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CERTAIN ARTINIAN RINGS ARE NOETHERIAN

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1. Introduction. Throughout this paper the word "ring" will mean an associative ring which need not have an identity element. There are Artinian rings which are not Noetherian, for example $C(p^{\infty})$ with zero multiplication. These are the only such rings in that an Artinian ring R is Noetherian if and only if R contains no subgroups of type $C(p^{\infty})$ [**1**, p. 285]. However, a certain class of Artinian rings is Noetherian. A famous theorem of C. Hopkins states that an Artinian ring with an identity element is Noetherian [**3**, p. 69]. The proofs of these theorems involve the method of "factoring through the nilpotent Jacobson radical of the ring". In this paper we state necessary and sufficient conditions for an Artinian ring (and an Artinian module) to be Noetherian. Our proof avoids the concept of the Jacobson radical and depends primarily upon the concept of the length of a composition series. As a corollary we obtain the result of Hopkins.

All conditions are on the right, that is, an Artinian ring means a right Artinian ring, etc. We say a module A is *embeddable* in a module B provided that A is isomorphic to some submodule of B. Our Theorem 3 states that *an Artinian ring R is Noetherian if and only if a cyclic Noetherian submodule of R is embeddable in some Noetherian factor module of R*. An example in the paper shows that in an Artinian ring R which is also Noetherian, a cyclic submodule need not be isomorphic to some factor module of R. However in a ring R with 1 a cyclic submodule xR of R is always isomorphic to a factor module of R, namely $xR \cong R/r(x)$ where $r(x) = \{y \in R: xy = 0\}$. Hence from Theorem 3 an Artinian ring with 1 is Noetherian.

Recall that a module has a composition series if and only if it is both Artinian and Noetherian [**3**, p. 23]. Suppose a module M has a composition series. Then there does exist a finite sequence of submodules $M_0, M_1, \ldots, M_n$ such that $(0) = M_0 \subset M_1 \subset \ldots \subset M_n = M$ and each factor module M_i/M_{i-1} is a nonzero simple module for $1 \leqq i \leqq n$. The integer n is called the length of the composition series for the module M. The length is well defined (Jordan-Hölder Theorem, [**3**, p. 21]) and we write $L(M) = n$.

In an Artinian module a Noetherian submodule has a composition series. *For an Artinian module M the following three conditions are equivalent*: (1) *M is Noetherian*; (2) *there does exist a cyclic Noetherian submodule A of M such that $L(B) \leqq L(A)$ for any cyclic Noetherian submodule B of M*; (3) *there is a bound for the set of lengths of the composition series for the cyclic Noetherian submodules of M* (Theorem 1). There is always a bound for the set of lengths of the composi-

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tion series for the Noetherian factor modules of an Artinian ring (Proposition 2). We use Theorem 1 and Proposition 2 to prove that an Artinian ring R is Noetherian provided that each cyclic Noetherian submodule of R is embeddable in some Noetherian factor module of R. In a more general setting we have Theorem 7. For a module M over an Artinian ring R let each cyclic submodule of M be embeddable in some Noetherian factor module of R. The following conditions are equivalent: (1) M is Noetherian; (2) M is Artinian; (3) each factor module of M does not contain an infinite direct sum of nonzero submodules.

2. Artinian modules. Let M denote a module over a ring R. For $m \in M$ we equate mR' with the (cyclic) submodule generated by m. We denote the *socle* of M by $s(M)$. Recall, if B is a submodule of M and both B and M/B are Noetherian then M is Noetherian [**3**, p. 22]. We use this fact below.

THEOREM 1. *An Artinian module M is Noetherian if and only if there is a bound for the set of lengths of the composition series for the cyclic Noetherian submodules of M.*

Proof. If a module M is Artinian and Noetherian then $L(B) \leqq L(M)$ for any submodule B of M. We prove the converse. Let M denote an Artinian module over a ring R. We define a recursive sequence $\{K_n\}$ of submodules as follows: let $K_1 = s(M)$ and let K_{n+1} be the submodule K_n with the property that $s(M/K_n) = K_{n+1}/K_n$. The socle is finitely generated (hence Noetherian) in a factor module of an Artinian module. Therefore, K_1 and K_2/K_1 are Noetherian. It follows that K_2 is Noetherian and also each K_i is Noetherian for $i \geqq 1$. If $K_j = K_{j+1}$ for some j, then $s(M/K_j) = (0) + K_j/K_j$ and $M = K_j$. Thus M is Noetherian. To complete the proof we will show that the condition $K_n \subset K_{n+1}$ for all $n \geqq 1$ leads to contradiction. It suffices to prove that $x \in K_{n+1} - K_n$ for all $n \geqq 1$ implies $L(xR') \geqq n + 1$ for this would contradict the hypothesis. We argue by induction. If $x \in K_1 - (0)$ then $L(xR') \geqq 1$. Assume $y \in K_n - K_{n-1}$ implies $L(yR') \geqq n$. Let $p \in K_{n+1} - K_n$. There is a $q \in pR'$ such that $q \in K_n - K_{n-1}$ because $s(M/K_{n-1})$ is essential in the Artinian module M/K_{n-1}. Hence $pR' \neq qR'$ and $L(qR') \geqq n$ by the induction step. Therefore, $L(pR') \geqq n + 1$ and this completes the proof.

