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Approximating the Riemann Zeta-function by Polynomials with Restricted Zeros

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Abstract. We approximate the Riemann Zeta-Function by polynomials and Dirichlet polynomials with restricted zeros.

The Riemann zeta-function ζ has zeros at the negative even integers (the so-called *trivial zeros*). The Riemann Hypothesis states that the remaining zeros (the *non-trivial zeros* all lie on the critical line $S = \{z : \Re z = 1/2\}$. A refinement of the Riemann Hypothesis claims that, moreover, the zeros are simple.

We wish to approximate ζ by sequences of polynomials whose zeros have these properties on larger and larger sets. Since Euler originally defined the zeta-function by the Dirichlet series

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad 1 < x < +\infty,$$

it seems natural to approximate ζ, not only by "ordinary" polynomials

$$P(z) = \sum_{n=0}^{m} a_n z^n,$$

but also by *Dirichlet* polynomials

$$D(z) = \sum_{n=1}^{m} \frac{a_n}{n^z}.$$

To clearly distinguish between Dirichlet polynomials and ordinary polynomials, we shall sometimes refer to the latter as *algebraic* polynomials.

While the theory of approximation by algebraic polynomials is a well developed classical subject, that of approximation by Dirichlet polynomials has received less attention. Recently [1, Lemma 4.1], the two theories were shown to be equivalent.

Theorem 1 *There exists an increasing sequence K_n of compact subsets of $\mathbb{C}$ whose union is $\mathbb{C}$, a sequence P_n of algebraic polynomials, and a sequence D_n of Dirichlet polynomials, with the following properties:*

$$\max\left\{ |P_n(z) - \zeta(z)|, |D_n(z) - \zeta(z)| \right\} \le 1/n, \quad \textit{for } z \in K_n \setminus \{1\}.$$
$$P_n(1) = D_n(1) = n.$$

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On $K_n \cap (\mathbb{R} \cup S)$,

$$P_n \text{ and } D_n \text{ have only simple zeros and they are the } \zeta\text{-zeros.}$$

On $K_n \setminus (\mathbb{R} \cup S)$,

$$P_n \text{ and } D_n \text{ have no zeros,}$$
$$P_n(\mathbb{R}) \subset \mathbb{R}, \quad D_n(\mathbb{R}) \subset \mathbb{R}.$$

It follows that $P_n \to \zeta$ and $D_n \to \zeta$ pointwise on all of $\mathbb{C}$, and, for each fixed m, the convergence is uniform with respect to the Euclidean distance on $K_m \setminus \{1\}$ and uniform with respect to the spherical distance on K_m.

Let K be a compact subset of $\mathbb{C}$. As usual, $A(K)$ denotes the space of functions continuous on K and holomorphic on the interior K^0, endowed with the sup-norm, and $P(K)$ denotes the uniform closure in $A(K)$ of the set of algebraic polynomials. Similarly, $D(K)$ will denote the closure in $A(K)$ of the set of Dirichlet polynomials.

Lemma 2 *For a compact set* $K \subset \mathbb{C}$ *the following are equivalent:*

(i) $\mathbb{C} \setminus K$ *is connected;*
(ii) $P(K) = A(K)$;
(iii) $D(K) = A(K)$.

The equivalence of (i) and (ii) is Mergelyan's Theorem, the most important theorem in polynomial approximation. The equivalence of (ii) and (iii) is a very recent result due to Aron et al. [1, Lemma 4.1].

For $A \subset \mathbb{C}$, we denote by A^* the set $\{\overline{z} : z \in A\}$, and we say that A is *real-symmetric* if $A = A^*$. We say that a function $f : A \to \mathbb{C}$ on a real-symmetric set A is real-symmetric if $f(z) = \overline{f(\overline{z})}$. For a class X of functions on a real-symmetric set A, we denote by $X_{\mathbb{R}}$ the class of functions in X that are real-symmetric. If X is a complex vector space, we note that $X_{\mathbb{R}}$ is a real vector space (even though the functions may be complex valued). We have a real-symmetric version of the previous lemma.

