ON CERTAIN ONTO MAPS

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Let Δ_n $(n > 0)$ denote the subset of the Euclidean $(n + 1)$-dimensional space defined by

$$\Delta_n = \{(t_0, t_1, \ldots, t_n) : 0 \leqslant t_i \leqslant 1 \text{ and } \sum_{i=0}^{n} t_i = 1\}.$$

A subset σ of Δ_n is called a *face* if there exists a sequence $0 \leqslant i_1 < i_2 < \ldots < i_m \leqslant n$ such that

$$\sigma = \Delta_n \cap \{(t_0, t_1, \ldots, t_n) : t_{i_1} = t_{i_2} = \ldots = t_{i_m} = 0\},$$

and the dimension of σ is defined to be $(n - m)$. Let $\Delta^{\cdot}_n$ denote the union of all faces of Δ_n of dimensions less than n. A topological space Y is called *solid* if any continuous map on a closed subspace A of a normal space X into Y can be extended to a map on X into Y. By Tietz's extension theorem, each face of Δ_n is solid. The present paper is concerned with a generalization of the following theorem which seems well known. However, since it is used in an essential way later, we include a sketch of a proof.

THEOREM 1. *Let f be a continuous map on Δ_n into Δ_n such that $f[\sigma] \subset \sigma$ for each face σ of Δ_n. Then f is onto.*

Proof. Using the fact that each face is solid, by a step by step process, starting from the lowest dimensional faces, one can construct a continuous map F on $\Delta_n \times [0, 1]$ into Δ_n such that $F(x, 1) = x$, $F(x, 0) = f(x)$, and $F(x, t) \in \sigma$ whenever σ is a face and $(x, t) \in \sigma \times [0, 1]$. Hence the map f, as a map of the pair† $(\Delta_n, \Delta^{\cdot}_n)$ into $(\Delta_n, \Delta^{\cdot}_n)$, is homotopic to the identity map. It follows that $f_* : H_n(\Delta_n, \Delta^{\cdot}_n) \to H_n(\Delta_n, \Delta^{\cdot}_n)$ is the identity homomorphism 1. If f is not onto, then there is a point x_0 such that $x_0 \notin f[\Delta_n]$.‡ Then by means of the radial projection through x_0, f is homotopic to a continuous map g such that $g[\Delta_n] \subset \Delta^{\cdot}_n$ and the points on $\Delta^{\cdot}_n$ are fixed during the homotopy. Hence§ $1 = f_* = g_* = 0 : H_n(\Delta_n, \Delta^{\cdot}_n) \to H_n(\Delta_n, \Delta^{\cdot}_n)$, which implies that $H_n(\Delta_n, \Delta^{\cdot}_n) = 0$. However, this contradicts the known result: $H_n(\Delta_n, \Delta^{\cdot}_n) \approx Z$.

The purpose of this paper is to prove a generalization of Theorem 1, namely

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†For the terminology and the results of the algebraic topology we are using in this proof, consult the first chapter of Eilenberg and Steenrod **(2)**.

‡The point x_0 can be chosen in $\Delta_n \sim \Delta^{\cdot}_n$. For otherwise, $f[\Delta_n] \supset \Delta_n \sim \Delta^{\cdot}_n$ and, since $f[\Delta_n]$ is compact, $f[\Delta_n] = f[\Delta_n]^- \supset (\Delta_n \sim \Delta^{\cdot}_n)^- = \Delta_n$ which implies that f is onto.

§$g : (\Delta_n, \Delta^{\cdot}_n) \to (\Delta_n, \Delta^{\cdot}_n)$ can be "factored "through $(\Delta^{\cdot}_n, \Delta^{\cdot}_n)$ and $H(\Delta^{\cdot}_n, \Delta^{\cdot}_n) = 0$; therefore $g_* = 0$.

Theorem 2, and to give one immediate application (Theorem 4 and 4′). A deeper application of Theorem 2 is made in the paper by Kiefer **(6)**. In fact, he conjectured Theorem 2, and I wish to acknowledge my indebtedness to him for many stimulating conversations we had on this subject.

The following notation is used. If Z is a subset of the product $X \times Y$ and x is a point of X, then $Z_x = \{y : (x, y) \in Z\}$. If X is a subset of a linear space, the smallest convex set containing X (that is, the convex hull of X), is denoted by $\langle X \rangle$. Our terminology agrees with Kelley **(5)**, and results in that book will be used freely.

