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THE RANGE OF INVARIANT MEANS ON LOCALLY COMPACT ABELIAN GROUPS

BY

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ABSTRACT. It has been shown by E. Granirer that for certain infinite amenable discrete groups G there exists a nested family of left almost convergent subsets of G on which every left invariant mean on $m(G)$ attains as its range the entire [0, 1] interval. This paper examines the range of left invariant means on $L^\infty(G)$ for infinite locally compact abelian groups G and demonstrates the existence in every such group of a nested family of left almost convergent Borel subsets on which every left invariant mean on $L^\infty(G)$ attains as its range the interval [0, 1].

0. **Introduction.** It is a consequence of Liapounoff's Convexity Theorem (see [5]) that for any finite, non-atomic measure μ on a measure space X, the set $\{\mu(A)$: A measurable subset of $X\}$ is closed and convex. Thus for non-atomic probability measures we have $\{\mu(A)$: A measurable subset of $X\}=[0, 1]$.

If φ is a left invariant mean (LIM) on $L^\infty(G)$ where G is a locally compact group, then φ can be considered as a finitely additive probability measure on the Borel subsets of G. It has been shown by the author in [4] that for such a φ, there exists a nested family $\{A(t):t \in [0, 1]\}$ of Borel subsets of G (which depend heavily on the mean φ being considered) with $\varphi(X_{A(t)})=t$ for each $t \in [0, 1]$. The difficulties which arise in proving this result are due to the fact that we have only finite additivity rather than the countably additive measures of Liapounoff's Theorem.

Granirer showed in [1] that when G is a member of a certain class of infinite amenable discrete groups (which includes all infinite discrete abelian groups) there exists a nested family $\{A(t):t \in [0, 1]\}$ of (Borel) subsets of G with $\varphi(X_{A(t)})=t$ for *any* LIM φ on $m(G)$ (the algebra of bounded real-valued functions on G).

In this paper we extend the results of [4] in the case of locally compact abelian groups to show that the nested family can be chosen independently of the mean being considered. This gives us the following

THEOREM A. *Let G be an infinite locally compact abelian group. Then there exists a nested family $\{A(t):t \in [0, 1]\}$ of Borel subsets of G for which $\varphi(X_{A(t)})=t$ for any* LIM *φ on $L^\infty(G)$.*

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In fact this result can be strengthened somewhat to give

THEOREM B. *Let G be an amenable locally compact group such that there exists a closed normal subgroup H of G for which G/H is infinite and abelian. Then there exists a nested family $\{A(t):t \in [0, 1]\}$ of Borel subsets of G with $\varphi(X_{A(t)})=t$ for any* LIM *φ on $L^{\infty}(G)$.*

The reader will note in the body of the paper that the major difficulties in obtaining these results lie in the case where G is a compact abelian group.

REMARKS 1. It is not known whether all infinite *discrete* amenable groups admit such a nested collection of left almost convergent sets. (See next section for terminology.)

2. Using a theorem proved by Wong ([6] Theorem 7.3) it can be shown that the sets $A(t)$ obtained in these theorems have the property that for each $t \in [0, 1]$, the constant function t belongs to the norm closure (in $L^{\infty}(G)$) of $P(G)^*X_{A(t)}$. (Where $P(G)=\{f \in L^1(G):f \geq 0, \int f\,d\mu=1\}$; here μ denotes the left Haar measure on G.)

1. **Preliminaries.**

For most of our terminology we shall follow Hewitt and Ross [2]. R, Z and C will denote the reals, integers and complex numbers and T represents the set

$$\{\exp(2\pi ix) \mid 0 \leq x < 1\} \quad \text{in } C.$$

Let G be a locally compact group. If H is a closed normal subgroup of G then G/H is a locally compact group and π_H will denote the canonical map from G onto G/H. H will be called a direct factor of G if $G=H\times G_1$ for some closed normal subgroup G_1. In this case it is clear that H is topologically isomorphic with G/G_1. If G is a finite group we denote its order by $o(G)$. If G/H is a finite group then the subset $\{b_1, b_2, \ldots, b_n\}$ of G is called a set of representatives for G/H if $b_iH \cap b_jH=\varnothing$ for $i\neq j$ and $G=\bigcup_{i=1}^{n} b_iH$.

