

Topological spaces and Abian structures

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This paper characterizes topological spaces which have an Abian structure - a uniform-like structure of the covering type.

In [1], Abian introduced a uniform-like structure of the covering type. More recently, Pu [6] has related such structures, which we call Abian structures following Pu's lead, to a uniform-like structure of the entourage type, which he calls a regular structure.

DEFINITION 1. Let X be a topological space and $\{A_\lambda : \lambda \in \Lambda\}$ a family of open covers for X . We define the λ -symmetric neighbourhood $S_\lambda(x)$ of x by $S_\lambda(x) = \cup\{A \in A_\lambda : x \in A\}$. We also order Λ by setting $\lambda \leq \mu$ iff A_λ refines A_μ . $\{A_\lambda : \lambda \in \Lambda\}$ is called an Abian structure for X if it satisfies:

- (A.i) for every point x and every open set V containing x there is some $\lambda \in \Lambda$ such that $\cup\{S_\lambda(y) : y \in S_\lambda(x)\} \subseteq V$;
- (A.ii) every two elements λ and μ of Λ are comparable; that is, given A_λ and A_μ , then either A_λ refines A_μ or A_μ refines A_λ .

(Note that Λ need not be totally ordered as \leq need not be anti-symmetric.)

Pu [6, Remark 4.2] showed that every topological space with an Abian structure must be regular, but was unable to settle the question as to

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whether every regular topological space has an Abian structure.

In [2], we gave an example of a regular topological space which does not possess an Abian structure.

Perhaps a simpler example might be the Alexandroff one-point compactification $Y = X \cup \{\infty\}$ of an uncountable discrete space X . Y is compact T_5 , but the point ∞ does not have a totally ordered neighbourhood base and hence [2, Proposition] Y does not possess an Abian structure.

In this paper we give a necessary and sufficient condition for a topological space to possess an Abian structure. We show, in fact, a condition much like that for pseudometrizable, except that it does not imply first countability, namely the following:

CONDITION 1. The topological space X is regular, and for some cardinal κ the following hold:

(B.i) X has a base $\mathcal{B} = \bigcup_{k \in K} \mathcal{B}_k$ where $\#K = \kappa$, and for each

$k \in K$ every point of X has a neighbourhood meeting less than $\max\{\kappa, \aleph_0\}$ members of \mathcal{B}_k ;

(B.ii) if \mathcal{C} is a collection of open sets and $\#\mathcal{C} < \kappa$ then $\bigcap\{C : C \in \mathcal{C}\}$ is open.

Furthermore we show that X has an Abian structure iff it has a compatible uniformity with a totally ordered base.

DEFINITION 2. We call a collection \mathcal{C} of subsets of X an L_α collection if every point of X has a neighbourhood meeting less than \aleph_α members of \mathcal{C} . Thus the L_0 collections are just the locally finite collections.

DEFINITION 3. We call a topological space X with the property that the intersection of less than \aleph_α open sets is open, a P_α space.

Sikorski [7] has considered some of the properties of such spaces, and used the term ω_α -additive.

It should be observed that the notion of P_α space used by Misra [4]

corresponds to our $P_{\alpha+1}$ space. It seems that Misra [4, footnote p. 349] is mistaken when he claims that his notion of P_α space coincides with Sikorski's ω_α -additivity. Thus Misra's P -spaces correspond to our P_1 -spaces.

DEFINITION 4. We inductively define

$$S_\lambda^1(x) = S_\lambda(x) ,$$

$$S_\lambda^{n+1}(x) = \cup \{ S_\lambda^n(y) : y \in S_\lambda^n(x) \} .$$

Note: $y \in S_\lambda^n(x)$ iff $x \in S_\lambda^n(y)$.

It may be easily shown that if $\{A_\lambda : \lambda \in \Lambda\}$ is an Abian structure for X , then if U is an open set containing the point x , for any positive integer n there is a $\lambda \in \Lambda$ such that $S_\lambda^n(x) \subseteq U$.