THEOREM 2. *Let A be a convex compact subset of a (real or complex) Hausdorff locally convex linear topological space, let G be a closed subset of $\Delta_n \times A$ such that, for each x in Δ_n, G_x is non-empty and convex, and let p be the projection of $\Delta_n \times A$ onto Δ_n. If q is a continuous map of G into Δ_n such that $q[G \cap p^{-1}[\sigma]] \subset \sigma$ whenever σ is a face of Δ_n, then q maps G onto Δ_n.*

Notice that Theorem 1 is a special case of Theorem 2 in which A is a single point. If there is a continuous map $h : \Delta_n \to G$ such that $p \circ h$ is the identity map, then Theorem 2 is an immediate consequence of Theorem 1. However, it can easily be seen that, in general, no such h exists, and this is the essential difficulty in the proof of Theorem 2. We shall prove it by establishing first that there is a continuous map h on Δ_n into $\Delta_n \times A$ such that $p \circ h$ is the identity and the range of h is arbitrarily near G. This is done in Theorem 3. We remark that, even in a simple case where $A = [0, 1]$, G may not be arcwise connected nor simply connected.

THEOREM 3. *Let A be a convex compact subset of a (real or complex) Hausdorff locally convex linear topological space E, and let X be a compact Hausdorff topological space. If G is a closed subset of $X \times A$ such that G_x is non-empty and convex for each x in X, and if U is an open neighbourhood of G in $X \times A$, then there is a continuous map $h : X \to U$ such that $p \circ h$ is the identity map, where p denotes the projection of $X \times A$ onto X.*

Proof. By restricting the domain of the multiplication by scalars, we can always make a complex linear topological space into a real linear topological space. Therefore, without loss of generality, we can assume that E is a real linear topological space.

Let $\mathfrak{V} = \{V : V \text{ is open and } G \subset V \subset U\}$; then $\mathfrak{V}$ is directed by $\subset$. We prove first that, for some $\tilde{V}$ in $\mathfrak{V}$, $\langle \tilde{V}_x \rangle \subset U_x$ for each x in X. Assume that no such V exists; then, for each V in $\mathfrak{V}$, there are points x_V and z_V such that $x_V \in X$ and $z_V \in \langle V_{x_V} \rangle \sim U_{x_V}$. Since X and A are compact there are converging subnets of $\{x_V, V \in \mathfrak{V}\}$ and $\{z_V, V \in \mathfrak{V}\}$. More precisely, there is a directed set $\{\Gamma, \geqq\}$ and a function T on Γ into $\mathfrak{V}$ such that, for each V_0 in $\mathfrak{V}$, there is a γ_0 in Γ with the property that $\gamma \geqq \gamma_0$ implies $T(\gamma) \subset V_0$ and furthermore new nets $\{x_{T(\gamma)}, \gamma \in \Gamma\}$ and $\{z_{T(\gamma)}, \gamma \in \Gamma\}$ converge to, say, x_0

and z_0 respectively. We assert that $z_0 \in G_{x_0}$. If not, by a standard separation theorem, there is a continuous linear functional f on E such that

$$\sup\{f(y) : y \in G_{x_0}\} < f(z_0)$$

(see, for instance (**1**, p. 22)). Pick a real number a so that

$$\sup\{f(y) : y \in G_{x_0}\} < a < f(z_0).$$

Let $W = \{y : y \in A \text{ and } f(y) < a\}$, then W is an open convex neighbourhood of G_{x_0} in A. There exists a neighbourhood N of x_0 such that† $G_x \subset W$ for each $x \in N$. Choose an open neighbourhood N_1 of s_0 such that $N_1^- \subset N$, and let

$$M = (N \times W) \cup (X \sim N_1^-) \times A.$$

Then M is an open neighbourhood of G in $X \times A$; hence one can choose γ_0 in Γ so that $\gamma \geqslant \gamma_0$ implies

$$x_{T(\gamma)} \in N_1 \quad \text{and} \quad T(\gamma) \subset M.$$

For simplicity, we shall write $x(\gamma)$ for $x_{T(\gamma)}$. Then, if $\gamma \geqslant \gamma_0$,

$$T(\gamma)_{x(\gamma)} \subset M_{x(\gamma)} = (N \times W)_{x(\gamma)} = W,$$

from which it follows that

$$z_{T(\gamma)} \in \langle T(\gamma)_{x(\gamma)} \rangle \subset \langle W \rangle = W.$$

Hence $\gamma \geqslant \gamma_0$ implies that $f(z_{T(\gamma)}) < a$, and, since $\lim\{z_{T(\gamma)}, \gamma \in \Gamma\} = z_0$, $f(z_0) \leqslant a$, which contradicts our choice of a. Therefore, we must accept that $z_0 \in G_{x_0}$.

