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SCALE FUNCTIONS AND TREE ENDS

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Abstract

A class of totally disconnected groups consisting of partial direct products on an index set is examined. For such a group, the scale function is found, and for automorphisms arising from permutations of the index set, the tidy subgroups are characterised. When applied to the case where the index set is a finitely-generated free group and the permutation is translation by an element x of the group, the scale depends on the cyclically reduced form of x and the tidy subgroup on the element which conjugates x to its cyclically reduced form.

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0. Introduction and notation

It was shown by van Dantzig in 1931 that each totally disconnected locally compact group has a base of neighbourhoods of the identity consisting of compact open subgroups, [13]. He also gave an example of a totally disconnected locally compact group which fails to have a *normal* compact open subgroup. This example is the semidirect product $G \rtimes_\alpha \mathbb{Z}$, where

$$G = \left\{ g \in \prod_{\mathbb{Z}} \mathbb{Z}/(2\mathbb{Z}) : \exists N \text{ such that } g(k) = 0 \text{ for } k < N \right\}$$

and the automorphism α is the translation defined by $\alpha(g)(k) = g(k+1)$. The subgroups $G_N = \{g : g(k) = 0 \text{ for } k < N\}$ form a base of neighbourhoods of the identity for a topology on G, in which each G_N is compact and open, and $G \rtimes_\alpha \mathbb{Z}$ is equipped with the product topology.

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Results in the paper [15] imply that van Dantzig's example is in fact typical of the way in which a totally disconnected locally compact group may fail to have a normal compact open subgroup. It was shown that, if G is any totally disconnected locally compact group and x is an element of G, then there is a compact open subgroup U of G such that:

T1. $U = U_+U_-$, where $U_\pm = \bigcap_{n\geq 0} x^{\pm n}Ux^{\mp n}$; and
T2. $U_{++} = \bigcup_{n\geq 0} x^n U_+ x^{-n}$ and $U_{--} = \bigcup_{n\geq 0} x^{-n}U_- x^n$ are closed subgroups of G.

A subgroup satisfying T1 and T2 is said to be *tidy* for x. Note that if $xUx^{-1} = U$, then U is tidy for x and U_+, U_-, U_{++} and U_{--} equal U. Conversely, if $U = U_+ = U_-$, then x normalises U.

It was further shown in [15] that the index $s(x) = [xU_+x^{-1} : U_+]$ is independent of the choice of subgroup tidy for x and defines a continuous function $s : G \to \mathbb{Z}^+$ such that

S1. $s(x) = 1 = s(x^{-1})$ if and only if there is a compact open subgroup U of G with $xUx^{-1} = U$.

Regarding α as an element of $G \rtimes_\alpha \mathbb{Z}$ in the van Dantzig example, $U_+ = G_0$ is tidy for α, $U_{++} = G$, $U_- = \{e\}$, $s(\alpha) = 2$ and $s(\alpha^{-1}) = 1$.

The function s is called the *scale function* of G. In the case when x is not periodic, $\langle x, U_+\rangle$ is closed and is isomorphic to $U_{++} \rtimes_\alpha \mathbb{Z}$, where α is the automorphism of U_{++} defined by $\alpha(u) = xux^{-1}$, $(u \in U_{++})$. Thus any element x of G which fails to normalise any compact open subgroup of G belongs to a closed subgroup $U_{++} \rtimes_\alpha \mathbb{Z}$ (or $U_{--} \rtimes_\alpha \mathbb{Z}$) which has the same form as van Dantzig's example.

These results do not completely answer the question of when totally disconnected locally compact groups have normal compact open subgroups. It can happen that each element of a group G normalises some compact open subgroup, but that G has no normal compact open subgroup. The scale function of a group in which each element normalises some compact open subgroup is identically 1 and so, following [10], we shall call such a group *uniscalar*. Examples of uniscalar groups having no normal compact open subgroup are given in [16] and [6, Section 6]. (Note that the main theorem in [16] was proved earlier in [7].) However in all known examples G is not compactly generated and it is an important question in the structure theory of totally disconnected locally compact groups to decide whether there are compactly generated uniscalar groups which have no normal compact open subgroups. A partial answer is given in [6] where it is shown that each compactly generated, uniscalar, rank 1 p-adic Lie group does have a compact open normal subgroup. This question is a special case of the problem of how the local, or element by element, structure described by the tidy subgroups of G may be assembled to give a global description of G.

The present paper generalises van Dantzig's construction with the aim of using the groups found to help to answer some of these global structure questions.

1. The extension of van Dantzig's construction and preliminary remarks

For the extension of van Dantzig's construction we consider groups of the form $G \rtimes A$, where A is a discrete group and G is a restricted product, indexed by a set X, of copies of a finite group K. The restricted product is defined as follows. We suppose X to be partitioned into subsets P and S and define

$$G = \mathop{-\!\!\!\!\prod}_{S,P} K = \mathop{-\!\!\!\!\prod}_{X} K = \sum_S K \times \prod_P K.$$

The topology on G is defined to be the product of the discrete topology on $\sum_S K$ and the product topology on $\prod_P K$. For $Y \subseteq X$, define $G_{[Y]} = \{f \in G : f = e \text{ off } Y\}$; then $G_{[Y]}$ is compact if and only if $Y \setminus P$ is finite and $G_{[Y]}$ is open if and only if $P \setminus Y$ is finite. As Y ranges over all sets of finite difference with P, the sets $G_{[Y]}$ form a base for the topology at $e \in G$. In a situation where the roles of the sets P and S are reversed, we will use the symbol $\prod\!\!\!\!-$ in place of $-\!\!\!\!\prod$.

The action of A by automorphisms of G is induced by an action of A on X. For each bijection $\alpha : X \to X$ and $f \in G$, define $\alpha(f)$ by $\alpha(f)(x) = f(\alpha^{-1}(x))$. It follows from the description of the topology of G above that $\alpha(f) \in G$ and $\alpha^{-1}(f) \in G$ if and only if the symmetric difference $P \,\triangle\, \alpha(P)$ is finite and in this case the map $f \mapsto \alpha(f)$ is a continuous automorphism of G. Hence, given an action of A on X such that $P \,\triangle\, \alpha(P)$ is finite for every α in A, there is an induced action of A by automorphisms of G. The *semidirect product* $G \rtimes A$ is defined to be the set $G \times A$ equipped with the product topology and the multiplication $(g_1, \alpha_1)(g_2, \alpha_2) = (g_1\alpha_1(g_2), \alpha_1\alpha_2)$. It is a totally disconnected locally compact group. The identity element in G will be denoted by e and that in A by ι. The identity in $G \rtimes A$ then is (e, ι).

This construction is our extension of van Dantzig's example. The original example may be retrieved by taking $X = \mathbb{Z} = A$, $P = \mathbb{Z}^+$ and $\alpha(n) : k \mapsto k - n$, $(n \in A,\ k \in X)$ in our construction. Note that: if S is finite, then $-\!\!\!\!\prod_X K$ is isomorphic to the compact group $\prod_X K$ and; if P is finite, then $-\!\!\!\!\prod_X K$ is isomorphic to the discrete group $\sum_X K$. In these cases we do not get any new types of totally disconnected groups and so we will usually consider cases in which both P and S are infinite.

The aim now is to analyse the examples $G \rtimes A$:

- to identify tidy subgroups for elements x in these groups;
- to describe the scale function for these groups; and
- investigate how the tidy subgroups depend on x.

PROPOSITION 1.1. *Let Y be a subset of X such that $Y \,\triangle\, P$ is finite and let (g, α) belong to $G \rtimes A$. Then $G_{[Y]}$ is tidy for (g, α) if and only if it is tidy for (e, α). There may be subgroups tidy for (e, α) which are not tidy for (g, α).*

PROOF. It is immediate from the definitions that for any $g \in G$, $(g, \alpha)G_{[Y]}(g, \alpha)^{-1}$

$= G_{[\alpha(Y)]}$. Taking $x = (g, \alpha)$ and $U = G_{[Y]}$, it follows that U_+, U_-, U_{++} and U_{--} are independent of g. Hence $G_{[Y]}$ satisfies T1 and T2 with $x = (g, \alpha)$ if and only if T1 and T2 are satisfied with $x = (e, \alpha)$.