If $f \in L^{\infty}(G)$ and $x \in G$ we define $L_xf \in L^{\infty}(G)$ by $L_xf(y)=f(xy)$ for all $y \in G$. An element φ in $L^{\infty}(G)^*$ is called a mean if $\|\varphi\|=1$ and $\varphi(f)\geq 0$ whenever $f\geq 0$ almost everywhere in G. φ is said to be a left invariant mean (LIM) if $\varphi(L_xf)=\varphi(f)$ for all $x \in G, f \in L^{\infty}(G)$. G is amenable if there exists a LIM on $L^{\infty}(G)$. If A is a Borel subset of G then X_A (the characteristic function of A) is in $L^{\infty}(G)$ and for convenience of notation we will write $\varphi(A)$ rather than $\varphi(X_A)$. The reader should note that, since φ is finitely additive, if $A_1, \ldots, A_n$ are disjoint Borel subsets of G we have $\varphi(\bigcup_{i=1}^{n} A_i)= \sum_{i=1}^{n} \varphi(A_i)$. A function $f \in L^{\infty}(G)$ is called left almost convergent to $c \in R$ if $\varphi(f)=c$ for any LIM φ on $L^{\infty}(G)$.

DEFINITION 1.1. If G is a locally compact group and Q is a subset of $[0, 1]$ then a collection $\{A(t):t \in Q\}$ of *Borel* subsets of G is called a nested collection with range Q if

(i) $s<t$ implies $A(s)\subset A(t)$ for all $s, t \in Q$ and
(ii) $\varphi(A(t))=t$ for any LIM φ on $L^{\infty}(G)$ and each $t \in Q$.

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2. Our main purpose is to show that for any infinite, locally compact abelian group G, there exists a nested collection in G with range $[0, 1]$. In this section we will prove that it is sufficient that there exist a closed normal subgroup H of G such that for some countable dense subset Q of $[0, 1]$, we have a nested collection in G/H with range Q.

The following lemma indicates the manner in which nested collections in G/H give rise to collections in G with the same range.

LEMMA 2.1. *Let G be an amenable locally compact group and H a closed, normal subgroup of G. If there exists a nested collection with range Q in G/H then there exists a nested collection with range Q in G.*

Proof. Let us first define a mapping $T:L^\infty(G/H)\to L^\infty(G)$ as follows—for $f\in L^\infty(G/H)$ let $Tf=f\circ\pi_H$. Since A is locally null (with respect to left Haar measure on G/H) iff $\pi_H^{-1}(A)$ is locally null in G (see [3] p. 66) and since $f\circ\pi_H$ is Borel measurable we have $Tf\in L^\infty(G)$ with $\|Tf\|_\infty=\|f\|_\infty$. Clearly $f\to Tf$ is linear so T is a linear isometry from $L^\infty(G/H)$ into $L^\infty(G)$. Thus the adjoint T^* is a linear mapping from $L^\infty(G)^*$ into $L^\infty(G/H)^*$. If φ is a mean on $L^\infty(G)$ then $T^*\varphi$ is clearly a mean on $L^\infty(G/H)$. If, in addition, φ is left invariant it is easily checked that $T^*\varphi$ is a LIM on $L^\infty(G/H)$. (Since $T(L_{xH}f)=L_xTf$ for any $xH\in G/H, f\in L^\infty(G/H)$).

Let $\mathscr{F}=\{A(t):t\in Q\}$ be a nested collection with range Q in G/H and let $B(t)=\pi_H^{-1}(A(t))$ for $t\in Q$. Since π_H is continuous, $B(t)$ is a Borel set in G for each $t\in Q$ and $\mathscr{F}'=\{B(t):t\in Q\}$ is clearly nested. If φ is a LIM on $L^\infty(G)$ then for any $t\in Q$ we have $t=T^*\varphi(X_{A(t)})=\varphi(TX_{A(t)})=\varphi(X_{B(t)})$ since $T^*\varphi$ is a LIM on $L^\infty(G/H)$. Therefore $\mathscr{F}'$ is a nested collection with range Q in G.