By our choice of x_V and z_V, $(x_V, z_V) \notin U$ for each V in $\mathfrak{V}$. Therefore, for each γ in Γ, $(x_{T(\gamma)}, z_{T(\gamma)}) \notin U$, and, since U is open, it follows that $(x_0, z_0) \notin U$. Hence $(x_0, z_0) \notin G$ or $z_0 \notin G_{x_0}$, which contradicts the conclusion of the last paragraph. This establishes that there is a member $\tilde{V}$ in $\mathfrak{V}$ such that, for each x in X, $\langle \tilde{V}_x \rangle \subset U_x$. For each $y \in A$, let $W(y) = \{x : (x, y) \in \tilde{V}\}$. Then $W(y)$ is an open subset of X and $\cup\{W(y) ; y \in A\} = X$. Since X is compact there are points $y_1, y_2, \ldots, y_k$ in A such that $W(y_1) \cup W(y_2) \cup \ldots \cup W(y_k) = X$. Hence there are continuous functions $h_1, \ldots, h_k$ on X into $[0, 1]$ such that $\sum_{i=1}^k h_i(x) = 1$ for all x and $h_i(x) = 0$ if $x \notin W(y_i)$. (See, for instance (**5**, 5.W, p. 171).) Set $h(x) = (x, \sum_{i=1}^k h_i(x) y_i)$. Then clearly $p \circ h(x) = x$ for each x in X. Since $h_i(x) \neq 0$ implies that $y_i \in \tilde{V}_x$, $\sum_{i=1}^k h_i(x) y_i \in \langle \tilde{V}_x \rangle \subset U_x$. Consequently, $h(x) \in U$ for all x in X, and Theorem 3 is proved.

Proof of Theorem 2. Assume that q is not onto. Then since the image of q is closed, there is a point x_0 in $\Delta_n \sim \Delta^{\cdot}_n$ which is not in the image‡ of q. Let

†This property of the point-set transformation $x \to G_x$ is known as the *upper semi-continuity* and is a consequence of the fact that G is closed and A is compact. See, for example, (**3**, Lemma 2, p. 123).

‡See preceding footnote †.

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r be a continuous map on $\Delta_n \sim \{x_0\}$ onto $\Delta^{\cdot}_n$ such that $r(x) = x$ for each x in $\Delta^{\cdot}_n$. (For instance, r can be defined by the radial projection from x_0.) Let $f = r \circ q$; then f is a continuous map of G into $\Delta^{\cdot}_n$ such that $f[G \cap p^{-1}[\sigma]] \subset \sigma$ for each face σ of Δ_n.

