then, if (8) is positive at the largest allowed value of σ_4, it is always positive. If (7) has distinct real zeroes, its value at each local minimum must be negative. We may increase σ_4 until the value at one of the local minima is zero; then two zeroes coincide and we have the case first considered. Thus, (8) is always positive.

To show that (8) is a decreasing function of σ_4, we need inequalities relating the symmetric functions. We shall use the method of Breusch [2] to show that

$$13\sigma_1^3 - 44\sigma_1\sigma_2 + 64\sigma_3 > 0. \tag{9}$$

By the arithmetic-geometric inequality, $\sigma_1^4 \geq 4^4\sigma_4 > 104\sigma_4$. Combining this with (9) gives $7\sigma_1^4 - 22\sigma_1^2\sigma_2 + 32\sigma_1\sigma_3 - 52\sigma_4 > 0$, showing that (8) is a decreasing function of σ_4.

Our last step is to prove (9). Since the left side is a homogeneous polynomial, we can set $\sigma_1 = 1$ without loss of generality. Since σ_3 is positive, (9) holds when $\sigma_2 < \frac{1}{4}$. Hence, we assume $\sigma_2 \geq \frac{1}{4}$. The polynomial (7) has four real zeroes. Its derivative,

$$4[(x-\tfrac{1}{4})^3 + (\tfrac{1}{2}\sigma_2 - \tfrac{3}{16})(x-\tfrac{1}{4}) + (\tfrac{1}{8}\sigma_2 - \tfrac{1}{4}\sigma_3 - \tfrac{1}{32})],$$

must have three real zeroes. We compute the discriminant of this cubic polynomial, as suggested by Breusch [2], obtaining

$$32(\sigma_2 - \tfrac{3}{8})^3 + 27(\sigma_2 - 2\sigma_3 - \tfrac{1}{4})^2 \leq 0. \tag{10}$$

But the quantity on the left is positive, unless $\sigma_2 \leq \frac{3}{8}$. This bound for σ_2 could be obtained more easily from Maclaurin's inequality [1, 4]. From (10), we have

$$13 - 44\sigma_2 + 64\sigma_3 \geq 5 - 12\sigma_2 - 8(1 - \tfrac{8}{3}\sigma_2)^{3/2}.$$

We now minimize the function on the right side, using the bounds for σ_2. Since this function has a negative second derivative at interior points of the interval $[\frac{1}{4}, \frac{3}{8}]$, the minimum is at one of the end points. The values at both end points are positive, which completes the proof.

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Proof of Theorem 3. We shall prove (3) and

$$D_{3,1}(x)D_{3,1}(y)\le\left[D_{3,1}\left(\frac{x+y}{2}\right)\right]^2 \tag{11}$$

by use of Jensen's inequality [6]. Most of the relevant properties of $D_{3,1}(x)$ are stated in two lemmas.

LEMMA 2. *If* $D_{3,1}(x)\neq 0$, *then*

$$\frac{d^2}{dx^2}\log|D_{3,1}(x)|\le 0, \tag{12}$$

with equality iff $a_1=a_2=a_3$.

Proof. We may assume $a_1\ge a_2\ge a_3$ without loss of generality. Since $D_{3,1}(x)\neq 0$, no vanishing denominators appear in

$$\begin{aligned}\frac{d^2}{dx^2}\log|D_{3,1}(x)|=&\frac{-a_1^xa_2^x\left(\log\frac{a_1}{a_2}\right)^2+a_2^xa_3^x\left(\log\frac{a_2}{a_3}\right)^2-a_3^xa_1^x\left(\log\frac{a_3}{a_1}\right)^2}{(-a_1^x+a_2^x+a_3^x)^2}\\ &+\frac{-a_1^xa_2^x\left(\log\frac{a_1}{a_2}\right)^2-a_2^xa_3^x\left(\log\frac{a_2}{a_3}\right)^2+a_3^xa_1^x\left(\log\frac{a_3}{a_1}\right)^2}{(-a_1^x+a_2^x+a_3^x)^2}\\ &+\frac{a_1^xa_2^x\left(\log\frac{a_1}{a_2}\right)^2-a_2^xa_3^x\left(\log\frac{a_2}{a_3}\right)^2-a_3^xa_1^x\left(\log\frac{a_3}{a_1}\right)^2}{(a_1^x+a_2^x-a_3^x)^2}\end{aligned} \tag{13}$$