Every subgroup is tidy for the identity (e, ι). We give an example of a compact open subgroup which is not tidy for (g, ι). For the example, let X be a single point, A be trivial and $K = S_3$ be the group of permutations of $\{1, 2, 3\}$. Let $H = \{e, (12)\}$ and $g = (123)$. Then $gHg^{-1} \cap H = \{e\}$ and it follows that $H \times \{\iota\}$ is not tidy for (g, ι). Although this example is discrete, even finite, it can be used as the basis of nondiscrete examples. □

If the finite group K happens to be abelian, then any subgroup $U \subset G \times \{\iota\}$ is tidy for (g, α) if and only if it is tidy for (e, α). We shall see that (g, α) always has tidy subgroups of the form $G_{[Y]}$, from which follows the

COROLLARY 1.2. *For each (g, α) in $G \rtimes A$ we have $s((g, \alpha)) = s((e, \alpha))$.*

Proposition 1.1 shows that in order to identify some tidy subgroups for arbitrary elements of $G \rtimes A$ it suffices to identify tidy subgroups of the form $G_{[Y]}$ for the elements (e, α). We may work inside G for this and consequently may simplify notation as follows: the compact open subgroup $U \subset G$ will be said to be *tidy for* α if $U \times \{\iota\}$ is tidy for (e, α). Observe that the criteria for U to be tidy for x are stated in terms of the inner automorphism $g \mapsto xgx^{-1}$ $(g \in G)$ and so U will be tidy for α if and only if U satisfies T1 and T2 with $U_\pm = \bigcap_{n\geq 0} \alpha^{\pm n}(U)$, $U_{++} = \bigcup_{n\geq 0} \alpha^n(U_+)$ and $U_{--} = \bigcup_{n\geq 0} \alpha^{-n}(U_-)$. Similarly, the *scale* of the automorphism α will be the scale of (e, α), which is $s(\alpha) = [\alpha(U_+) : U_+]$.

The identification of tidy subgroups for individual elements will be seen to reduce to van Dantzig's example and so we begin with a complete description of this case. It is necessary only to identify the subgroups tidy for the single automorphism α induced by the translation $k \mapsto k - 1$ of $\mathbb{Z}$.

PROPOSITION 1.3. *Suppose U is a compact open subgroup of $\mathop{+\!\!\prod}_{\mathbb{Z}} K$. Then U is a tidy subgroup for α if and only if*

$$U = \sum_{j<m} \{e\} \times U_0 \times \prod_{j\geq n} K,$$

where $m, n \in \mathbb{Z}$, $n \geq m$ and $U_0 \subseteq K^{n-m}$ is such that $\{e\} \times U_0 \subseteq U_0 \times K$.

PROOF. Since U is compact and open, there exist m, n such that $\prod_{j\geq n} K \subseteq U \subseteq \prod_{j\geq m} K$. Put $U_0 = U \bigcap \prod_{j=m}^{n-1} K \subseteq K^{n-m}$. Note that

$$U_- = \bigcap_{k\geq 0} \alpha^{-k}(U) \subseteq \bigcap_{k\geq 0} \alpha^{-k}\Big(\prod_{j\geq m} K\Big) = \bigcap_{k\geq 0} \prod_{j\geq m+k} K = \{e\}.$$

If U is tidy, then $U = U_+U_- = U_+ = \bigcap_{k\geq 0} \alpha^k(U)$, and so $U \subseteq \alpha(U)$. Consequently $\{e\} \times U_0 \subseteq U_0 \times K$. Conversely, if $\{e\} \times U_0 \subseteq U_0 \times K$ then $\alpha(U) \supseteq U$, so $U_+ = U$. Moreover, $\alpha^k(U_+) \supseteq \prod_{j\geq n-k} K$ so $\bigcup_{k\geq 0} \alpha^k(U_+) = \prod'_{\mathbb{Z}} K$, which is closed, as is $\bigcup_{k\geq 0} \alpha^{-k}(U_-) = \{e\}$. Hence U is tidy. $\square$

There are several ways in which such a subgroup $U_0 \subseteq K^{n-m}$ may arise.

(a) If $m = n$ then $U = \prod_{j\geq n} K = G_{[[n,\infty)]}$.

(b) If $\{K_i\}_{\mathbb{Z}}$ is an increasing sequence of subgroups of K varying from $\{e\}$ to K, then $\prod_{\mathbb{Z}} K_i$ is a tidy subgroup for α.

(c) If $\varphi : K \to K$ is a group homomorphism such that φ^{n-m} is the trivial homomorphism, then

$$U_0 = \{(\varphi^{n-m-1}(k), \varphi^{n-m-2}(k), \dots, \varphi(k), k) : k \in K\}$$

has the desired property.

The group $G \rtimes \mathbb{Z}$ thus has subgroups, as in (a), which are tidy for every element of the group. That is not the case in general. In the next section we characterise tidy subgroups of individual elements in $G \rtimes A$ and then investigate global properties of particular examples in later sections.

2. Scale functions of automorphisms of $\prod'_X K$

The purpose of the present section is to determine $s(\alpha)$ where α is an automorphism of $\prod'_X K$ as considered in the introduction. Beyond this, we will see a characterisation of the compact open subgroups that are tidy for α. There are few surprises here—this case reduces to a finite product of fundamental cases, including the groups of the type considered in Section 1.

Before proceeding, we have a lemma, whose proof follows directly from the definitions of 'tidy' and 'scale function'.

LEMMA 2.1. *Suppose for $i = 1, 2$ that G_i is a totally disconnected group and $\alpha_i \in \mathrm{Aut}(G_i)$ has a tidy subgroup U_i. Put $G = G_1 \times G_2$, $U = U_1 \times U_2$, a compact open subgroup of G and $\alpha = \alpha_1 \otimes \alpha_2 \in \mathrm{Aut}(G)$. Then U is a tidy subgroup for α and $s(\alpha) = s(\alpha_1)s(\alpha_2)$.*

The action of $\{\alpha^n : n \in \mathbb{Z}\}$ on X defines orbits $\mathcal{O}_i = \{\alpha^n(z_i)\}_{n\in\mathbb{Z}}$ for i in some index set $\mathbf{I}$. Define $\mathcal{O}_i^+ = \{\alpha^n(z_i) : n \geq 0\}$ and $\mathcal{O}_i^- = \mathcal{O}_i \setminus \mathcal{O}_i^+$. Each element of the finite set $P \,\Delta\, \alpha(P)$ can be written as either $\alpha^k(z_i) \in P$ with $\alpha^{k-1}(z_i) \notin P$ or vice-versa. Consequently, there are only finitely many places where orbits cross from P into S or from S into P. For a single infinite orbit $\mathcal{O}_i$, this means that either $\alpha^k(z_i) \in P$ for all

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sufficiently large k or $\alpha^k(z_i) \in S$ for all sufficiently large k. Equivalently, precisely one of $\mathcal{O}_i^+ \setminus P$ or $\mathcal{O}_i^+ \setminus S$ is finite. A similar dichotomy holds for $\mathcal{O}_i^-$.

We partition $\mathbf{I}$ into 6 parts: $\mathbf{I} = \mathbf{I}_P \cup \mathbf{I}_S \cup \mathbf{I}_{P,P} \cup \mathbf{I}_{S,P} \cup \mathbf{I}_{P,S} \cup \mathbf{I}_{S,S}$ where:

(i) $\mathbf{I}_P$ and $\mathbf{I}_S$ are those i for which $\mathcal{O}_i$ is finite, with $\mathbf{I}_P$ being those for which $\mathcal{O}_i \subseteq P$ and $\mathbf{I}_S$ being those for which $\mathcal{O}_i \cap S \neq \emptyset$, and

(ii) each $\mathbf{I}_{Q,R}$ consists of those i for which $\mathcal{O}_i$ is infinite and $\mathcal{O}_i^- \setminus Q$ and $\mathcal{O}_i^+ \setminus R$ are finite.