3. Artinian Rings.

PROPOSITION 2. *There is a bound for the set of lengths of the composition series for the Noetherian factor modules of an Artinian ring.*

Proof. Let R denote an Artinian ring and let $H = \{K$: K is a right ideal in R and R/K is a Noetherian module$\}$. Suppose A and B are in H. We claim that the intersection $A \cap B$ is in H. Clearly R/A is Noetherian implies the submodule $A + B/A$ is Noetherian. Therefore, $B/A \cap B$ which is isomorphic to $A + B/A$ and R/B are Noetherian and thus $R/A \cap B$ is Noetherian. Hence $A \cap B$ is in H. Since R is Artinian there does exist a minimal element K in H. If D is in H then

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$D \cap K$ is in H and $D \supseteq K$ by the minimality of K. We conclude $L(R/D) \leqq L(R/K)$ for all D in H. This completes the proof.

THEOREM 3. *An Artinian ring R is Noetherian if and only if each cyclic Noetherian submodule of R is embeddable in some Noetherian factor module of R.*

Proof. Let R denote an Artinian and Noetherian ring. For $x \in R$ the cyclic submodule xR' is isomorphic to the submodule $xR' + (0)/(0)$ of the Noetherian factor module $R/(0)$. For the converse, the hypothesis and Proposition 2 imply that there is a bound for the set of lengths of the composition series for the cyclic Noetherian submodules of the ring. Theorem 1 forces the Artinian ring to be Noetherian.

We point out that if a ring is Artinian and Noetherian then a cyclic submodule need not be isomorphic to some factor module of the ring. Let R denote the ring generated by a and b subject to $a^2 = b^2 = a + a = b + b = ab + ab = 0$ and $ab = ba \neq 0$. Thus $R = \{0, a, b, ab, a + b, a + ab, a + b + ab, b + ab\}$. The submodule $abR' = A$ is the only submodule with two elements and R/A is not cyclic. The cyclic submodule aR' has four elements and hence aR' is not isomorphic to R/A.

COROLLARY 4. *An Artinian ring, in which a principal right ideal is isomorphic to a factor module of the ring, is also a Noetherian ring.*

Proof. The proof follows from Theorem 3.

COROLLARY 5. (Hopkins) *An Artinian ring with* 1 *is Noetherian.*

Proof. The proof follows from Corollary 4.

4. Modules over Artinian rings. It is known that a module over an Artinian ring with 1 is Noetherian if and only if it is Artinian [**3**, p. 69]. We give a similar result for modules over Artinian rings without 1.

Recall a module is said to be (Goldie) *finite dimensional* if it does not contain an infinite direct sum of nonzero submodules.

LEMMA 6. *A module is Artinian if and only if each cyclic submodule is Artinian and each factor module is finite dimensional.*

Proof. The necessary condition is clear. In order to prove the sufficiency, suppose that M is a module with the properties of the hypothesis. The proof is by contradiction. Suppose $A_1 \supset A_2 \supset \ldots$ is a strictly decreasing sequence of submodules of M and let $A = \bigcap A_i$ for all $i \geqq 1$. In $M/A = \bar{M}$ we have $s(\bar{M}) \cap \bar{A}_i \neq (\bar{0})$ for all $i \geqq 1$ because cyclic modules in $\bar{M}$ are Artinian and thus contain simple submodules. Recall that factor modules are finite dimensional. Thus, $s(\bar{M})$ is Artinian and the sequence $s(\bar{M}) \cap \bar{A}_1 \supseteq s(\bar{M}) \cap A_2 \ldots$ becomes stationary. For some integer n we have $s(\bar{M}) \cap \bar{A}_n = s(\bar{M}) \cap \bar{A}_{n+1} \neq (\bar{0})$ for all $i \geqq 1$. This contradicts the fact that $\bigcap A_i = A$ for all $i \geqq 1$. Thus we conclude that the module is Artinian.

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THEOREM 7. *For a module M over an Artinian ring R let each cyclic submodule of M be embeddable in some Noetherian factor module of R. The following conditions are equivalent:*

(1) *The module M is Artinian.*

(2) *The module M is Noetherian.*

(3) *Each factor module is finite dimensional in M.*

Proof. The implication, (1) implies (2), follows directly from Proposition 2 and Theorem 1. The implication, (2) implies (3), is clear. We show (3) implies (1). The hypothesis forces each cyclic module to be Artinian. The result follows from Lemma 6.

COROLLARY 8. (Hopkins [**3**, p. 69]). *A unital module over an Artinian ring with* 1 *is Artinian if and only if it is Noetherian.*

Proof. The proof follows from Theorem 7.

Let M denote an arbitrary R-module. Let $M(p^k)$ be the collection of all elements of the additive subgroup M whose orders are at most p^k where p is a prime and k is a positive integer. Clearly, $M(p^k)$ is also a submodule and $M(p^k) \subseteq M(p^{k+1})$. It follows that if M is a Noetherian module, then M has no subgroups of type p^∞.

PROPOSITION 9 [**1**, Theorems 73.2 and 73.3]. *Let R be an Artinian ring. The following conditions are equivalent:*

(1) *R is Noetherian.*

(2) *R contains no subgroups of type p^∞.*

(3) *R can be embedded into an Artinian ring with an identity in such a manner that its right ideals remain right ideals of the new ring.*

Proof. The implication, (1) implies (2), is clear by the remark preceding this proposition. The remark that follows Theorem 73.2 of [**2**, p. 285] shows that (2) implies (3). If (3) holds, then R is Noetherian via Corollary 5 and hence (1) holds. This completes the proof.

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