ISOMORPHISMS OF MULTIPLIER ALGEBRAS

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1. Introduction. Let A and B be semisimple Banach algebras, and let $M_l(A)$ (resp. $M_l(B)$) be the algebra of left multipliers on A (resp. B). Suppose that A is an abstract Segal algebra in B. We find conditions on A and B which imply that $M_l(A)$ is topologically algebra isomorphic to $M_l(B)$. As a special case we obtain the result of [**8**] which states that if A is an A^*-algebra that is a *-ideal in its B^*-algebra completion B and A^2 is dense in A then $M_l(A)$ is topologically algebra isomorphic to $M_l(B)$. We make an application of our main result to right complemented Banach algebras.

2. Preliminaries. Let A be a semisimple Banach algebra. A linear mapping $T:A\to A$ is called a left multiplier if $T(xy)=T(x)y$, for all $x, y\in A$. Let $M_l(A)$ be the algebra of all left multipliers on A. Since every left multiplier on A is continuous [**7**], $M_l(A)$ is a Banach algebra under the operator bound norm. For each $a\in A$, let L_a be the operator given by $L_a(x)=ax$, for all $x\in A$. Then $L_a\in M_l(A)$, for all $a\in A$, and the mapping $a\to L_A$ is a norm-decreasing algebra isomorphism of A into $M_l(A)$. In what follows we will identify A as a subalgebra of $M_l(A)$.

PROPOSITION 2.1. *Let A be a semisimple Banach algebra. Let B be a closed subalgebra of $M_l(A)$ which contains A. Then*

(i) *A is a left ideal of B,*

(ii) *B is a semisimple Banach algebra.*

Proof. (i) Let $T\in B$ and $a\in A$. Then $T(a)\in A$ and $TL_a(x)=T(ax)=T(a)x=L_{T(a)}(x)$, for all $x\in A$. Hence, $TL_a=L_{T(a)}$.

(ii) Let J be the radical of B. Since A is a left ideal of B, $J\cap A$ is also a left ideal of B. Every $x\in J\cap A$ is left quasi-regular in B and so has a left quasi-inverse in A. Therefore $J\cap A=(0)$ as A is semisimple, so that also $JA=(0)$. Let $T\in J$. Then $0=TL_x=L_{T(x)}$. From the semisimplicity of A we see that $T(x)=0$ for all x in A and so $T=0$. This completes the proof.

Let L_A be the closure of A in $M_l(A)$. By Proposition 2.1, L_A is a semisimple Banach algebra and contains A as a dense left ideal. In the terminology of [**9**], A is an abstract Segal algebra in L_A. We call L_A the left regular representation of A.

NOTATION. Let A and B be Banach algebras such that A is an abstract Segal algebra in B. We will denote the norm on $A(B)$ *by* $\|\cdot\|_A$ ($\|\cdot\|_B$). If T is a linear map on B, then $T\mid A$ will denote the restriction of T to A.

PROPOSITION 2.2. *Let A be an abstract Segal algebra in a C^*-algebra B. Then there exists a topological algebra isomorphism of B onto L_A which maps a onto L_a, for all $a\in A$.*

Proof. By [**9**, p. 303, Theorem 3.3], A is semisimple and therefore the mapping

$a \to L_a$ is an algebra isomorphism of A into L_A. By [**9**, p. 299, Theorem 1.6], there exists a constant $C>0$ such that $\|ba\|_A \leq C\,\|b\|_B\,\|a\|_A$, for all $b \in B$ and $a \in A$. This shows that $\|L_b\| \leq C\,\|b\|_B$ for all $b \in A$. Hence the mapping $\varphi\colon a \to L_a$ of A with the norm $\|\cdot\|_B$ into L_A is continuous and extends to a continuous algebra homomorphism ψ of B into L_A. Let K be the kernel of ψ. Then K is a closed ideal of B and $K \cap A = (0)$. Since $KA \subset K \cap A$, $KA=(0)$ and therefore $KB=(0)$. By the semisimplicity of B, $K=(0)$ and so ψ is an isomorphism. By [**4**, p. 1104, Lemma 5.3], $\psi(B)$ is closed in L_A. Since $\psi(B)$ is dense in L_A, we obtain $\psi(B)=L_A$. Thus ψ is onto L_A and therefore bicontinuous. Clearly, $\psi(a)=L_a$, for all $a \in A$.

