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A NOTE ON BRAUER CHARACTER DEGREES OF SOLVABLE GROUPS

YOU-QIANG WANG

ABSTRACT. Let G be a finite solvable group. Fix a prime integer p and let t be the number of distinct degrees of irreducible Brauer characters of G with respect to the prime p. We obtain the bound $3t-2$ for the derived length of a Hall p'-subgroup of G. Furthermore, if $|G|$ is odd, then the derived length of a Hall p'-subgroup of G is bounded by t.

1. **Introduction.** All groups considered in this paper are finite and solvable. Let p be a prime. We denote by H a Hall p'-subgroup of G and by $\mathrm{IBr}_p(G)$ the set of irreducible Brauer characters of G with respect to the prime p. Let $t_p(G) = |\{\varphi(1) \mid \varphi \in \mathrm{IBr}_p(G)\}|$. We obtain a linear bound for the derived length of H in terms of $t_p(G)$. The key point in our proof is to reduce the modular case to the ordinary case for which we can apply the results in Berger [1] and Isaacs [2]. Consequently, our result is a generalization of Berger [1, Theorem 2.4] and Isaacs [2, Corollary 7] (by taking p not to divide $|G|$).

Let $\varphi \in \mathrm{IBr}_p(G)$ and $\mathcal{X}$ be a F-representation of G affording φ. We define $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\mathcal{X}$. Since any two F-representations of G affording φ are similar, $\mathrm{Ker}\,\varphi$ is well-defined. The following proposition may seem innocuous, but it is the key to reduce the proof of our main results.

PROPOSITION. *Let $\varphi \in \mathrm{IBr}_p(G)$. Then, for any p-regular element $g \in G$,*

$$g \in \mathrm{Ker}\,\varphi \text{ if and only if } \varphi(g) = \varphi(1).$$

PROOF. Let g be a p-regular element of G. By Fong-Swan Theorem, there exists $\chi \in \mathrm{Irr}(G)$ such that $\varphi = \hat{\chi}$ (the restriction of χ to the set of p-regular elements of G). If $\varphi(g) = \varphi(1)$, the $\chi(g) = \chi(1)$, and hence $g \in \mathrm{Ker}\,\chi$ by Isaacs [3, Lemma 2.19]. Furthermore, by Isaacs [3, Theorem 15.8], $g \in \mathrm{Ker}\,\chi \leq \mathrm{Ker}\,\varphi$. Conversely, assume that $g \in \mathrm{Ker}\,\varphi$. Let $\mathcal{X}$ be an F-representation of G affording φ. Then $\mathcal{X}(g)$ is the $\varphi(1) \times \varphi(1)$ identity matrix over F. Hence $1 \in F$ is the only eigenvalue of $\mathcal{X}(g)$, which has the multiplicty $\varphi(1)$. By the definition of φ, $\varphi(g) = \varphi(1)$. ∎

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2. **Main Results.**

THEOREM. *Suppose that G is solvable. Let* $\varphi \in \mathrm{IBr}_p(G)$ *and* $M \leq G$ *such that* $M \leq \operatorname{Ker}\psi$ *whenever* $\psi \in \mathrm{IBr}_p(G)$ *with* $\psi(1) < \varphi(1)$. *Then*

(1) $\big(O^p(M)\big)''' \leq \operatorname{Ker}\varphi$;

(2) $\big(O^p(M)\big)'' \leq \operatorname{Ker}\varphi$ *if* $2 \not| \varphi(1)$;

(3) $\big(O^p(M)\big)' \leq \operatorname{Ker}\varphi$ *if* $2 \not| |G|$.

PROOF. Let $N = \bigcap_{\psi \in \mathrm{IBr}_p(G), \psi(1) < \varphi(1)} \operatorname{Ker}\psi$. Then $M \leq N$ and $N \triangle G$. Without loss of generality, we can assume that $M = N$.