Again using the finiteness of $P \triangle \alpha(P)$, we see that there are only finitely many $i \in \mathbf{I}_S$ with $\mathcal{O}_i \cap P$ nonempty, and likewise for $i \in \mathbf{I}_{S,S}$ having $\mathcal{O}_i \cap P$ nonempty and $i \in \mathbf{I}_{P,P}$ having $\mathcal{O}_i \cap S$ nonempty. Also the cardinalities $n_+ = |\mathbf{I}_{P,S}|$ and $n_- = |\mathbf{I}_{S,P}|$ are finite. Consequently,

$$P' = \bigcup_{i \in \mathbf{I}_P} \mathcal{O}_i \cup \bigcup_{i \in \mathbf{I}_{P,P}} \mathcal{O}_i \cup \bigcup_{i \in \mathbf{I}_{S,P}} \mathcal{O}_i^+ \cup \bigcup_{i \in \mathbf{I}_{P,S}} \mathcal{O}_i^-$$

has finite difference with P, making $G_{[P']}$ a compact open subgroup of G. We will use the following decomposition of G

$$\prod_{i \in \mathbf{I}_P}^{\text{finite}} \prod_{\mathcal{O}_i} K \times \sum_{i \in \mathbf{I}_S}^{\text{finite}} \prod_{\mathcal{O}_i} K \times \prod_{i \in \mathbf{I}_{P,P}} \prod_{\mathcal{O}_i} K \times \prod_{i \in \mathbf{I}_{S,P}}^{\text{finite}} \coprod_{\mathcal{O}_i} K \times \prod_{i \in \mathbf{I}_{P,S}}^{\text{finite}} \coprod_{\mathcal{O}_i} K \times \sum_{i \in \mathbf{I}_{S,S}} \sum_{\mathcal{O}_i} K$$
$$= G_P \times G_S \times G_{P,P} \times G_{S,P} \times G_{P,S} \times G_{S,S}$$

where each $G_{*,*}$ is the obvious factor. This type of subscripting will also be used to denote sets in a partition of X consisting of the union of the corresponding orbits. This gives us, for instance, $X_{P,P} = \bigcup_{i \in \mathbf{I}_{P,P}} \mathcal{O}_i$ and $G_{P,P} = G_{[X_{P,P}]}$.

THEOREM 2.2. *With notation as above, $G_{[P']}$ is a tidy subgroup for α, $s(\alpha) = |K|^{n_+}$ and $s(\alpha^{-1}) = |K|^{n_-}$.*

PROOF. Clearly, $G_{[P']} = G_P \times \{e\} \times G_{P,P} \times \prod_{i \in \mathbf{I}_{S,P}} G_{[\mathcal{O}_i^+]} \times \prod_{i \in \mathbf{I}_{P,S}} G_{[\mathcal{O}_i^-]} \times \{e\}$, corresponding to the decomposition of G above. Since the first three and the last factors of $G_{[P']}$ are α-invariant, they are tidy for the corresponding restriction $\alpha_{*,*}$ with $s(\alpha_{*,*}) = s(\alpha_{*,*}^{-1}) = 1$. Next note that for each $i \in \mathbf{I}_{P,S}$, the action of α on the invariant subgroup $G_{[\mathcal{O}_i]}$ is a shift as in Section 1. Consequently $G_{[\mathcal{O}_i^-]}$ is tidy for α_i, the restriction of α to $G_{[\mathcal{O}_i]}$, with $s(\alpha_i) = |K|$ and $s(\alpha_i^{-1}) = 1$. The situation for $i \in \mathbf{I}_{S,P}$ is similar, but with $s(\alpha_i) = 1$ and $s(\alpha_i^{-1}) = |K|$. The result now follows from Lemma 2.1. $\square$

To simplify what follows, we will assume for the rest of this section that $P = P'$. This does not cause any loss of generality, P and P' have finite difference, and so define

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the same totally disconnected locally compact group G. Moreover, the partitioning of the orbits and resulting decomposition of G are identical.

The subgroups U of G that are tidy for α can be characterised using a similar strategy to the above, and again this reduces to a product of subgroups of the type considered in Section 1. However, the decomposition of G on which the structure of U is based need not be as fine as that above. To obtain an appropriate decomposition, we reindex $G_{S,P}$ and $G_{P,S}$ to each be a single partial direct product. For instance

$$G_{S,P} = \prod_{i \in \mathrm{I}_{S,P}}^{\text{finite}} \coprod_{\mathcal{O}_i} K \cong \coprod_{\mathbb{Z}} \prod_{i \in \mathrm{I}_{S,P}}^{\text{finite}} K = \coprod_{\mathbb{Z}} K^{n_+},$$

on which α acts by translation. The tilde $\sim$ will be used to denote the isomorphisms $G_{S,P} \to \coprod_{\mathbb{Z}} K^{n_+}$ and $G_{P,S} \to \coprod_{\mathbb{Z}} K^{n_-}$, as appropriate.

Much of the analysis of a tidy subgroup U relies on establishing relationships between the orbits $\mathcal{O}_i$ and the support set of U. In the following, Q is this support set, and R is the largest subset of X such that $G_{[R]} \subseteq U$. Then since $G_{[R]} \subseteq U \subseteq G_{[Q]}$, with U being compact and open in the product topology, the sets P, Q and R differ in only finitely many points.

LEMMA 2.3. *With U a tidy subgroup with support set Q as above,*

(i) *finite orbits are either totally contained in Q or disjoint from Q,*
(ii) $Q \cap X_{S,S} = \emptyset$,
(iii) $\alpha^k(Q \cap X_{S,P}) \subseteq Q$ *for all* $k \geq 0$, *and*
(iv) $\alpha^{-k}(Q \cap X_{P,S}) \subseteq Q$ *for all* $k \geq 0$.

PROOF. The support sets of U_+ and U_- are subsets of $\bigcap_{n=0}^{\infty} \alpha^n(Q)$ and $\bigcap_{n=0}^{\infty} \alpha^{-n}(Q)$ respectively. Since $U = U_+U_-$, we have that

$$Q = \bigcap_{n=0}^{\infty} \alpha^n(Q) \cup \bigcap_{n=0}^{\infty} \alpha^{-n}(Q).$$

Each of the conclusions follows immediately. $\square$

An immediate consequence of this is that any tidy subgroup U is contained within $G_P \times G_S \times G_{P,P} \times G_{S,P} \times G_{P,S}$. With $G_{P,P}$ we can actually do better, and obtain $G_{P,P}$ as a factor of U, and consequently of U_+ and U_-.

LEMMA 2.4. *$X_{P,P}$ is a subset of R.*

PROOF. By definition of the topology on G, $Z = X_{P,P} \setminus R$ is finite. Put $Y = X_{P,P} \setminus \left(\bigcup_{k \geq 0} \alpha^k(Z)\right)$. Then $G_{[Y]} < U_+$ and so $\bigcup_{k \geq 0} G_{[\alpha^k(Y)]} < U_{++}$. Since α is just a shift on each orbit $\mathcal{O}_i \subset X_{P,P}$, $\bigcup_{k \geq 0} \alpha^k(Y) = X_{P,P}$. Hence $\bigcup_{k \geq 0} G_{[\alpha^k(Y)]}$ is dense in $G_{[X_{P,P}]}$. Since U_{++} is closed, we have $G_{[X_{P,P}]} < U_{++}$.

However, $G_{[X_{P,P}]}$ is compact and so there is m such that $\alpha^{-m}(G_{[X_{P,P}]}) < U_+ < U$. Since $G_{[X_{P,P}]}$ is invariant under α, it follows that $G_{[X_{P,P}]} < U$. $\square$

To complete the classification of tidy subgroups, we need to consider the finite orbits and those orbits passing from P to S or vice-versa.

THEOREM 2.5. *A subgroup U of G is tidy for α if and only if $U = U_0 \times G_{P,P} \times U_{S,P} \times U_{P,S} \times \{e\}$ where U_0 is a compact open subgroup of $G_P \times G_S$ invariant under α and α^{-1}, $U_{S,P}$ is supported on $X_{S,P}$ and satisfies $\alpha(U_{S,P}) < U_{S,P}$ and $U_{P,S}$ is supported on $X_{P,S}$ and satisfies $\alpha(U_{P,S}) > U_{P,S}$.*

PROOF. By Lemma 2.1, the stated conditions are sufficient for U to be tidy for α. Conversely, suppose U is tidy for α. Put

$$U_0 = \{g \in G_P \times G_S : \exists h \in U \text{ such that the restriction of } h \text{ to } X_P \times X_S \text{ equals } g\},$$

$$U_{S,P} = \{g \in G_{S,P} : \exists h \in U \text{ such that the restriction of } h \text{ to } X_{S,P} \text{ equals } g\}$$

and

$$U_{P,S} = \{g \in G_{P,S} : \exists h \in U \text{ such that the restriction of } h \text{ to } X_{P,S} \text{ equals } g\}.$$

Since Lemma 2.4 shows that $G_{P,P} < U$, to prove the Theorem it will suffice to show that each of $U_{S,P}$ and $U_{P,S}$ is a subgroup of U and that U_0 is a subgroup of $U_+ \cap U_-$.