The first numerator is never positive, and $(-a_1^x+a_2^x+a_3^x)^{-2}\ge(a_1^x-a_2^x+a_3^x)^{-2}$. Hence, $-2a_1^xa_2^x\left(\log\frac{a_1}{a_2}\right)^2(a_1^x-a_2^x+a_3^x)^{-2}$ is an upper bound for the first two terms on the right side of (13). Further estimation gives

$$\frac{d^2}{dx^2}\log|D_{3,1}(x)|\le\frac{-a_1^xa_2^x\left(\log\frac{a_1}{a_2}\right)^2-a_2^xa_3^x\left(\log\frac{a_2}{a_3}\right)^2-a_3^xa_1^x\left(\log\frac{a_3}{a_1}\right)^2}{(a_1^x+a_2^x-a_3^x)^2},$$

which is negative unless $a_1=a_2=a_3$. We note that (12) is also valid when $x\le 0$.

The uniqueness of the positive real zero of $D_{3,1}(x)$ is essential for the proof of Theorem 3. This uniqueness follows from

LEMMA 3. $D_{3,1}(x)/(a_1a_2a_3)^x$ *is a monotonic decreasing function of x, unless* $a_1=a_2=a_3$.

Proof. If x is less than the first positive zero of $D_{3,1}(x)$, we note that $\frac{d}{dx}\log\frac{D_{3,1}(x)}{(a_1a_2a_3)^x}$ vanishes at $x=0$, and use Lemma 2 to show that $\log\frac{D_{3,1}(x)}{(a_1a_2a_3)^x}$

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is decreasing, unless $a_1=a_2=a_3$. If $D_{3,1}(x)\le 0$, we may assume $a_1>a_2\ge a_3$ without loss of generality. Then

$$-\frac{D_{3,1}(x)}{(a_1a_2a_3)^x}=\left[1-\left(\frac{a_2}{a_1}\right)^x-\left(\frac{a_3}{a_1}\right)^x\right]\left[\left(\frac{a_1}{a_2}\right)^x+1-\left(\frac{a_3}{a_2}\right)^x\right]\left[\left(\frac{a_1}{a_3}\right)^x-\left(\frac{a_2}{a_3}\right)^x+1\right]$$

is the product of three increasing functions; the first factor is non-negative and the other two are positive.

We can now prove (11). If $D_{3,1}\left(\frac{x+y}{2}\right)=0$ and $x\neq y$, then Lemma 3 implies $D_{3,1}(x)D_{3,1}(y)<0$. If $D_{3,1}\left(\frac{x+y}{2}\right)\neq 0$, then either $D_{3,1}(x)$ or $D_{3,1}(y)$ has the same sign as $D_{3,1}\left(\frac{x+y}{2}\right)$. If all three of them have the same sign, we obtain (11) from Lemma 2 and Jensen's inequality; if not, (11) still holds.

Theorem 3 is easily verified if $D_{3,1}(x)D_{3,1}(y)D_{3,1}(x+y)=0$. If $D_{3,1}(x+y)>0$, then $D_{3,1}(x)$ and $D_{3,1}(y)$ are positive. Using Lemma 2 and Jensen's inequality, we find

$$\log D_{3,1}(x)\ge\frac{x}{x+y}\log D_{3,1}(x+y)+\frac{y}{x+y}\log D_{3,1}(0)=\frac{x}{x+y}\log D_{3,1}(x+y) \tag{14}$$

and, similarly,

$$\log D_{3,1}(y)\ge\frac{y}{x+y}\log D_{3,1}(x+y). \tag{15}$$

In both cases, equality holds iff $a_1=a_2=a_3$. Addition of (14) and (15) gives (3). If $D_{3,1}(x+y)<0$, the proof is trivial unless $D_{3,1}(x)$ and $D_{3,1}(y)$ have opposite signs. we assume $D_{3,1}(x)<0<D_{3,1}(y)$, and use Lemma 3 to show

$$\frac{|D_{3,1}(x)|}{(a_1a_2a_3)^x}<\frac{|D_{3,1}(x+y)|}{(a_1a_2a_3)^{x+y}}$$

and

$$\frac{D_{3,1}(y)}{(a_1a_2a_3)^y}<D_{3,1}(0)=1.$$

Therefore, $|D_{3,1}(x)|\,D_{3,1}(y)<|D_{3,1}(x+y)|$, which completes the proof.