THEOREM 1. If X has an Abian structure $\{A_\lambda : \lambda \in \Lambda\}$ then for some cardinal κ which is either 1 or infinite regular¹, X satisfies Condition 1. Furthermore, $B = \cup_{k \in K} B_k$ may be chosen such that each B_k is discrete.

Proof. We may suppose that Λ is well-ordered by the reverse ordering \geq , and that $\text{ord}(\Lambda)$ is either 1 or infinite regular. For if not, let Λ' be a coinital subset of Λ well-ordered by \geq , and suppose that $\text{ord}(\Lambda')$ has the minimum value possible. Then $\text{ord}(\Lambda')$ is either 1 or infinite regular, and $\{A_\lambda : \lambda \in \Lambda'\}$ is an Abian structure for X .

Furthermore it is equivalent to $\{A_\lambda : \lambda \in \Lambda\}$ in that id_X is a uniform isomorphism between them; that is, is uniformly continuous [1,

¹ The regular cardinals are those cardinals κ which are not the sum of less than κ cardinals, each of which is less than κ .

The regular ordinals are the initial ordinals whose corresponding cardinal is regular.

The infinite regular ordinals are characterized as being those infinite ordinals σ with the property that if S is a subset of σ and $\text{ord}(S) < \sigma$ then S is not cofinal in σ .

Definition 2] in both directions. Let $\kappa = \#\Lambda$.

That topological spaces which have Abian structures, are regular was stated by Pu [6]. Alternatively we may note that if U is open and $x \in U$, then $S_\lambda^2(x) \subseteq U$ for some $\lambda \in \Lambda$. Then

$$x \in S_\lambda(x) \subseteq X - U\{S_\lambda(y) : y \in X-U\} \subseteq U,$$

$S_\lambda(x)$ is open, and $X - U\{S_\lambda(y) : y \in X-U\}$ is closed, as required.

If $\kappa = 1$, then every point has a smallest neighbourhood. By regularity the topology is generated by a clopen partition P of X . Clearly P is the base required in (B.i). (B.ii) is trivially satisfied. Note that in fact in this case the arbitrary intersection of open sets is open. Spaces with this property have been considered by Pears [5] who calls them quasi-ordered spaces.

We now suppose $\kappa = \aleph_\alpha$ is infinite regular.

Let $K = \Lambda \times \Lambda$; then $\#K = \aleph_\alpha$. Let $\lambda \in \Lambda$, and irreflexively well-order A_λ by $<'$. For each $A \in A_\lambda$ and each $\mu \in \Lambda$ let

$$\begin{aligned} A_\mu &= \{x \in A : S_\mu^4(x) \subseteq A\}, \\ A'_\mu &= A_\mu - U\{B \in A_\lambda : B <' A\}, \\ A_\mu^* &= U\{S_\mu(x) : x \in A'_\mu\}. \end{aligned}$$

Then $\{A_\mu^* : A \in A_\lambda, \mu \in \Lambda\}$ is an open refinement of A_λ . We claim that for each $\mu \in \Lambda$, $B_{\lambda\mu} = \{A_\mu^* : A \in A_\lambda\}$ is discrete, and, in particular, that for each $x \in X$, $S_\mu(x)$ meets at most one A_μ^* .

For suppose $S_\mu(x)$ meets A_μ^* and B_μ^* . Then $S_\mu^2(x)$ meets A_μ and B_μ . We may suppose $A <' B$. Then let $y \in S_\mu^2(x) \cap A'_\mu$, it follows that $S_\mu^4(x)$ meets B'_μ . But $y \in A_\mu$, so $S_\mu^4(x) \subseteq A$, and yet $B'_\mu \cap A = \emptyset$. This is a contradiction, and hence $S_\mu(x)$ meets at most one A_μ^* .

We now claim $\mathcal{B} = \bigcup_{\lambda, \mu \in \Lambda} B_{\lambda\mu}$ is a base. Suppose $x \in U$ and U is open; then $S_\lambda(x) \subseteq U$ for some λ . Thus for some $\mu \in \Lambda$, $x \in A_\mu^* \in B_{\lambda\mu}$ where $A \in A_\lambda$, and so $x \in A_\mu^* \subseteq A \subseteq S_\lambda(x) \subseteq U$.