The next lemma uses a technique of Granirer in [1] to show that it is sufficient to find a nested collection whose range is countable and dense in $[0, 1]$.

LEMMA 2.2. *Let G be an amenable locally compact group and Q a countable, dense subset of* $[0, 1]$ *such that there exists a nested collection in G with range Q. Then there exists a nested collection in G with range* $[0, 1]$.

Proof. Let $\{A(t):t\in Q\}$ be nested with range Q. For $t\in[0, 1]$ let

$$A'(t)=\bigcap_{\substack{s\in Q\\ t\leq s}} A(s) \quad \text{and} \quad A'(1)=G$$

Since Q is countable we know that $A'(t)$ is a Borel set in G and $\mathscr{F}=\{A'(t):t\in[0, 1]\}$ is clearly nested. If $s_1<t<s_2$ with $s_1, s_2\in Q$ we have $A(s_1)\subseteq A'(t)\subseteq A(s_2)$ so for any LIM φ on $L^\infty(G)$, $s_1=\varphi(A(s_1))\leq\varphi(A'(t))\leq\varphi(A(s_2))=s_2$ and since Q is dense in $[0, 1]$, this implies $\varphi(A'(t))=t$ so $\mathscr{F}$ is nested with range $[0, 1]$.

3. **Nested collections in T, R and Z.** In the next section we will require the following lemma

LEMMA 3.1. *There exists a nested collection with range* [0, 1] *in each of the locally compact abelian groups T, R and Z.*

Proof. (i) Let $Q=\{k/n: 0\leq k\leq n; k, n\in Z^+\}$ and for $k/n\in Q$ define $A(k/n)=\{\exp(2\pi ix):0\leq x<k/n\}$. Then $\mathcal{F}=\{A(k/n):k/n\in Q\}$ is a nested family of Borel subsets of T.

If we let $A_n=\{\exp(2\pi ix):0\leq x<1/n\}$ we have $T=\bigcup_{m=0}^{n-1}\exp(2\pi im/n)\cdot A_n$ (disjoint union) so if φ is a LIM on $L^\infty(T)$ we have $1=\varphi(T)=n\cdot\varphi(A_n)$ and thus $\varphi(A_n)=1/n$. Since $A(k/n)=\bigcup_{m=0}^{k-1}\exp(2\pi im/n)\cdot A_n$ (disjoint union) this implies that $\varphi(A(k/n))=k\cdot\varphi(A_n)=k/n$. Therefore $\mathcal{F}$ is a nested collection in T with range Q and (since Q is a countable dense subset of [0, 1]) by Lemma 2.2 there exists a nested collection in T with range [0, 1].

(ii) Note that Z is a closed normal subgroup of R and $T=R/Z$. By part (i) there exists a nested collection with range [0, 1] in R/Z so by lemma 2.1 such a collection also exists in R.

(iii) Since Z is a discrete right cancellation, left amenable group which is abelian (and hence not an "AB Group" as defined in [1]) the result follows from Theorem 3 of [1].

Note. Combined with the results in section 2 this implies that any locally compact amenable group G with T, R or Z as a direct factor contains a nested collection with range [0, 1] for we must have G/H topologically isomorphic with T, R or Z for some closed normal subgroup H of G.

4. In this section we use the following two structure theorems to show that any locally compact, compactly generated abelian group has a nested collection with range [0, 1].

THEOREM 4.1. ([2] *page* 90, *Theorem* 9.8) *Every locally compact, compactly generated abelian group G is topologically isomorphic with* $R^m\times Z^n\times F$ *for some non-negative integers m and n and some compact abelian group F.*

THEOREM 4.2. ([2] *page* 89, *Theorem* 9.5) *Let G be a compact abelian group and U a neighbourhood of e in G. Then there exists a closed subgroup H of G with* $H\subseteq U$ *and G/H topologically isomorphic with* $T^k\times F$ *where k is a non-negative integer and F is a finite abelian group.*

It is clear from Theorem 4.1 that if G is abelian and not compact it must have either R or Z as a direct factor and therefore contains a nested collection with range [0, 1].