For a more complete treatment of L_A see [**13**].

3. Main result. We first prove the following.

PROPOSITION 3.1. *Let A be a semisimple Banach algebra. Then every left multiplier S on A has a unique extension to a left multiplier T on L_A and $\|T\| \leq \|S\|$.*

Proof. For convenience of notation let $B=L_A$. Then, for every $a \in A$,

$$\|S(a)\|_B = \|L_{S(a)}\| = \|SL_a\| \leq \|S\|\,\|L_a\| = \|S\|\,\|a\|_B.$$

Therefore S is bounded on A with respect to the norm $\|\cdot\|_B$ and so has a unique extension T to B with $\|T\| \leq \|S\|$. Clearly, $T \in M_l(B)$ and $T \mid A = S$.

A left (right) approximate identity $\{u_\alpha\}$ in a Banach algebra A is said to be *quasi-bounded* if the set $\{L_{u_\alpha}\}$ is bounded in L_A. It is easy to see that if $\{u_\alpha\}$ is a quasi-bounded left (right) approximate identity in A then $\{L_{u_\alpha}\}$ is a bounded left (right) approximate identity in L_A.

THEOREM 3.2. *Let A be a semisimple Banach algebra with a quasi-bounded left approximate identity. Then $M_l(A)$ is topologically algebra isomorphic to $M_l(L_A)$.*

Proof. For convenience of notation, let $B=L_A$. Let $T \in M_l(B)$. Since A has a quasi-bounded left approximate identity, we see that A^2 is dense in A and therefore, by the Hewitt–Cohen factorization theorem [**6**, p. 268, Theorem 32.22], $A = B \cdot A = \{ba : b \in B \text{ and } a \in A\}$. Hence $T(A) \subseteq A$ and so $T \mid A \in M_l(A)$ [**8**, p. 316]. Let $T' = T \mid A$, and let $\{u_\alpha\}$ be a quasi-bounded left approximate identity in A. Since T is continuous, there is a constant $D>0$ such that $\|T(u_\alpha)\|_B \leq D$ for all α. By [**9**, p. 299, Proposition 1.6], there exists a constant $C>0$ such that $\|ba\|_A \leq C\,\|b\|_B\,\|a\|_A$ for all $b \in B$ and $a \in A$. Thus, for each $a \in A$,

$$\begin{aligned}\|T'(a)\|_A &= \lim_\alpha \|T'(u_\alpha a)\|_A = \lim_\alpha \|T(u_\alpha)a\|_A \\ &\leq \sup_\alpha C\,\|T(u_\alpha)\|_B\,\|a\|_A \leq CD\,\|T\|\,\|a\|_A\end{aligned}$$

whence $\|T'\| \leq CD\,\|T\|$. Now, by Proposition 3.1, every $S \in M_l(A)$ has a unique extension T to B, $T \in M_l(B)$ and $\|T\| \leq \|S\|$. Hence the mapping $T \to T'$ is a continuous algebra isomorphism from $M_l(L_A)$ onto $M_l(A)$.

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COROLLARY 3.3. *Let A be an abstract Segal algebra in B. Assume that* (i) *A^2 is dense in A and* (ii) *B is semisimple and has a bounded left approximate identity contained in A. If B is topologically algebra isomorphic to L_A (in the sense of Proposition* 2.2), *then $M_l(A)$ is topologically algebra isomorphic to $M_l(B)$.*

Proof. Let $\{u_\alpha\}$ be a bounded left approximate identity of B contained in A. Since A^2 is dense in A, by [**3**, p. 5, Proposition 3.3], $\{u_\alpha\}$ is a left approximate identity of A. If the mapping ψ of Proposition 2.2 takes B onto L_A, then $\{u_\alpha\}$ is also a quasi-bounded left approximate identity of A and $M_l(L_A)$ is topologically algebra isomorphic to $M_l(B)$. The conclusion now follows from Theorem 3.2.

COROLLARY 3.4. *Let A be a Banach algebra which is a dense two-sided ideal in a B^*-algebra B. Assume that A^2 is dense in A. Then $M_l(A)$ is topologically algebra isomorphic to $M_l(B)$.*

Proof. By [**5**, p. 15, 1.7.1], B has a bounded approximate identity contained in A. We may now apply Proposition 2.2 and Corollary 3.3 to complete the proof.