To show this for U_0, let g be in U_0 and choose $h \in U$ whose restriction to $X_P \times X_S$ equals g. Factor h as h^+h^- where $h^+ \in U_+$ and $h^- \in U_-$. Since $G_{P,P} < U$, it may be supposed that h equals e on $X_{P,P}$ and, by Lemma 2.3, that h^+ and h^- equal e on $X_{P,P} \cup X_{S,P}$ and $X_{P,P} \cup X_{P,S}$ respectively. Let $g^+ \in U_0$ agree with h^+ on $X_S \cup X_P$. Then $h^+ = g^+f^+$, where $f^+ \in G_{P,S}$. Now $\alpha^{-n}(f^+)$ converges to the identity as $n \to \infty$ and, since g^+ is supported on finite α-orbits, $\alpha^{-n!}(g^+) \to g^+$ as $n \to \infty$. It follows that $g^+ = \lim_{n\to\infty} \alpha^{-n!}(h^+)$, which belongs to $U_+ \cap U_-$. Similarly, $g_- = \lim_{n\to\infty} \alpha^{n!}(h_-)$ belongs to $U_+ \cap U_-$. Therefore $g = g^+g^-$ belongs to $U_+ \cap U_-$.

Next let g be in $U_{S,P}$ and choose $h \in U$ whose restriction to $X_{S,P}$ equals g. Factor h as h^+h^- where $h^+ \in U_+$ and $h^- \in U_-$. Then it may be supposed that h, h^+ and h^- equal e on $X_P \cup X_S \cup X_{P,P}$. Now h^+ also equals e on $X_{S,P}$ and h_- does on $X_{P,S}$. Hence we have $g = h_-$ and thus belongs to $U_- < U$. Similarly, $U_{P,S} < U_+ < U$. $\square$

The structure of the subgroups $U_{S,P}$ and $U_{P,S}$ may be described more explicitly with the aid of Proposition 1.3 since $\widetilde{U}_{S,P} \subseteq \coprod_{\mathbb{Z}} K^{n_+}$ and $\widetilde{U}_{P,S} \subseteq \coprod_{\mathbb{Z}} K^{n_-}$ are tidy under translation.

3. Groups acting on graphs

Now suppose that we have a finitely-generated group $G = \langle a_1, \dots, a_n \rangle$, and a right G-set X. Let Γ be the Cayley graph of the action of G on X, so that vertices

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$x, y \in X$ are adjacent when there is some k with $xa_k = y$ or $xa_k^{-1} = y$. In this section we consider partial direct sums as previously, where the index set is a graph and the mapping α is given as right-translation by an element $a \in G$. Note that this is *not* an automorphism of the graph G. We find that the possibilities for the set P in the construction of the partial direct sum can be related to the structure of the graph Γ.

In a graph Γ with vertex set X, we define:

(i) a *path* to be a 1-1 mapping $\xi : \mathbb{N} \to X$ such that consecutive terms are adjacent vertices in Γ,

(ii) two vertices x and y to be *in the same component of* Γ if there is a sequence of vertices $x_0, x_1, \dots, x_n$ with $x_0 = x$ and $x_n = y$ and such that x_{k-1} is adjacent to x_k for $k = 1, \dots, n$,

(iii) the *components* of Γ to be the equivalence classes under the equivalence relation on vertices of being in the same component,

(iv) two paths ξ and ζ to be *disconnected* by a set $Y \subseteq X$ when there exist arbitrarily large i, j such that ξ_i and ζ_j lie in different components of $\Gamma \setminus Y$,

(v) two paths to be equivalent, $\zeta \sim \xi$, when there is no finite set disconnecting them, and

(vi) an *end* of Γ to be an equivalence class of paths in Γ.

The set of all ends is denoted Ω and the set $\overline{X} = X \cup \Omega$ can be endowed with a natural topology so that X is dense. If X is a connected graph, which occurs if G acts transitively, then $\overline{X}$ is compact. In this topology, points $x \in X$ are isolated and for a finite set $Y \subseteq X$, the closure of a component Z of $\Gamma \setminus Y$ includes precisely those ends whose paths eventually lie in Z. Then a base of neighbourhoods of $\omega \in \Omega$ can be taken to be $\{\overline{Z}_Y : Y \text{ finite}\}$, where Z_Y is the component of $\Gamma \setminus Y$ in which the paths of ω eventually lie. It can be shown that $\overline{X}$ is metrisable—see for example [17, equation (2.1)] or [3, Proposition IV.6.7]. For a Cayley graph, it follows immediately from the definition that each vertex x is only a finite distance from its right translate xa. Hence, if $\{x_i\}_1^\infty \subseteq X$ is a path converging to $\omega \in \Omega$, then the sequence $\{x_i a\}_1^\infty$ (which need not be a path) will also converge to ω.

PROPOSITION 3.1. *If $P \subseteq X$ then $P \setminus Pa$ is finite for all $a \in G$ if and only if $\overline{P}$ is an open set in $\overline{X}$.*

PROOF. Suppose $P \setminus Pa$ is finite for all $a \in G$, then so is $Pa \setminus P = (P \setminus Pa^{-1})a$. Put $F = \bigcup_1^n (Pa_k \cup Pa_k^{-1}) \setminus P$, a finite set with $P \subseteq X \setminus F$. Now, any $y \in X$ connected by an edge to some $x \in P$ is either in P or in F. It follows that each component of $X \setminus F$ is either wholly contained in P or does not meet P and so P is composed of components. Hence $\overline{P}$ is open.

Now suppose $P \setminus Pa$ is infinite for some $a \in G$. Let $\omega \in \Omega$ be a limit point of $P \setminus Pa$, so that $\omega \in \overline{P}$. If $\{x_i\}_1^\infty \subseteq P \setminus Pa$ is a sequence converging on ω, then

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$\{x_i a^{-1}\}_1^\infty \subseteq Pa^{-1}\backslash P$ is a sequence also converging on ω. However, $\{x_i a^{-1}\}_1^\infty \subseteq X\backslash P$, and so ω is not an interior point of $\overline{P}$. Hence $\overline{P}$ is not open. □

In the situation where $\Gamma = G$ and $\alpha = \rho_x$, translation by an element $x \in G$, the structure of the space of ends of G can yield information on the scale function.

PROPOSITION 3.2. *Suppose H is a subgroup of G such that H has a single end. Then $s(\rho_x) = 1$ for all $x \in H$.*

PROOF. Let $y \in G$ be such that $\mathcal{O}_i = \{yx^n\}_{-\infty}^\infty$ is infinite. We show that $\mathcal{O}_i$ is an orbit with $i \in \mathbf{I}_{P,P}$ or $i \in \mathbf{I}_{S,S}$, by the classification scheme in Section 2.

Since H has only one end, so does the coset yH, say $\omega \in \Omega$. Supposing $\omega \in \overline{P}$, a closed and open set in $G \cup \Omega$, we have that both sequences $\{yx^n\}_0^\infty$ and $\{yx^{-n}\}_0^\infty$ lie in P after a finite number of terms. Consequently $i \in \mathbf{I}_{P,P}$. A similar argument gives $i \in \mathbf{I}_{S,S}$ in the case when $\omega \notin \overline{P}$. □

Now a finitely generated infinite group G has either one, two or infinitely many ends, see [3, Theorem IV.6.10]. If G has two ends, then it has an infinite cyclic subgroup of finite index, see [3, Theorem IV.6.12], and the scale function and tidy subgroups may be computed using the techniques of Section 1. If G has infinitely many ends, then it is essentially a non-trivial free product, see [3, Theorem IV.6.10]. The ends of groups were studied in [4] and crucial steps towards the results on the number of ends taken in [8, 12, 1].

4. Scale functions on free groups and free products

We now consider several examples where $A = X$ is a group, with the action being right multiplication, that is, the action of x in A is given by $y \mapsto yx^{-1}$. With this action we have that $(e, x)G_{[Y]}(e, x)^{-1} = G_{[Yx^{-1}]}$, $(x \in A,\ Y \subset X)$. Typically, P will consist of a finite number of branches of the Cayley graph of A.