Lemma 3 *For a real-symmetric compact set* $K \subset \mathbb{C}$, *the following are equivalent:*

(i) $\mathbb{C} \setminus K$ *is connected;*
(ii) $P_{\mathbb{R}}(K) = A_{\mathbb{R}}(K)$;
(iii) $D_{\mathbb{R}}(K) = A_{\mathbb{R}}(K)$.

Proof Suppose $\mathbb{C} \setminus K$ is not connected. Then K has a bounded complementary component U. Fix $a \in U$. The function $f(z) = (z-a)^{-1}(z-\overline{a})^{-1}$ is in $A_{\mathbb{R}}(K)$. Suppose, to obtain a contradiction, that there is a sequence p_n of real-symmetric polynomials such that

$$|p_n(z) - (z-a)^{-1}(z-\overline{a})^{-1}| < 1/n, \quad \text{for all } z \in K.$$

Then

$$|p_n(z)(z-a)(z-\overline{a}) - 1| < |(z-a)(z-\overline{a})|/n, \quad \text{for all } z \in \partial(U \cup U^*) \subset K.$$

By the maximum principle,

$$|p_n(z)(z-a)(z-\overline{a}) - 1| < \max_{w \in \partial(U \cup U^*)} |(w-a)(w-\overline{a})|/n, \quad \text{for all } z \in U \cup U^*.$$

In particular, at $z = a$, we have

$$1 < \max_{w \in \partial(U \cup U^*)} |(w - a)(w - \overline{a})|/n, \quad \text{for all } n,$$

which is a absurd. Therefore, (ii) implies (i). A similar argument shows that (iii) implies (i).

Now suppose that $\mathbb{C} \setminus K$ is connected and $f \in A_{\mathbb{R}}(K)$. By Lemma 2, there are algebraic polynomials P_n and Dirichlet polynomials D_n that converge uniformly to f on K. Since f is real-symmetric, it is easy to see that the real-symmetric algebraic polynomials $(P_n(z) + \overline{P_n(\overline{z})})/2$ and the real-symmetric Dirichlet polynomials $(D_n(z) + \overline{D_n(\overline{z})})/2$ also converge uniformly to f on K. Thus, (i) implies (ii) and (iii). ■

The next lemma, due to Frank Deutsch [2], generalizes a result of Walsh and states that if we can approximate, we can simultaneously interpolate.

Lemma 4 *Let Y be a dense (real or complex) linear subspace of the (respectively real or complex) linear topological space X and let $L_1, \dots, L_n$ be continuous linear functionals on X. Then for each $x \in X$ and each neighbourhood U of x, there is a $y \in Y$ such that $y \in U$ and $L_i(y) = L_i(x)$, $i = 1, \dots, n$.*

With the help of these lemmas, we now prove the theorem.

Proof of Theorem 1 Our construction of the sets K_n is inspired by a construction in [3].

First, we prove the theorem for algebraic polynomials P_n. Set $t_0 = 0$ and let t_k, $k \in \mathbb{N}$, be the imaginary parts of the zeros of ζ in the upper half-plane, arranged in increasing order. For each $t_k, k > 0$, there are at most finitely many corresponding zeros of ζ. Choose $0 < \lambda_1 < \lambda_2 < \cdots < 1$, such that $\lambda_i \nearrow 1$. Let $s_0 = 1$ and for $k \in \mathbb{Z} \setminus \{0\}$, let $s_k = 2k$. For each $i \in \mathbb{N}$ and for each $-i \le j \le i$, and $k = 0, 1, \dots, i$, set

$$Q_{ijk} = \left\{ z : s_j \le \Re z \le s_j + \lambda_i(s_{j+1} - s_j),\ t_k \le |\Im z| \le t_k + \lambda_i(t_{k+1} - t_k) \right\},$$
$$Q_i = \bigcup Q_{ijk}, \quad -i \le j \le i, \quad k = 0, 1, \dots, i.$$

The compact set Q_i is real-symmetric and is the union of disjoint closed rectangles, so $\mathbb{C} \setminus Q_i$ is connected.