Let K^m be the union of all faces of Δ_n of dimensions $\leqslant m$, and let $G^m = G \cup p^{-1}[K^m]$; then G^m is a closed subset of $\Delta_n \times A$ with the property that $(G^m)_x$ is convex for each x in Δ_n. For convenience $G^m = G$ if $m = -1$. By induction on m, $m \leqslant n - 1$, a continuous map f^m on G^m into $\Delta^{\cdot}_n$ will be defined so that $f^{m-1} = f^m|G^{m-1}$ and, for each face σ of Δ_n, $f^m[G^m \cap p^{-1}[\sigma]] \subset \sigma$. For $m = -1$, we take $f^{-1} = f$. Now assume that f^m has been defined $(-1 < m < n - 1)$. Let σ be an $(m + 1)$-dimensional face. Then $p^{-1}[\sigma] = \sigma \times A$ is a normal space, and f^m maps $G^m \cap p^{-1}[\sigma]$ into σ. Since σ is solid, $f^m|G^m \cap p^{-1}[\sigma]$ can be extended to a map f_σ^{m+1} on $p^{-1}[\sigma]$ into σ. If σ and τ are two distinct $(m + 1)$-dimensional faces, then $\sigma \cap \tau \subset K^m$. Hence f_σ^{m+1} and f_τ^{m+1} agree on $p^{-1}[\sigma] \cap p^{-1}[\tau] = p^{-1}[\sigma \cap \tau] \subset G^m$. Therefore, we can define f^{m+1} as follows: $f^{m+1}(x) = f^m(x)$ if $x \in G^m$, and $f^{m+1}(x) = f_\sigma^{m+1}(x)$ if $x \in p^{-1}[\sigma]$ and σ is an $(m + 1)$-dimensional face of Δ_n. It is clear that $f^{m+1}[G^{m+1} \cap p^{-1}[\sigma]] \subset \sigma$ for each face σ of Δ_n. Now f^{n-1} maps G^{n-1} into $\Delta^{\cdot}_n$ and $\Delta^{\cdot}_n$ is an absolute neighbourhood retract (for the definition of absolute neighbourhood retract and the relevant facts used in this proof see (**4**, I, Ex. C, J, and L)); therefore, there is an open neighbourhood U of G^{n-1} in $\Delta_n \times A$ and an extension $\bar{f}$ of f^{n-1} on U into $\Delta^{\cdot}_n$. Now by Theorem 3, there is a map h on Δ_n into U such that $p \circ h$ is the identity map. Set $g = \bar{f} \circ h$; then g is a continuous map on Δ_n into $\Delta^{\cdot}_n$ such that $g[\sigma] \subset \sigma$ for each face σ of Δ_n. For, if σ is an m-dimensional face and $m < n$, then $h[\sigma] \subset p^{-1}[\sigma] \subset G^{n-1}$ and hence $g[\sigma] = \bar{f}[h[\sigma]] \subset f^{n-1}[p^{-1}[\sigma]] \subset \sigma$. If σ is n-dimensional, then trivially $g[\sigma] \subset \sigma$. But by Theorem 1, g is necessarily onto, which contradicts the statement $g[\Delta_n] \subset \Delta^{\cdot}_n$. Therefore, q must map G onto Δ_n, and the proof of theorem is complete.

A real-valued function f on a convex subset A of a real or complex linear space is called *convex* (resp. *concave*) if

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda) f(y)$$

(resp. $f(\lambda x) + (1 - \lambda)y) \geqslant \lambda f(x) + (1 - \lambda) f(y))$, whenever $x, y \in A$ and $0 \leqslant \lambda \leqslant 1$. The function f is called *quasi-convex* (resp. *quasi-concave*) if the set $\{x : f(x) \leqslant r\}$ (resp. $\{x : f(x) \geqslant r\}$) is convex for each real number r. A convex (resp. concave) function is necessarily quasi-convex (resp. quasi-concave). A real-valued function h on a subset P of the Euclidean $(n + 1)$-dimensional space is *non-decreasing*, if $(x_0, \ldots, x_n)$, $(x_0', \ldots, x_n') \in P$ and $x_i \leqslant x_i'$ for $i = 0, 1, \ldots, n$, then

$$h(x_0, \ldots, x_n) \leqslant h(x_0', \ldots, x_n').$$

A real valued function is *strictly-positive* if the range is contained in the interval $(0, \infty)$.

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THEOREM 4. *Let A be a convex compact subset of a Hausdorff locally convex linear topological space, let $f_0, \ldots f_n$ be strictly-positive continuous convex functions on A, and let h be a non-decreasing continuous quasi-convex function on the subset $\{(x_0, \ldots, x_n) : x_i \geqslant 0\}$ of the $(n + 1)$-dimensional Euclidean space. Then there are positive numbers $\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_n$ and a point x_0 in A such that $\sum_{i=0}^{n} \bar{\lambda}_i = 1$, $h(\bar{\lambda}_0 f_0(x_0), \ldots, \bar{\lambda}_n f_n(x_0)) = \inf\{h(\bar{\lambda}_0 f_0(x), \ldots, \bar{\lambda}_n f_n(x)) : x \in A\}$, and $\bar{\lambda}_0 f_0(x) = \bar{\lambda}_1 f_1(x_0) = \ldots = \bar{\lambda}_n f_n(x_0)$.*

THEOREM 4′. *Let A be a convex compact subset of a Hausdorff locally convex linear topological space, let $f_0, \ldots, f_n$ be strictly-positive continuous concave functions on A, and let h be a non-decreasing continuous quasi-concave function on the subset $\{(x_0, \ldots, x_n) : x_i \geqslant 0\}$ of the $(n + 1)$-dimensional Euclidean space. Then there are positive numbers $\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_n$ and a point x_0 in A such that $\sum_{i=0}^{n} \bar{\lambda}_i = 1$, $h(\bar{\lambda}_0 f_0(x_0), \ldots, \bar{\lambda}_n f_n(x_0)) = \sup\{h(\bar{\lambda}_0 f_0(x), \ldots, \bar{\lambda}_n f_n(x)) : x \in A\}$, and $\bar{\lambda}_0 f_0(x) = \bar{\lambda}_1 f_1(x_0) = \ldots = \bar{\lambda}_n f_n(x_0)$.*