COROLLARY 3.5. *Let A be an A^*-algebra of the first kind and let B be its B^*-algebra completion. Then $M_l(A)$ is topologically algebra isomorphic to $M_l(B)$.*

We will now consider an application of Theorem 3.2 to right complemented Banach algebras. For the definition and basic properties of right (left) complemented Banach algebras see [**11**]. (See also [**1**], [**14**]).

THEOREM 3.6. *Let A be a semisimple annihilator right complemented Banach algebra. Then A has a quasi-bounded left approximate identity.*

Proof. Let p denote the right complementor on A and let $\{e_\alpha : \alpha \in \Omega\}$ be a maximal family of mutually orthogonal minimal p-projections in A. We recall that an idempotent e in A is called a minimal p-projection if e is a minimal idempotent and $(eA)^p = (1-e)A$. Since A is an annihilator algebra, every non-zero closed right ideal of A contains a minimal p-projection. Moreover, if e is a minimal p-projection in a closed right ideal I and f is a minimal p-projection in I^p then $ef = fe = 0$ (see [**11**, p. 654]). It follows that the family $\{e_\alpha A : \alpha \in \Omega\}$ has a dense linear span in A and, for each $\alpha' \in \Omega$, $e_{\alpha'}A \cap \mathrm{cl}_A\left(\sum_\alpha e_\alpha A\right) = (0)$. Furthermore, by [**12**, p. 268, Theorem 5.9], for every $x \in A$, $x = \sum_{\alpha \neq \alpha'} e_\alpha x$, where convergence is with respect to the net of finite partial sums. Thus the family $\{e_\alpha A : \alpha \in \Omega\}$ forms an unconditional decomposition for A and, in particular, the directed set E of all finite sums $e_{\alpha_1} + \ldots + e_{\alpha_n}$, $\alpha_1, \ldots, \alpha_n \in \Omega$, is a left approximate identity of A. Therefore, by [**2**, p. 231, Theorem 3.4], there exists a constant $K > 0$ such that, for any $\alpha_1, \ldots, \alpha_n \in \Omega$,

$$\left\| \sum_{i=1}^{n} e_{\alpha_i} x \right\| \leq K \, \|x\|$$

for all $x \in A$. Hence if we let $L_\alpha = L_{e_\alpha}$, $\alpha \in \Omega$, then the set of all finite sums $L_{\alpha_1} + \ldots +$

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L_{α_n}, $\alpha_1, \ldots, \alpha_n \in \Omega$, is bounded and consequently E is a quasi-bounded left approximate identity of A.

Corollary 3.7. *Let A be a semisimple annihilator right complemented Banach algebra. Then $M_l(A)$ is topologically algebra isomorphic to $M_l(L_A)$.*

Proof. This follows immediately from Theorems 3.2 and 3.6.

Theorem 3.8. *Let A be a topologically simple, semisimple annihilator right complemented Banach algebra. Then there exists a Hilbert space H such that $M_l(A)$ is topologically algebra isomorphic to $L(H)$, the algebra of all bounded linear operators on H.*

Proof. By [**1**, p. 40, Theorem 1], A can be continuously embedded as an abstract Segal algebra in the algebra $LC(H)$ of all compact linear operators on a Hilbert space H. (If A is finite dimensional then H is finite dimensional and the embedding is onto $LC(H) = L(H)$.) Since $LC(H)$ is a B^*-algebra, by Proposition 2.2, L_A is topologically algebra isomorphic to $LC(H)$. Therefore, by Corollary 3.7, $M_l(A)$ is topologically algebra isomorphic to $M_l(LC(H))$. Observing that $M_l(LC(H))$ is topologically algebra isomorphic to $L(H)$ [**10**, p. 506, Lemma 2.1] completes the proof.

Let A be the algebra of trace-class operators or the algebra of Hilbert–Schmidt operators on a Hilbert space H. Then A is a dual A^*-algebra which is a dense *-ideal in $LC(H)$. We note that A is also a topologically simple, semi-simple right complemented Banach algebra. Hence $M_l(A)$ is topologically algebra isomorphic to $M_l(LC(H))$ and consequently topologically algebra isomorphic to $L(H)$.

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