In all these examples, the group A is a free group or a free product of groups. In the case of free groups, elements of the group will be written as words in the generators and their inverses, and we will assume that the words are *reduced*—that is, of minimal length. For two words u and x, with u non-empty, we will say the *count* of u in x is the number of times u appears as a subword of x. As we are interested in the orbits $\{yx^n\}$, we will also need to know the asymptotic behaviours of the count of a word u in the reduced word of yx^n. For this we define a *cyclic reduction* of an element x, this being a word w of shortest length in the conjugacy class of x. Note that the cyclic reduction of a word is defined only up to cycling of the letters, for example abc and $bca = a^{-1}abca$ are both cyclic reductions of abc. Then the *cyclic count* of a word

u in x is the number of *cyclic occurrences* of u in the cyclic reduction w of x—this includes normal instances of u as a subword of w and those instances of u that wrap around from the end to the beginning of w. This means that the cyclic count is the number c such that for all sufficiently large n, the difference between the count of u in x^n and the count of u in x^{n+1} is c. If $S \subseteq H$, then the *total count* of S in x is the sum of the counts of all elements $u \in S$ in the word x and the *total cyclic count* is defined similarly.

For instance, if a, b, c are generators of $\mathbb{F}_3$, then

(i) a cyclic reduction of $w = abcbabca^2bac^{-1}b^{-1}a^{-1}$ is $babca^2ba$ and the cyclic count of ab in w is 3;

(ii) the cyclic count of $ababab$ in ab is 1; and

(iii) the total cyclic count of $\{(ab)^n\}_1^\infty$ in ab is infinite.

In the cases dealt with below, the total cyclic count is finite. The computation of $s(x)$ in terms of a cyclic count begins by assuming that x is cyclically reduced. This does not affect the value of the scale function, as it is invariant under conjugation.

In the case of the free product $B * C$, a non-empty reduced word consists of a word with symbols alternating between $B \setminus \{e\}$ and $C \setminus \{e\}$ and the length of such a word will be the number of symbols. The count of a word u of length one is not well-defined because B and C need not be free but for u with length at least two the number of occurrences of u in x can be counted. A cyclically-reduced word will either have length one or have even length because any word $c_1b_1c_2\cdots b_nc_{n+1}$ of length $2n+1$, $n \geq 1$, may be cyclically reduced to the word $b_1c_2\cdots b_n(c_{n+1}c_1)$ of length $2n$.

The first example to be discussed is the mixed case where A is the free product $B * C$, with B a free group and C a discrete group. In this case the letters in reduced words will be generators of B or their inverses and elements of C. 'Length', 'count' and so on are defined accordingly.

EXAMPLE 4.1. Take $B = \{b^n\} \cong \mathbb{Z}$ and C any discrete group. Let $A = X = B * C$ and let $P = \{y \in X :$ the reduced word for y is of the form $bw\}$. Let x be in A and write $x = zwz^{-1}$ where w is cyclically reduced. Then:

(i) $\log_{|K|} s(x)$ equals the cyclic count of b in x; and

(ii) a tidy subgroup for x is $G_{[Pz^{-1}]}$.

PROOF. (i) Suppose to begin with that x is cyclically reduced. For $x = b$, the only orbit passing from P to S is $\{b^n\}$, and so $s(b) = |K|$. Consequently for $x = b^n$, $s(b^n) = |K|^n$ if $n > 0$ and $s(b^n) = 1$ if $n \leq 0$.

For other cyclically reduced words we can suppose that $x = b^{q_1}c_1b^{q_2}c_2\cdots b^{q_k}c_k$. The cyclic count of b in x is $\sum_{q_i>0} q_i$. Then $y \in P$ satisfies $yx^{-1} \notin P$ if and only if $y = b^ic_jb^{q_{j+1}}c_{j+1}\cdots b^{q_k}c_k$ where $1 \leq j \leq k$ and $1 \leq i \leq q_j$. The value of $\log_{|K|} s(x)$ is at most the number of such y, which is $\sum_{q_i>0} q_i$. Since x is cyclically reduced we have for such y that $yx^{-n} \in S$ when n is positive and $yx^{-n} \in P$ when n is negative. Hence $\log_{|K|} s(x)$ is exactly $\sum_{q_i>0} q_i$ as required. The result now follows for general x because the scale function is invariant under conjugation.

(ii) It was shown in the first part that, if x is cyclically reduced, then all x-orbits enter or leave P exactly once or not at all. Hence $G_{[P]}$ is tidy for x when x is cyclically reduced. If $x = zwz^{-1}$, where w is cyclically reduced, then $(e, z)G_{[P]}(e, z)^{-1} = G_{[Pz^{-1}]}$ is tidy for x. $\square$

It has been seen in other examples that tidy subgroups are a type of normal form for elements of totally disconnected locally compact groups. In [15] it was seen that, in automorphism groups of trees, identifying tidy subgroups for an automorphism x corresponds to identifying the unique path in the tree such that x is a translation along the path. In [5] it was seen that, in p-adic Lie groups, identifying tidy subgroups for x corresponds to finding the Jordan canonical form for the adjoint representation of x on the Lie algebra. In the present case we see that identifying tidy subgroups corresponds to finding the cyclically reduced form for a word x.

Another characterisation of the scale function and tidy subgroups is given in [14]. It is shown that for any totally disconnected locally compact group G

$$s(x) = \min\{[xUx^{-1} : U \cap xUx^{-1}] : U \text{ is a compact open subgroup of } G\} \quad (x \in G)$$

and the minimum is attained at precisely those compact open subgroups which are tidy for x. In view of this characterisation, and of the fact that the scale function is invariant under conjugation, it is not surprising that identifying tidy subgroups in this example involves minimising the cyclic word length.

The proof of (i) shows that all cyclically reduced elements of A have $G_{[P]}$ as a tidy subgroup. However, as we shall see, there is no subgroup of G which is tidy for every element of A. For a pair of elements x_1, x_2 in a totally disconnected locally compact group define

$$d(x_1, x_2) = \min\{[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] : U_i \text{ tidy for } x_i,\ i = 1, 2.\}.$$

Then $d(x_1, x_2)$ is a measure of how far x_1 and x_2 are from having a common tidy subgroup.

The computation of this value involves the cyclic reduction of pairs of elements of $B * C$. Among the conjugates of a pair (x_1, x_2) are pairs (w_1, zw_2z^{-1}) where w_1 and w_2 are cyclically reduced and z has minimum length. Define a *cyclic reduction* of (x_1, x_2) to be such a pair. The element z will be said to be a *comparator* of x_1 and x_2. Some properties of the comparator and notation will be required in the following discussion.

Suppose that the comparator $z = b^{q_1}c_1b^{q_2}c_2 \cdots b^{q_k}c_k$ is a reduced word, where we may have $q_1 = 0$ or $c_k = \iota$ but $q_j \neq 0$ and $c_j \neq \iota$ otherwise. There are further restrictions on z depending on w_1 and w_2. The possibilities and restrictions for cyclically reduced w_2 are:

(1) $w_2 = b^n$, in which case $c_k \neq \iota$ (otherwise b^{q_k} can be commuted past b^n and the length of z reduced);
(2) $w_2 \in C$, in which case $c_k = \iota$ (otherwise c_k and c_k^{-1} can be absorbed into w_2 and the length of z reduced); and
(3) $w_2 = b^r c' \cdots c'' b^s$, where r and s are not both 0 and have the same sign, in which case either: (a) $c_k \neq \iota$, or (b) $c_k = \iota$ and $s = 0$ if q_k has the same sign as r and $r = 0$ if q_k has the opposite sign as s.

In all cases except (3a) when either r or s is zero, $zw_2^n z^{-1}$ is in reduced form for all non-zero n. In case (3a) with $r = 0$, c_k and c' can be combined to reduce the length of $zw_2^n z^{-1}$ by 1. Note however that $c_k c' \neq \iota$ because then w_2 could be replaced with a cyclically equivalent word and the length of z reduced. Thus in this case there is a contraction in $zw_2^n z^{-1}$ but no cancellation. No further reduction of $zw_2^n z^{-1}$ can be made and similarly if $s = 0$. Thus in all cases there is no cancellation possible in $zw_2^n z^{-1}$. It may be seen in the same way that there is no cancellation in $z^{-1} w_1^n z$ for any $n \neq 0$.

(iii) Let $x_1, x_2 \in A$. Then $\log_{|K|} d(x_1, x_2)$ equals the total count of $\{b, b^{-1}\}$ in a comparator of x_1 and x_2.

PROOF. The function d is conjugation invariant and so we have that $d(x_1, x_2) = d(w_1, zw_2z^{-1})$ where (w_1, zw_2z^{-1}) is a cyclic reduction of (x_1, x_2). We compute this latter value.