Thus $\mathcal{B} = \bigcup_{\lambda, \mu \in \Lambda} B_{\lambda\mu}$ is the required base.

It remains to show that X is a P_α -space. Let \mathcal{C} be a family of open sets with $\#\mathcal{C} < \aleph_\alpha$. If $\bigcap \mathcal{C} \neq \emptyset$, consider an arbitrary point $x \in \bigcap \mathcal{C}$. For each $C \in \mathcal{C}$ define $\lambda_C = \max\{\lambda \in \Lambda : S_\lambda(x) \subseteq C\}$. Then $\{\lambda_C : C \in \mathcal{C}\}$ cannot be coinitial in Λ as ω_α is a regular ordinal, and hence there is some $\mu \in \Lambda$ such that $S_\mu(x) \subseteq C$ for all $C \in \mathcal{C}$. Hence $\bigcap \mathcal{C}$ is a neighbourhood of x .

Note 1. If X satisfies Condition 1 then it does so for some cardinal κ' which is either 1 or infinite regular. For if κ is finite then \mathcal{B} itself is L_0 and so satisfies (B.i) with $\kappa' = 1$. Hence Condition 1 is satisfied with $\kappa' = 1$.

We need now only consider κ infinite, but not regular.

Let $\kappa = \aleph_\alpha$. Let K' be an index set with $K \subseteq K'$ and $\#K' = \aleph_{\alpha+1}$. Extend $\{B_k : k \in K\}$ to $\{B_k : k \in K'\}$ by taking repetitions. Then $\mathcal{B} = \bigcup_{k \in K'} B_k$ is a base for X . Furthermore, the B_k are L_α ; hence $L_{\alpha+1}$.

As $\aleph_{\alpha+1}$ is regular, it remains to show that X is $P_{\alpha+1}$, for then Condition 1 will be satisfied with $\kappa' = \aleph_{\alpha+1}$.

Let $\{U_\gamma : \gamma \in \Gamma\}$ be a family of less than $\aleph_{\alpha+1}$ open sets. We may assume $\#\Gamma = \aleph_\alpha$. Well-order Γ so that $\text{ord}(\Gamma) = \omega_\alpha$, and let Δ be a cofinal subset of Γ of cardinal $\aleph_\beta < \aleph_\alpha$. For each $\delta \in \Delta$ let

$$V_\delta = \bigcap \{U_\gamma : \gamma \in \Gamma, \gamma < \delta\}.$$

Then each V_δ is open as it is the intersection of less than \aleph_α open

sets. Furthermore,

$$\bigcap_{\gamma \in \Gamma} U_\gamma = \bigcap_{\delta \in \Delta} V_\delta,$$

and $\bigcap_{\delta \in \Delta} V_\delta$, being the intersection of \aleph_β ($< \aleph_\alpha$) open sets is open.

Thus we see X is also $P_{\alpha+1}$.

THEOREM 2. *If X satisfies Condition 1 then it is normal.*

Proof. If κ is finite, the topology is generated by a clopen partition of X , and so every closed set is also open. Hence X is normal.

We now suppose $\kappa = \aleph_\alpha$. Let F and G be disjoint closed sets, and well-order κ by \leq so that $\text{ord}(K) = \omega_\alpha$. Let

$$U = \bigcup_{k \in K} \left(\bigcup \{B \in \mathcal{B}_k : \bar{B} \cap G = \emptyset\} - \bigcup_{l \leq k} \{\bar{B} : B \in \mathcal{B}_l, \bar{B} \cap F = \emptyset\} \right),$$

$$V = \bigcup_{k \in K} \left(\bigcup \{B \in \mathcal{B}_k : \bar{B} \cap F = \emptyset\} - \bigcup_{l \leq k} \{\bar{B} : B \in \mathcal{B}_l, \bar{B} \cap G = \emptyset\} \right).$$

By regularity, $F \subseteq U$, $G \subseteq V$.