Proof. Since the proof of Theorem 4′ is completely analogous to that of Theorem 4, it suffices to prove Theorem 4. For $\lambda = (\lambda_0, \ldots, \lambda_n) \in \Delta_n$, let $m_\lambda = \inf\{h(\lambda_0 f_0(x), \ldots, \lambda_n f_n(x)) : x \in A\}$ and $M_\lambda = \{x : h(\lambda_0 f_0(x), \ldots, \lambda_n f_n(x)) = m_\lambda\}$. Then, for each λ, M_λ is non-empty and convex. It is convex, because, if $x, x' \in M_\lambda$, $0 \leqslant \mu, \mu' \leqslant 1$ and $\mu + \mu' = 1$, then $m_\lambda \leqslant h(\lambda_0 f_0(\mu x + \mu' x'), \ldots) \leqslant h(\mu\lambda_0 f_0(x) + \mu'\lambda_0 f_0(x'), \ldots) \leqslant m_\lambda$. Let M be a subset of $\Delta_n \times A$ defined by $M = \bigcup\{\{\lambda\} \times M_\lambda : \lambda \in \Delta_n\}$; then we assert that M is closed in $\Delta_n \times A$. For, if $(\lambda, x) \in \Delta_n \times A \sim M$ (that is, $x \in A \sim M_\lambda$), then there is a number r such that $h(\lambda_0 f_0(x), \ldots, \lambda_n f_n(x)) > r > m_\lambda$. Choose neighbourhoods U and V of λ and x respectively so that $\lambda' \in U$ and $x' \in V$ imply that

$$h(\lambda_0' f_0(x'), \ldots, \lambda_n' f_n(x')) > r > m_{\lambda'}.$$

Then the neighbourhood $U \times V$ of (λ, x) is disjoint from M; hence M is closed.

Now define a continuous map q on M into Δ_n by $q(\lambda, x) = (\sum_{i=0}^{n} \lambda_i f_i(x))^{-1} (\lambda_0 f_0(x), \ldots, \lambda_n f_n(x))$. By Theorem 2, q maps M onto Δ_n; in particular, there is a point $(\bar{\lambda}, x_0)$ in M such that

$$q(\bar{\lambda}, x_0) = \frac{1}{n+1}(1, \ldots, 1),$$

from which the theorem follows.

ADDENDUM TO THEOREM 4′—THEOREM 4″. *If in Theorem 4′ we assume, in addition, that $h(x_0, \ldots, x_n) = 0$ if and only if $x_0 = \ldots = x_n = 0$, then the assumption that $f_0, \ldots, f_n$ be strictly positive can be replaced by that $f_0, \ldots, f_n$ be non-negative and not identically zero.*

Proof of 4″. The only place in the proof of Theorem 4 (and 4′) where the assumption of strict positiveness of $f_0, \ldots, f_n$ is used is in the definition of the map q. Let $\lambda = (\lambda_0, \ldots, \lambda_n)$ be an arbitrary point in Δ_n, then because of the new assumption on h,

$$\sup\{h(\lambda_0 f_0(x), \ldots, \lambda_n f_n(x)) : x \in A\} = m_\lambda > 0.$$

Therefore, if x is a point in A such that $h(\lambda_0 f_0(x), \ldots, \lambda_n f_n(x)) = m_\lambda$, then $\sum_{i=0}^{n} \lambda_i f_i(x) > 0$. Hence under the new set of conditions the map q can still be defined.

COROLLARY. *Let X be a compact Hausdorff space, and let $f_0, \ldots, f_n$ be non-negative and not identically zero continuous functions on X. Then there are positive numbers $\lambda_0, \ldots, \lambda_n$ and a positive Baire measure μ of total mass* 1 *such that the function $x \to g(x) = \lambda_0 f_0(x) + \lambda_1 f_1(x) + \ldots + \lambda_n f_n(x)$ is almost everywhere* $[\mu]$ *equal to* $\sup\{g(x) : x \in X\}$ *and*

$$\lambda_0 \int f_0 d\mu = \lambda_1 \int f_1 d\mu = \ldots = \lambda_n \int f_n d\mu.$$

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