Since w_1 and w_2 are cyclically reduced, $G_{[P]}$ is tidy for w_1 and $G_{[Pz^{-1}]}$ is tidy for zw_2z^{-1}. Denote these groups as U_1 and U_2 respectively. Then

$$\log_{|K|}[U_1 : U_1 \cap U_2] = \#(P \setminus Pz^{-1}) \quad \text{and} \quad \log_{|K|}[U_2 : U_1 \cap U_2] = \#(Pz^{-1} \setminus P).$$

An element $y \in P$ satisfies $yz^{-1} \notin P$ if and only if $y = b^i c_j b^{q_{j+1}} c_{j+1} \cdots b^{q_k} c_k$ where $1 \leq j \leq k$ and $1 \leq i \leq q_j$. Hence $\#(Pz^{-1} \setminus P)$ equals $\sum_{q_j > 0} q_j$, which is the count of b in z. Now $\#(P \setminus Pz^{-1}) = \#(Pz \setminus P)$ which, by the same argument, equals the count of b in z^{-1}. This is just the count of b^{-1} in z and so $\log_{|K|} d(w_1, zw_2z^{-1})$ is at most the total count of $\{b, b^{-1}\}$ in z.

Now let U_1 be any subgroup tidy for w_1 and U_2 be any subgroup tidy for zw_2z^{-1}. Suppose at first that both w_1 and w_2 have infinite order. Let $y \in P$ be such that $yz^{-1} \notin P$. Then $y = b^i c_j b^{q_{j+1}} c_{j+1} \cdots b^{q_k} c_k$ as above, so that y is a right subword of z. Since there is no cancellation in $zw_2^n z^{-1}$ for any non-zero n, it follows that there is no cancellation in yw_2^n for any n and hence that $yw_2^n \in P$ for every n. It follows that $yz^{-1}(zw_2z^{-1})^n \in Pz^{-1}$ for every n. Since this orbit is infinite, we have $G_{[yz^{-1}]} \subset U_2$. Similarly, $yz^{-1} = b^{i-q_j} c_{j-1} b^{-q_{j-1}} \cdots c_1^{-1} b^{-q_1}$ is a right subword of z^{-1}. Since there

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is no cancellation in $z^{-1}w_1^n z$ for $n \neq 0$, $yz^{-1}w_1^n \notin P$ for every n. Since this orbit is infinite, we have $G_{[yz^{-1}]} \cap U_1 = \{e\}$. We have shown then that $G_{[Pz^{-1}\setminus P]} \cap U_1 = \{e\}$ and $G_{[Pz^{-1}\setminus P]} \subset U_2$ and it follows that $\log_{|K|}[U_2 : U_1 \cap U_2] \geq \#(Pz^{-1} \setminus P)$. It may be shown that in a similar way that $\log_{|K|}[U_1 : U_1 \cap U_2] \geq \#(P \setminus Pz^{-1})$. Therefore $\log_{|K|} d(w_1, zw_2z^{-1})$ is at least the total count of $\{b, b^{-1}\}$ in z when w_1 and w_2 have infinite order.

The remaining case is when either w_1 or w_2 has finite order. Suppose, without loss of generality, that w_1 has finite order. Then, by [9, Corollary 4.1.4], w_1 is conjugate to an element of C and so, since w_1 is cyclically reduced, it follows that in fact w_1 belongs to C. The above discussion of the comparator shows that $q_1 \neq 0$ in this case. The w_1-orbits are finite and U_1 is an infinite product of finite groups invariant under w_1.

As a first subcase, consider when w_2 has infinite order. Then, as seen above, $G_{[Pz^{-1}\setminus P]} \subset U_2$ and $G_{[P\setminus Pz^{-1}]} \cap U_2 = \{e\}$. It cannot be shown that $G_{[Pz^{-1}\setminus P]} \cap U_1$ *must* be the trivial subgroup but we can show that this intersection may be *assumed* to be trivial without increasing $[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2]$. To this end, let yz^{-1} be in $Pz^{-1} \setminus P$ and note that the equation we are trying to prove holds if w_1 is the identity, so that we may suppose that $\langle w_1 \rangle$ has at least two elements. Then, for $w \in \langle w_1 \rangle \setminus \{\iota\}$, the element $yz^{-1}wzw_2^n$ has no cancellations because $q_1 \neq 0$, $w \in C$ and zw_2^n has no cancellations. Hence $yz^{-1}wzw_2^n$ is not in P, because yz^{-1} isn't, and it follows that $yz^{-1}w(zw_2z^{-1})^n$ is not in Pz^{-1} for any n. Since this orbit is infinite, it follows that $G_{[yz^{-1}w]} \cap U_2 = \{e\}$. Denote $U_1 \cap G_{[yz^{-1}\langle w_1 \rangle]}$ and $U_2 \cap G_{[yz^{-1}\langle w_1 \rangle]}$ by U_1' and U_2' respectively. Then we have seen that U_1 is the product over the orbit of copies of some subgroup, L say, of K and $U_2' = G_{[yz^{-1}]}$. Hence $[U_1' : U_1' \cap U_2'] \geq |L|$ and $[U_2' : U_1' \cap U_2'] = |K|/|L|$. It follows that if U_1 is now replaced by the group which is trivial on $yz^{-1}\langle w_1 \rangle$ and agrees with U_1 elsewhere, then the new group is also tidy for w_1 and $[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2]$ is not increased. Doing this for each $yz^{-1} \in Pz^{-1} \setminus P$ we arrive at a group U_1 which is tidy for w_1 and satisfies $G_{[Pz^{-1}\setminus P]} \cap U_1 = \{e\}$ without increasing $[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2]$. Similarly, it may be shown that U_1 may be assumed to satisfy $G_{[P\setminus Pz^{-1}]} \subset U_1$ without increasing $[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2]$. For this, show that for each $y \in P \setminus Pz^{-1}$ and $w \in \langle w_1 \rangle \setminus \{\iota\}$ we have $yw(zw_2z^{-1})^n \in Pz^{-1}$ for every n, from which it follows that $U_2'' = U_2 \cap G_{[y\langle w_1 \rangle]} = G_{[y\langle w_1 \rangle \setminus \{y\}]}$. Hence $[U_1'' : U_1'' \cap U_2''] = |L|$ and $[U_2'' : U_1'' \cap U_2''] \geq |K|/|L|$, where $U_1'' = U_1 \cap G_{[y\langle w_1 \rangle]}$ is the product of copies of L. Then U_1 may be replaced by a group whose intersection with $G_{[y\langle w_1 \rangle]}$ is the product of copies of K without increasing $[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2]$. Therefore, $\log_{|K|} d(w_1, zw_2z^{-1})$ is at least the total count of $\{b, b^{-1}\}$ when w_2 has infinite order.

The second subcase is when w_1 and w_2 both have finite order. It may be *assumed* that $G_{[Pz^{-1}\setminus P]} \cap U_1 = \{e\}$, $G_{[P\setminus Pz^{-1}]} \subset U_1$, $G_{[Pz^{-1}\setminus P]} \subset U_2$ and $G_{[P\setminus Pz^{-1}]} \cap U_2 = \{e\}$ and this suffices to show that $\log_{|K|} d(w_1, zw_2z^{-1})$ is at least the total count of $\{b, b^{-1}\}$.

To see, for example, that we may assume that $G_{[P\setminus Pz^{-1}]} \subset U_1$, let $y \in P \setminus Pz^{-1}$, let $L_1 = U_1 \cap G_{[y]}$ and $L_2 = U_2 \cap G_{[y]}$. Then for each $u \in \langle w_1 \rangle \setminus \{\iota\}$ we have $yu \in P$ because $w \in C$, so that there is no cancellation. Furthermore, $q_1 \neq 0$ because $w_1 \in C$ and so there is no cancellation in yuz. It follows that $yu \in Pz^{-1}$ as well. Similar arguments show that

$$S_y = \{yu_1v_1\cdots u_l,\ yu_1v_1\cdots u_lv_l : u_j \in \langle w_1\rangle \setminus \{\iota\},\ v_j \in \langle zw_2z^{-1}\rangle \setminus \{\iota\}\}$$

is contained in $P \cap Pz^{-1}$. By construction, S_y is partitioned into (zw_2z^{-1})-cosets and $\{y\} \cup S_y$ into w_1-cosets. It may be shown that, if $U_1' = U_1 \cap G_{[\{y\}\cup S_y]}$ and $U_2' = U_2 \cap G_{[\{y\}\cup S_y]}$, then $[U_1' : U_1' \cap U_2'][U_2' : U_1' \cap U_2'] \geq |K|/|L_2|$. Replacing U_1 by the group which agrees with $G_{[\{y\}\cup S_y]}$ on $\{y\} \cup S_y$ and with U_1 elsewhere, and U_2 by the group which agrees with $G_{[S_y]}$ on S_y and with U_2 elsewhere, the new groups are still tidy for w_1 and zw_2z^{-1} respectively and we now have $[U_1' : U_1' \cap U_2'][U_2' : U_1' \cap U_2'] = |K|/|L_2|$. Repeating for each $y \in P \setminus Pz^{-1}$, we have $G_{[P\setminus Pz^{-1}]} \subset U_1$ and $[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2]$ has not been increased. Similar arguments show that the other assumptions may also be made and so $\log_{|K|} d(w_1, zw_2z^{-1})$ is at least the total count of $\{b, b^{-1}\}$ when w_1 and w_2 both have finite order. □

It follows from the above discussion that if x has infinite order and is in C, then x normalises a *unique* compact open subgroup of G.