Proof of Theorem 4. If a_1 and a_2 are positive, then $D_{2,s}(0)=(1-s)^2$,

$$\frac{D_{2,s}(x)}{(a_1a_2)^x}=1+s^2-s\left[\left(\frac{a_1}{a_2}\right)^x+\left(\frac{a_2}{a_1}\right)^x\right]$$

is a non-increasing function of x, and the proof of (4) is similar to that of (3). If $a_1a_2\ge 0$, or if $a_1+a_2=0$, (4) is easily verified. Thus, we assume $a_1a_2<0$ and $a_1+a_2\neq 0$. If x is an odd integer, then $D_{2,s}(x)<0$ and

$$\frac{|D_{2,s}(x)|}{|a_1a_2|^x}=1+s^2+s\left(\left|\frac{a_1}{a_2}\right|^x+\left|\frac{a_2}{a_1}\right|^x\right)$$

is an increasing function of x. If y is even, this gives

$$\frac{|D_{2,s}(x)|}{|a_1a_2|^x}<\frac{|D_{2,s}(x+y)|}{|a_1a_2|^{x+y}},$$

and we also have

$$\frac{D_{2,s}(y)}{(a_1a_2)^y}<D_{2,s}(0)=(1-s)^2.$$

Hence, $|D_{2,s}(x)|\,D_{2,s}(y)<(1-s)^2\,|D_{2,s}(x+y)|$. In this way, (4) is proved when x and y are integers of opposite parity. If x and y are both odd, $D_{2,s}(x+y)$ is unchanged when we replace a_1 and a_2 by their absolute values, while $D_{2,s}(x)D_{2,s}(y)$ decreases, so (4) must hold. Finally, (4) holds when x and y are both even.

Proof of Theorem 5. Since a_1^{x+y}, a_2^{x+y}, a_3^{x+y} form a triangle, $D_{3,1}(x+y)$ is positive; but (5) remains valid if it vanishes. To prove this, we shall use

$$0\le D_{3,1}(x+y)\le(a_1a_2a_3)^{x+y}. \tag{16}$$

To prove the second half of this inequality, let $x+y=1$; then

$$2a_1a_2a_3-2D_{3,1}(1)=(-a_1+a_2+a_3)(a_2-a_3)^2+(a_1-a_2+a_3)(a_1-a_3)^2 \\ +(a_1+a_2-a_3)(a_1-a_2)^2$$

is non-negative.

The inequality published by Jensen [6] and Pringsheim [8], and attributed to Lüroth, is useful. It implies that both triples (a_1^x, a_2^x, a_3^x) and (a_1^y, a_2^y, a_3^y) satisfy triangle inequalities. Hence,

$$D_{3,1}(x)>0 \quad\text{and}\quad D_{3,1}(y)>0, \tag{17}$$

and (5) holds when $s=1$.

If $s\le 0$, we note that the sign of the inequality in Lemma 2 is reversed. Jensen's inequality gives $D_{3,s}(x)D_{3,s}(y)\ge\left[D_{3,s}\left(\frac{x+y}{2}\right)\right]^2$. To prove (5), we shall show that

$$\left[D_{3,s}\left(\frac{x+y}{2}\right)\right]^2>2(1-s)^2D_{3,s}(x+y).$$

As in the previous paragraph, we have $D_{3,s}\left(\frac{x+y}{2}\right)>0$. We set $x+y=2$, without loss of generality. Then

$$D_{3,s}(1)=-s\sigma_1^3+(s+1)^2\sigma_1\sigma_2-(s+1)^3\sigma_3$$

and

$$D_{3,s}(2)=-s(\sigma_1^2-2\sigma_2)^3+(s+1)^2(\sigma_1^2\sigma_2^2-2\sigma_2^3\sigma_3+4\sigma_1\sigma_2\sigma_3)-(s+1)^3\sigma_3^2,$$

where σ_1, σ_2, σ_3 are the elementary symmetric functions of a_1, a_2, a_3. We shall use $s\le 0$ and