If G is compact and G/H is isomorphic with $T^k\times F$ for some finite abelian group F where $k>0$ then the result follows as well from sections 2 and 3.

It is therefore sufficient to consider the case where G is an infinite compact abelian group such that there is no closed subgroup H of G for which G/H has T as a direct factor. In this case we are able to use an idea of Granirer in [1] to obtain the result. In order to apply the method we first require the following lemma.

LEMMA 4.3. *Let G be an infinite compact abelian group such that there is no closed subgroup H of G for which G/H has T as a direct factor. Then there exists a strictly increasing sequence of positive integers* $\{p_n\}$ *and a decreasing sequence of open-closed subgroups* $\{H_n\}$ *of G such that, for each n,* G/H_n *is a finite group of order* p_n.

Proof. Let $p_1=1$ and $H_1=G$ and assume that p_k and H_k are defined for $k\le n$ and satisfy the required properties.

Choosing a neighbourhood U of e with $U\subsetneqq H_n$ and using Theorem 4.2 we obtain a closed subgroup H of G with $H\subsetneqq U\subsetneqq H_n$ and such that G/H is topologically isomorphic with $T^k\times F$ for some non-negative integer k and finite group F. The conditions on G imply that $k=0$ so G/H is finite. Let $H_{n+1}=H$ and $p_{n+1}=o(G/H)$. Then H_{n+1} is open-closed since G/H_{n+1} is finite; $H_{n+1}\subsetneqq H_n$ and

$$p_{n+1}=o(G/H)>o(G/H_n)=p_n \quad \text{since} \quad H\subsetneqq H_n.$$

The result follows by induction.

This lemma allows us to take care of the remaining case by proving

LEMMA 4.4. *Let G be an amenable locally compact group for which there exists a strictly decreasing sequence* $\{H_n\}$ *of closed normal subgroups for which* G/H_n *is a finite group for each n. Then G contains a nested collection with range* [0, 1].

Proof. Let $p_n=o(G/H_n)$ for each n. Then $\{p_n\}$ is a strictly increasing sequence of integers and p_n devides p_{n+1} for each n. Let $Q_n=\{k/p_n:k=1,2,\ldots,p_n\}$ and note that $Q_n\subseteq Q_{n+1}$ for all n.

We wish to show that, for any n, if $\{a_1,\ldots,a_{p_n}\}$ is a set of representatives of G/H_n, we can find a set of representatives $\{b_1,\ldots,b_{p_{n+1}}\}$ of G/H_{n+1} such that on defining

$$A(k/p_n)=\bigcup_{i=1}^{k}a_i\cdot H_n\subseteq G \quad \text{for} \quad 1\le k\le p_n$$

and

$$B(k/p_{n+1})=\bigcup_{i=1}^{k}b_i\cdot H_{n+1}\subseteq G \quad \text{for} \quad 1\le k\le p_{n+1}$$

we can show that whenever k_1/p_n, $k_2/p_n\in Q_n$ and $k/p_{n+1}\in Q_{n+1}$ with $k_1/p_n\le k/p_{n+1}\le k_2/p_n$ we have $A(k_1/p_n)\subseteq B(k/p_{n+1})\subseteq A(k_2/p_n)$. This implies that for $x\in Q_{n+1}\cap Q_n$ we have $A(x)=B(x)$ so we have extended a nested collection over Q_n to a nested collection over Q_{n+1}. Also since $G=\bigcup_{i=1}^{n+1}b_i\cdot H_{n+1}$ (disjoint union), for any LIM φ on $L^\infty(G)$ we have $\varphi(H_{n+1})=1/p_{n+1}$ so

$$\varphi(B(k/p_{n+1}))=k\cdot\varphi(H_{n+1})=k/p_{n+1}.$$

Similarly $\varphi(A(k/p_n))=k/p_n$ for $k/p_n\in Q_n$ so we have actually extended a nested collection with range Q_n to a nested collection with range Q_{n+1}.