It may seem that this example is rather special but in fact the same discussion applies whenever we take $P = \{y \in X :$ the reduced word for y has the form $sw\}$ where s is a reduced word ending in b, because, if z is any reduced word, then $\#(P \bigtriangleup Pz^{-1})$ equals the total count of $\{b, b^{-1}\}$ in z. We now give some examples where P has several branches.

EXAMPLE 4.2. Let $A = X = \mathbb{F}_k = \langle a_1, \dots, a_k\rangle$ and let P be those $w \in \mathbb{F}_k$ whose representations as reduced words begin with one of $a_1, a_2, \dots, a_k$.

(i) $\log_{|K|} s(x) = ($ total cyclic count of $\{a_i : 1 \leq i \leq k\}$ in x $)$
$\qquad - \big($ total cyclic count of $\{a_j^{-1}a_i : 1 \leq i \neq j \leq k\}$ in $x\big)$.

(ii) Let $x = zwz^{-1}$ where w is the cyclic reduction for x. If $w \in P \cap P^{-1}$, then $G_{[(P\cup\{\iota\})z^{-1}]}$ is tidy for x and $G_{[Pz^{-1}]}$ is not. If $w \in P \bigtriangleup P^{-1}$, then $G_{[Pz^{-1}]}$ and $G_{[(P\cup\{\iota\})z^{-1}]}$ are both tidy for x. If $w \notin P \cup P^{-1}$, then $G_{[Pz^{-1}]}$ is tidy for x and $G_{[(P\cup\{\iota\})z^{-1}]}$ is not.

(iii) Let x_1 and x_2 be in A, and let (w_1, zw_2z^{-1}) be the cyclic reduction of the pair (x_1, x_2). Then

$$\begin{aligned}\log_{|K|} d(x_1, x_2) = &\big(\text{total count of } \{a_i,\ a_i^{-1} : 1 \leq i \leq k\} \text{ in } z\big)\\ &- \big(\text{total count of } \{a_j^{-1}a_i,\ a_ia_j^{-1} : 1 \leq i \neq j \leq k\} \text{ in } z\big)\\ &+ \varepsilon(w_1, w_2, z),\end{aligned}$$

where $-2 \leq \varepsilon(w_1, w_2, z) \leq 2$.

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PROOF. (i) We may suppose that x is cyclically reduced and write $x = a_{j_1}^{m_1} a_{j_2}^{m_2} \cdots a_{j_n}^{m_n}$, where $j_i \neq j_{i+1}$, $i = 1, \dots, n-1$ and, if $n > 1$, $j_n \neq j_1$. Define

$$P_i = \{y \in A : \text{ the reduced word for } y \text{ has the form } a_i w\},$$

so that $P = \bigcup_{i=1}^{k} P_i$.

Let $y \in P_i$ for some i and suppose that $yx^{-1} \notin P_i$. As in the previous example, the number of such y equals the count of a_i in x, which is also the total cyclic count. As before, it follows from the fact that x is cyclically reduced that $yx^{-p} \notin P_i$ for $p \geq 0$ and $yx^{-p} \in P_i$ for $p < 0$. Hence the orbit $\{yx^{-p}\}$ will contribute to the value of $s(x)$ unless it enters P_j for some $j \neq i$. Since x is cyclically reduced, the words $(yx^{-1})x^{-p}$, $p > 0$, are reduced. Hence there are two ways in which the orbit may enter P_j: a) $yx^{-1} = \iota$ and $x^{-1} \in P_j$ or b) yx^{-1} is in P_j. Now the first possibility occurs if and only if $x \in P_i$ and $x^{-1} \in P_j$, which means that $j_1 = i$ and $m_1 > 0$ and $j_n = j$ and $m_n < 0$. The second possibility occurs when y has the form $a_{j_r}^{m_r} a_{j_{r+1}}^{m_{r+1}} \cdots a_{i_n}^{m_n}$, where $r > 1$, $j_r = i$ and $m_r > 0$ and $j_{r-1} = j$ and $m_{r-1} < 0$. These possibilities coincide with the cyclic occurrences of $a_j^{-1}a_i$ in X, so that the number of times the orbit leaves P_i and enters P_j is the cyclic count of $a_j^{-1}a_i$ in x. Hence $\#(Px^{-1} \setminus P)$ is equal to the difference between the total cyclic count of $\{a_i : 1 \leq i \leq k\}$ in x and the total cyclic count of $\{a_j^{-1}a_i : 1 \leq i \neq j \leq k\}$ in x. Since, as we have seen, each orbit which leaves $P \cup \{\iota\}$ does not return this number equals $\log_{|K|} s(x)$.

(ii) Let w be cyclically reduced. Then the argument in the previous paragraph shows that a w-orbit which leaves P does not return unless $w \in P \cap P^{-1}$, in which case one orbit leaves P, passes through ι and then returns to P. Hence if $w \notin P \cap P^{-1}$, then $G_{[P]}$ is tidy for w and, if $w \in P \cap P^{-1}$, then $G_{[P]}$ is not tidy for w but $G_{[P \cup \{\iota\}]}$ is.

If $w \in P$, then $\iota \in Pw^{-1}$ and, if $w \in P^{-1}$, then $\iota \in Pw$. Hence in both cases $G_{[P \cup \{\iota\}]}$ is tidy for w. However $G_{[P \cup \{\iota\}]}$ is not tidy for w if $w \notin P \cup P^{-1}$ because in that case $w^n \notin P$ for every n.

Therefore, for cyclically reduced w: if $w \in P \cap P^{-1}$, then $G_{[P \cup \{\iota\}]}$ is tidy for w but $G_{[P]}$ is not; if $w \in P \triangle P^{-1}$, then $G_{[P \cup \{\iota\}]}$ and $G_{[P]}$ are both tidy for w; and if $w \notin P \cup P^{-1}$, then $G_{[P]}$ is tidy for w but $G_{[P \cup \{\iota\}]}$ is not. Conjugating w by z yields the claim for x.

(iii) Let (w_1, zw_2z^{-1}) be the cyclic reduction of (x_1, x_2). Then $z^{-1}w_1z$ is a reduced word. It follows, as in the previous example, that $\{yw_1^n\}_{n \in \mathbb{Z}} \subset P$ for every $y \in P \setminus Pz^{-1}$ and that $\{yz^{-1}w_1^n\}_{n \in \mathbb{Z}} \cap P = \emptyset$ for every $yz^{-1} \in Pz^{-1} \setminus \{P \cup \{\iota\}\}$. Hence, for every subgroup U_1 tidy for w_1, $G_{[P \setminus Pz^{-1}]} \subset U_1$ and $G_{[Pz^{-1} \setminus (P \cup \{\iota\})]} \cap U_1 = \{e\}$. Similarly, since zw_2z^{-1} is a reduced word, for every set U_2 tidy for zw_2z^{-1} we have $G_{[Pz^{-1} \setminus P]} \subset U_2$ and $G_{[P \setminus (Pz^{-1} \cup \{z^{-1}\})]} \cap U_2 = \{e\}$. These observations may be used as in the previous example to show that $d(x_1, x_2)$ is attained when U_1 and U_2 are among the subgroups tidy for x_1 and x_2 given in (ii).