Let Z_i^1 be the zeros of ζ in $Q_i \cap (\mathbb{R} \cup S)$ and let Z_i^2 be the zeros of ζ in $Q_i \setminus (\mathbb{R} \cup S)$. Then $Z_i = Z_i^1 \cup Z_i^2$ is the set of zeros of ζ in Q_i. Denoting by $B(z, r)$ (resp. $\overline{B}(z, r)$) the open (resp. closed) disc of center z and radius r, set

$$\mathcal{B}_i = \bigcup_{z \in Z_i \cup \{1\}} B\Big(z, \frac{1}{i}\Big),$$
$$K_i = (Q_i \setminus \mathcal{B}_i) \cup Z_i \cup \{1\}.$$

Then $K_1, K_2, \dots,$ is an increasing sequence of compact sets whose union is $\mathbb{C}$, and the complement of each K_n is connected. Moreove, since Z_i^1 and Z_i^2 are real-symmetric, so are the K_i. Now, for $n = 1, 2, \dots,$ set

$$\mathcal{K}_n = K_n \cup \bigcup_{z \in Z_n \cup \{1\}} \overline{B}\Big(z, \frac{1}{2n}\Big) = (Q_n \setminus \mathcal{B}_n) \cup \bigcup_{z \in Z_n \cup \{1\}} \overline{B}\Big(z, \frac{1}{2n}\Big).$$

For each n, the complement of $\mathcal{K}_n$ is connected and K_n is real-symmetric, and so, by Lemma 3, the real-symmetric algebraic polynomials are dense in the space of real-symmetric holomorphic functions on (neighbourhoods of) $\mathcal{K}_n$. By Lemma 4, for every real-symmetric function f holomorphic on $\mathcal{K}_n$, and finitely many points $a_1, \ldots, a_m \in \mathcal{K}_n$ and for each $\epsilon > 0$, there is a real-symmetric polynomial P, such that $|f - P| < \epsilon$ on $\mathcal{K}_n$, and $P(a_j) = f(a_j)$, $j = 1, \ldots, m$. Moreover, for each $k \in \mathbb{N}$, there is such a polynomial P such that, for each $a_j \in \mathcal{K}_n^0$, $P^{(\ell)}(a_j) = f^{(\ell)}(a_j)$, for $\ell = 0, 1, \ldots, k$.

We apply this approximation and interpolation procedure to the following function, holomorphic, and real-symmetric on $\mathcal{K}_n$:

$$f_n(z) = \begin{cases} \zeta(z) & \text{for } z \in Q_n \setminus \mathcal{B}_n, \\ n & \text{for } z \in \overline{B}\big(1, 1/(2n)\big), \\ z - a & \text{for } z \in \overline{B}\big(a, 1/(2n)\big), \quad a \in Z_n^1, \\ 1/n & \text{for } z \in \overline{B}\big(a, 1/(2n)\big), \quad a \in Z_n^2. \end{cases}$$

Set $\delta_n = \min |f_n(z)|$ for $z \in Q_n \setminus \mathcal{B}_n$. Since ζ has no zeros on this compact set, $\delta_n > 0$. Choose $\epsilon_n < \min\{\delta_n/2, 1/n\}$. Invoking the approximation-interpolation procedure, for each n, there is a real-symmetric polynomial P_n such that

$$|P_n(z) - f_n(z)| < \epsilon_n, \quad \text{for all} \quad z \in \mathcal{K}_n;$$
$$P_n(1) = n;$$
$$P_n(a) = 0,\ P_n'(a) = 1, \quad \text{for all} \quad a \in Z_n^1;$$
$$P_n(a) = 1/n, \quad \text{for all} \quad a \in Z_n^2.$$

Since P_n is real-symmetric, $P_n(\mathbb{R}) \subset \mathbb{R}$. This completes the proof for algebraic polynomials.

The proof for Dirichlet polynomials is identical (thanks to Lemmas 3 and 4). ∎

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