$$D_{3,1}(1)=4\sigma_1\sigma_2-\sigma_1^3-8\sigma_3>0 \tag{18}$$

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to show that

$$[D_{3,s}(1)]^2-2(1-s)^2D_{3,s}(2)$$
$$=[(s-1)^4\sigma_2^2+s^2\sigma_1^2\sigma_2-s(2s^2-3s+2)\sigma_1^2(\sigma_1^2-3\sigma_2)](4\sigma_2-\sigma_1^2)$$
$$+2(s+1)^2(3s^2-3s+2)(\sigma_1^2-3\sigma_2)\sigma_1\sigma_3$$
$$+2(s+1)^2(-s^3+2s^2-4s+1)D_{3,0}(1)\sigma_3$$
$$+(s+1)^2(s^4+4s^3+8s^2-6s+5)\sigma_3^2 \tag{19}$$

is positive. From (18), $4\sigma_2-\sigma_1^2$ must be positive. Also, $\sigma_1^2-3\sigma_2$ is non-negative and $D_{3,0}(1)$ is positive. Therefore, (19) is always positive and (5) holds when $s\le 0$. In particular,

$$D_{3,0}(x)D_{3,0}(y)>2D_{3,0}(x+y). \tag{20}$$

Also, $\sigma_1\sigma_2\ge 9\sigma_3$ gives

$$D_{3,0}(1)=\sigma_1\sigma_2-\sigma_3\ge 8\sigma_3,$$

which can be written in the useful form

$$D_{3,0}(x)\ge 8(a_1a_2a_3)^x \quad\text{and}\quad D_{3,0}(y)\ge 8(a_1a_2a_3)^y. \tag{21}$$

We now use

$$D_{3,s}(x)=sD_{3,1}(x)+(s-1)^2D_{3,0}(x)-s(s-1)(s+3)(a_1a_2a_3)^x$$

to generate a lengthy expression for

$$\delta=D_{3,s}(x)D_{3,s}(y)-2(1-s)^2D_{3,s}(x+y).$$

If $0<s<1$, (3) and (20) give

$$\delta>s(2s-1)(2-s)D_{3,1}(x+y)+s(s-1)^2[D_{3,1}(x)D_{3,0}(y)+D_{3,0}(x)D_{3,1}(y)]$$
$$+s^2(1-s)(s+3)[D_{3,1}(x)(a_1a_2a_3)^y+D_{3,1}(y)(a_1a_2a_3)^x]$$
$$+s(1-s)^3(s+3)[D_{3,0}(x)(a_1a_2a_3)^y+D_{3,0}(y)(a_1a_2a_3)^x]$$
$$+s(s-1)^2(s+3)(s^2+5s-2)(a_1a_2a_3)^{x+y}.$$

Then (17) and (21) give

$$\delta>s(2s-1)(2-s)D_{3,1}(x+y)+s(s-1)^2(s+3)(s^2-11s+14)(a_1a_2a_3)^{x+y}.$$

Finally, (16) is used to show that δ is positive when $0<s<1$.

If $s>1$, and $s\ne 2$ then $D_{3,s}(x)$ can change sign more than once as x increases from 0 to $+\infty$. If $s>1$, (5) cannot hold for all positive x and y. But a generalization from (4) and (5) to larger n appears possible. Suppose a_1^{x+y}, a_2^{x+y}, a_3^{x+y}, a_4^{x+y} are the sides of a quadrangle. If x and y are positive and $s\le 1$, we conjecture that

$$D_{4,s}(x)D_{4,s}(y)>4(1-s)^2D_{4,s}(x+y).$$

FINAL REMARK. The reader may have some difficulty to see that Theorems 1, 2, 3, and 5, are valid when a_1 is negative. We have the following cases in which the inequalities are reversed. In Theorem 1, $a_1=-2$, $a_2=1$, $n=2$, $s=-\frac{1}{4}$, $x=1$ and $y=3$. In Theorem 2, $a_1=-1$, $a_2=a_3=a_4=1$. In Theorem 3, $a_1<0$, $x=1$, $y=2$; then let $a_2\to 0$ and $a_3\to 0$. In Theorem 5, $a_1=-1$, $a_2=a_3=1$, $s=0$, $x=y=1$.

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