If we let $Q=\bigcup_{n=1}^{\infty}Q_n$ then, starting with $H_1=G$ and $a_1=e$ and proceeding inductively, we obtain a nested collection in G with range Q. Since Q is countable and dense in [0, 1] (because p_n is strictly increasing) the result follows from lemma

2.2 if we can complete the induction step by finding a set of representatives of G/H_{n+1} with the properties mentioned above.

Given the set $\{a_1, \ldots, a_{p_n}\}$ of representatives of G/H_n we choose the representatives of G/H_{n+1} as follows:

Let $\{c_1, \ldots, c_m\}$ be a set of representatives of H_n/H_{n+1} (which is a subgroup of G/H_{n+1} and therefore finite). Since $p_{n+1}=m \cdot p_n$, for every $1 \leq i \leq p_{n+1}$ we can write i uniquely as $i=\alpha(i) \cdot m+\beta(i)$ where $0 \leq \alpha(i) \leq p_n$, $1 \leq \beta(i) \leq m$. Defining $b_i=a_{\alpha(i)+1} \cdot c_{\beta(i)}$ we have a collection $\{b_1, \ldots, b_{p_{n+1}}\}$ of elements of G and it is easily checked that this collection is a set of representatives of G/H_{n+1}. If $A(k/p_n)$ and $B(k/p_{n+1})$ are defined as they were earlier in the proof and we choose k_1/p_n, $k_2/p_n \in Q_n$ and k/p_{n+1} with $k_1/p_n \leq k/p_{n+1} \leq k_2/p_n$ then we must show that $A(k_1/p_n) \subseteq B(k/p_{n+1}) \subseteq A(k_2/p_n)$. Since $p_{n+1}=m \cdot p_n$ we must have $m \cdot k_1 \leq k \leq m \cdot k_2$. Note that the manner in which the b_i's were defined gives us

$$\{a_i \cdot c_j : 1 \leq i \leq k_1, 1 \leq j \leq m\} = \{b_i : 1 \leq i \leq k_1 \cdot m\}$$

so

$$\begin{aligned} A(k_1/p_n) &= \bigcup_{i=1}^{k_1} a_i H_n = \bigcup_{i=1}^{k_1} a_i \left(\bigcup_{j=1}^{m} c_j H_{n+1} \right) = \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{m} a_i c_j H_{n+1} \\ &= \bigcup_{i=1}^{k_1 m} b_i H_{n+1} \subseteq \bigcup_{i=1}^{k} b_i H_{n+1} = B(k/p_{n+1}). \end{aligned}$$

The other inclusion follows in the same manner and thus the result is proven.

5. Main results.

THEOREM 5.1. *Let G be an infinite locally compact abelian group. Then there exists a nested collection in G with range* [0, 1].

Proof. In section 4 the theorem is established in the case that G is compactly generated. If G is not compactly generated, let U be a compact symmetric neighbourhood of e and H the open-closed subgroup generated by U (i.e. $H=\bigcup_{n=1}^{\infty} U^n$ which is open and closed (see [2] p. 34 Theorem 5.7)). Since G is not compactly generated, G/H is infinite and abelian. Since H is open, G/H is a discrete group. As an abelian group, G/H cannot be an "AB group" so by Theorem 3 of [1] there exists a nested collection in G/H with range [0, 1] and the required collection exists in G by lemmas 2.1 and 2.2.

In view of section 2 we can write this result in a stronger form as

THEOREM 5.2. *Let G be an amenable locally compact group such that there exists a closed normal subgroup H of G for which G/H is infinite and abelian. Then G has a nested collection with range* [0, 1].

Proof. This result follows immediately from Theorem 5.1 and section 2.

REMARK. Theorem 5.2 guarantees the existence of a nested collection with range [0, 1] in any infinite, amenable, non-unimodular locally compact group for if we let $H=\{x \in G:\Delta(x)=1\}$ then H is a closed normal subgroup of G and G/H is infinite abelian due to the fact that Δ is continuous and the positive reals with multiplication are commutative.

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