If $U_1 = G_{[P]}$ and $U_2 = G_{[Pz^{-1}]}$, then

$$\log_{|K|}[U_1 : U_1 \cap U_2][U_2 : U_1 \cap U_2] = \#(P \triangle Pz^{-1})$$

and this number equals

$$\big(\text{total count of } \{a_i,\ a_i^{-1} : 1 \leq i \leq k\} \text{ in } z\big) \\ - \big(\text{total count of } \{a_j^{-1}a_i,\ a_ia_j^{-1} : 1 \leq i \neq j \leq k\} \text{ in } z\big).$$

The $\varepsilon(w_1, w_2, z)$ term arises because, depending on w_1, w_2 and z, $d(x_1, x_2)$ may be attained when $U_1 = G_{[P\cup\{\iota\}]}$ or $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$. There are numerous cases to be considered.

(1) When $w_1,\ w_2 \notin P \cup P^{-1}$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}]}$ and $\varepsilon(w_1, w_2, z) = 0$.
(2) When $w_1 \notin P \cup P^{-1}$ and $w_2 \in P \triangle P^{-1}$ and:

a) $z^{-1} \in P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = -1$;
b) $z^{-1} \notin P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}]}$ and $\varepsilon(w_1, w_2, z) = 0$.

(3) When $w_1 \notin P \cup P^{-1}$ and $w_2 \in P \cap P^{-1}$ and:

a) $z^{-1} \in P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = -1$;
b) $z^{-1} \notin P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = 1$.

(4) When $w_1,\ w_2 \in P \triangle P^{-1}$ and:

a) $z^{-1} \in P$ and $z \in P$, then $U_1 = G_{[P\cup\{\iota\}]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = -2$;
b) $z^{-1} \in P$ and $z \notin P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = -1$;
c) $z^{-1} \notin P$ and $z \in P$, then $U_1 = G_{[P\cup\{\iota\}]}$, $U_2 = G_{[Pz^{-1}]}$ and $\varepsilon(w_1, w_2, z) = -1$;
d) $z^{-1} \notin P$ and $z \notin P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}]}$ and $\varepsilon(w_1, w_2, z) = 0$.

(5) When $w_1, \in P \triangle P^{-1}$ and $w_2 \in P \cap P^{-1}$ and:

a) $z^{-1} \in P$ and $z \in P$, then $U_1 = G_{[P\cup\{\iota\}]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = -2$;
b) $z^{-1} \in P$ and $z \notin P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = -1$;
c) $z^{-1} \notin P$ and $z \in P$, then $U_1 = G_{[P\cup\{\iota\}]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = 0$;
d) $z^{-1} \notin P$ and $z \notin P$, then $U_1 = G_{[P]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and $\varepsilon(w_1, w_2, z) = 1$.

(6) When $w_1,\ w_2 \in P\cap P^{-1}$, then $U_1 = G_{[P\cup\{\iota\}]}$, $U_2 = G_{[Pz^{-1}\cup\{z^{-1}\}]}$ and when:

a) $z^{-1} \in P$ and $z \in P$, we have $\varepsilon(w_1, w_2, z) = -2$;
b) $z^{-1} \in P$ and $z \notin P$, we have $\varepsilon(w_1, w_2, z) = 0$;
c) $z^{-1} \notin P$ and $z \in P$, we have $\varepsilon(w_1, w_2, z) = 0$;
d) $z^{-1} \notin P$ and $z \notin P$, we have $\varepsilon(w_1, w_2, z) = 2$.

The remaining cases may be obtained from these by interchanging the roles of w_1 and w_2. $\square$

A similar analysis applies whenever P has several branches, that is, when $P_{s_1,\dots,s_n} = \{y \in X :$ the reduced word for y has the form $s_j w$ for some $j\}$. Let J denote the family of last letters of the s_j's counted according to multiplicity and let $D = \{s_j^{-1}s_i :$

$1 \leq i \neq j \leq n\}$. Then

$$\log_{|K|} s(x) = (\text{total cyclic count of } J \text{ in } x) - (\text{total cyclic count of } D \text{ in } x).$$

An extreme case is when $\{s_j\}$ is just the set of generators and their inverses, so that $P = \mathbb{F}_k \setminus \{\iota\}$. Then $J = \{s_j\}$ and the cyclic count of J in x is just the length of the cyclic reduction of x. D is the set of all length 2 words and the cyclic count of D in x is also just the length of the cyclic reduction of x. Hence the scale function is identically 1, which could have been seen immediately in this case because $\prod_X K$ is a normal compact open subgroup.

EXAMPLE 4.3. For any discrete groups B and C, let $A = X = B * C$ and let P consist of those reduced words beginning with a symbol from B. Then $\log_{|K|} s(x)$ is the total cyclic count of words cb in x, where $b \in B$ and $c \in C$.

We first check that this is a valid example, in that $P \triangle Px$ is finite for all x. Suppose $y \in P$ and $y \notin Px$. Then y begins with $b \in B$ but yx^{-1} does not begin with a symbol from B. Hence either $y = x$ or y must be a proper right subword of x, of which there are only finitely many. Hence $P \setminus Px$ is always finite. Also $Px \setminus P = (P \setminus Px^{-1})x$, which is finite so $P \triangle Px$ is finite.

PROOF. By cyclic reduction we can suppose that either $x \in B$, $x \in C$, or $x = b_1c_1 \cdots b_kc_k$ with alternating symbols from B and C. In the first case $Px^{-1} = P \cup \{\iota\}$ and $P \cup \{\iota\}$ is invariant under x, so that (e, x) normalises $G_{[P\cup\{\iota\}]}$ and $s(x) = 1$. In the second case $Px^{-1} = P$, (e, x) normalises $G_{[P]}$ and $s(x) = 1$.

In the third case, suppose y is such that $y \in P$ but $yx^{-1} \notin P$. Then we must have $y = b_jc_j \cdots b_kc_k$, where $1 \leq j \leq k$. Moreover, $yx^n \in P$ and $yx^{-n} \notin P$ for all $n \geq 1$. Therefore there are exactly k orbits $\{yx^{-n}\}$ such that for all sufficiently large n, $yx^n \in P$ and $yx^{-n} \notin P$. Hence $s(x) = |K|^k$, as required. □

5. Construction of uniscalar groups

The preceding examples have non-trivial scale function but it seems possible that the construction described in the first section could be used to construct a compactly generated uniscalar group without compact open normal subgroups and thus to answer the question discussed in the introduction. In order to say how this might be done, we first formulate some properties of group actions.

DEFINITION 5.1. Let the group A act on the set X and let $P \subset X$.

(i) P is *almost invariant* if $P \triangle a.P$ is finite for every $a \in A$.

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(ii) P is *locally nearly invariant* if for each $a \in A$ there is $Q_a \subset X$ such that $P \triangle Q_a$ is finite and $a.Q_a = Q_a$.

(iii) P is *nearly invariant* if there is $Q \subset X$ such that $P \triangle Q$ is finite and $a.Q = Q$ for every $a \in A$.

It is clear that, if P is nearly invariant, then it is locally nearly invariant and that if P is locally nearly invariant it is almost invariant. The condition that P be almost invariant is necessary for the group $G \rtimes A$ constructed in Section 1 to be a topological group. This group is uniscalar if and only if P is locally nearly invariant and has a compact open normal subgroup if, and only if, P is nearly invariant. Further, $G \rtimes A$ is compactly generated provided that A is finitely generated and its action on X is transitive. Hence a positive answer to the following question would produce an example of a compactly generated uniscalar group which does not have a compact open normal subgroup.

QUESTION 5.2. *Is there a finitely generated group A acting transitively on a set X with a $P \subset X$ which is locally nearly invariant but not nearly invariant*?

It is clear that X must be countable and that, if P is to be not nearly invariant, that both P and $X \setminus P$ must be infinite. In this case P is called a *moiety*.

If $P \subset X$ is almost invariant for an action of A on X and if for each $a \in A$ and $x \in X$ the orbit $\{a^n.x : n \in \mathbb{Z}\}$ is finite, then P is locally nearly invariant. It would be particularly interesting to find an action of this type where P is not nearly invariant because then for each $a \in A$ the group $G \rtimes A$ would have a base of neighbourhoods of the identity consisting of compact open subgroups normalised by a but would have no compact open normal subgroup.

In answer to these questions, Meenaxi Bhattacharjee and Dugald Macpherson have constructed an example satisfying these conditions in [2]. It follows then that there is a compactly generated uniscalar totally disconnected locally compact group which does not have a compact open normal subgroup. On the other hand, it is shown by Anne Parreau in [11] that every compactly generated, uniscalar p-adic Lie group has a compact open normal subgroup.

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