By Fong-Swan Theorem, there exists $\chi \in \mathrm{Irr}(G)$ such that $\hat{\chi} = \varphi$. For any $\theta \in \mathrm{Irr}(G)$ with $\theta(1) < \chi(1)$, $\hat{\theta}$ is a Brauer character of G, and hence $\hat{\theta} = \sum_{i=1}^{k} n_i\psi_i$, where $\psi_i \in \mathrm{IBr}_p(G)$ and n_i is a non-negative integer for $i = 1, \ldots, k$. For any i, since $\psi_i(1) \leq \hat{\theta}(1) = \theta(1) < \chi(1) = \varphi(1)$, $M \leq \operatorname{Ker}\psi_i$. Let g be a p-regular element of M. By the Proposition, $\psi_i(g) = \psi_i(1)$. Thus $\hat{\theta}(g) = \sum_{i=1}^{k} n_i\psi_i(g) = \sum_{i=1}^{k} n_i\psi_i(1) = \hat{\theta}(1)$. Hence $\theta(g) = \theta(1)$. This yields that $g \in \operatorname{Ker}\theta$. Since $O^p(M)$ is generated by all the p-regular elements of M, $O^p(M) \leq \operatorname{Ker}\theta$. Notice that $M \triangle G$ implies that $O^p(M) \triangle G$. Hence, by Isaacs [2, Theorem 6] and Berger [1, Theorem 2.2], we have that

(1) $\big(O^p(M)\big)''' \leq \operatorname{Ker}\chi$;

(2) $\big(O^p(M)\big)'' \leq \operatorname{Ker}\chi$ if $2 \not| \chi(1)$;

(3) $\big(O^p(M)\big)' \leq \operatorname{Ker}\chi$ if $2 \not| |G|$.

By Issacs [3, Theorem 15.8], $\operatorname{Ker}\chi \leq \operatorname{Ker}\hat{\chi} = \operatorname{Ker}\varphi$, and hence we have the conclusions. ■

Let $1 = f_1 < f_2 < \cdots < f_{t_p(G)}$ be the distinct irreducible Brauer character degrees of G. For $1 \leq r \leq t_p(G)$, let $\alpha_H(r)$ denote

$$\max\{\, dl(H \operatorname{Ker}\varphi / \operatorname{Ker}\varphi) \mid \varphi \in \mathrm{IBr}_p(G), \varphi(1) \leq f_r \}.$$

We notice that $\alpha_H(1) = 1$ and $\alpha_H\big(t_p(G)\big) = dl(H)$.

As a corollary of our theorem, we obtain a linear bound for the derived length of Hall p'-subgroups of G in terms of $t_p(G)$.

COROLLARY. *Let G be solvable and H be a Hall* p'*-subgroup of G. Then we have that*

(1) $\alpha_H(r) \leq 3r - 2$, *and*

(2) if $2 \not| |G|$, $\alpha_H(r) \leq r$.

In particular, we have that

(1) $dl(H) \leq 3t_p(G) - 2$, *and*

(2) if $2 \not| |G|$, $dl(H) \leq t_p(G)$.

PROOF. Use induction on r. Suppose $\varphi \in \mathrm{IBr}_p(G)$ with $\varphi(1) \leq f_r$ so that

$$H^{\alpha_H(r-1)} \leq \operatorname{Ker}\psi$$

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for all $\psi \in \mathrm{IBr}_p(G)$ with $\psi(1) < \varphi(1)$. By (1) and (3) of the Theorem, we have that

$$\left(O^p\left(H^{\alpha_H(r-1)}\right)\right)''' \leq \operatorname{Ker}\varphi,$$

and if $2 \not| \, |G|$, $\left(O^p\left(H^{\alpha_H(r-1)}\right)\right)' \leq \operatorname{Ker}\varphi$. Since H is a Hall p'-subgroup of G, $O^p\left(H^{\alpha_H(r-1)}\right) = H^{\alpha_H(r-1)}$. Hence, $H^{\alpha_H(r-1)+3} \leq \operatorname{Ker}\varphi$, and if $2 \not| \, |G|$, $H^{\alpha_H(r-1)+1} \leq \operatorname{Ker}\varphi$. Thus $\alpha_H(r) \leq \alpha_H(r-1)+3$, and if $2 \not| \, |G|$, $\alpha_H(r) \leq \alpha_H(r-1)+1$. Since $\alpha_H(1) = 1$ and $\alpha_H\left(t_p(G)\right) = dl(H)$, we have the conclusions by induction. ∎

REMARK. In his Ph.D. thesis at the University of Mainz, Dr. Frank Bernhardt obtains the same bound $3t_p(G) - 2$ for the derived length of H and the $2t_p(G) - 1$ bound for the $p = 2$ case and the odd order case. In addition, he obtains the $3t_p(G) - 2$ bound and the $t_p(G) - 1$ bound for the nilpotent length and the p-length of G respectively.

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Department of Mathematics
Ohio University
P.O.Box 5688
Athens, Ohio
USA 45701