X is a P_α -space and \mathcal{B}_k is L_α ; hence $\bigcup \{\bar{B} : B \in \mathcal{B}_k, \bar{B} \cap F = \emptyset\}$ is closed, and hence $\bigcup_{l \leq k} \bigcup \{\bar{B} : B \in \mathcal{B}_l, \bar{B} \cap F = \emptyset\}$ is closed. Therefore

U is open. Similarly V is open. Also, by construction, $U \cap V = \emptyset$. Hence X is normal.

THEOREM 3. *If X satisfies Condition 1 then it has a compatible uniformity with a totally ordered base.*

Proof. We suppose κ is either 1 or infinite regular.

If $\kappa \leq \aleph_0$ then X is pseudometrizable, and the pseudometric uniformity has a (countable) totally ordered base.

If $\kappa > \aleph_0$, say $\kappa = \aleph_\alpha$: for each $k \in K$ and $B \in \mathcal{B}_k$ let $\mathcal{B}_l = \bigcup \{\bar{C} : C \in \mathcal{B}_l, \bar{C} \subseteq B\}$. By regularity and the fact that \mathcal{B} is a base we see that $\bigcup \{\mathcal{B}_l : l \in K\} = \mathcal{B}$.

X is a P_α space and \mathcal{B}_l is an L_α collection, so we may show that

B_L is closed. By Sikorski [7, §2 (v)] and the normality of X we may find a clopen set Q_{BL} with $B_L \subseteq Q_{BL} \subseteq B$. Then $Q_{kL} = \{Q_{BL} : B \in \mathcal{B}_k\}$ is L_α (but not necessarily a cover). Furthermore, $Q = \cup\{Q_{kL} : k, l \in K\}$ is a base for the topology on X . Well-order $K \times K$ so that $\text{ord}(K \times K) = \omega_\alpha$. Let $Q_{kL}^* = \cup\{Q_{k',l'} : (k', l') \leq (k, l)\}$.

If $x \in X$, x has a neighbourhood $W_{k',l'}$ meeting no more than $\aleph_{\beta(k',l')}$ members of $Q_{k',l'}$ where $\aleph_{\beta(k',l')} < \aleph_\alpha$. Then as $\#\{(k', l') : (k', l') \leq (k, l)\} < \aleph_\alpha$ and X is a P_α -space, $\cap\{W_{k',l'} : (k', l') \leq (k, l)\}$ is a neighbourhood of x meeting no more than $\sum\{\aleph_{\beta(k',l')} : (k', l') \leq (k, l)\}$ members of Q_{kL}^* . As \aleph_α is a regular cardinal, $\sum\{\aleph_{\beta(k',l')} : (k', l') \leq (k, l)\} < \aleph_\alpha$. Thus Q_{kL}^* is an L_α collection.

Let $V_{kL} = \{(x, y) : \forall Q \in Q_{kL}^*, x \in Q \iff y \in Q\}$. Then each V_{kL} is transitive, and $V = \{V_{kL} : (k, l) \in K \times K\}$ is a totally ordered base for a uniformity on X . We claim that V generates the topology on X .

Now $Q = \cup\{Q_{kL}^* : k, l \in K\}$, and is a base for the topology on X , and if $x \in Q \in Q_{kL}^*$ then $V_{kL}[x] \subseteq Q$. Hence it will suffice to show $V_{kL}[x]$ is a neighbourhood of x .

As Q_{kL}^* is L_α , let S be an open neighbourhood of x meeting less than \aleph_α members of Q_{kL}^* . Then $\{Q : Q \in Q_{kL}^*, x \in Q\}$ and $\{X-Q : Q \in Q_{kL}^*, Q \cap S \neq \emptyset, x \notin Q\}$ are two families of open neighbourhoods of x of cardinal less than \aleph_α . Let

$$S^* = S \cap (\cap\{Q : Q \in Q_{kL}^*, x \in Q\}) \cap (\cap\{X-Q : Q \in Q_{kL}^*, Q \cap S \neq \emptyset, x \notin Q\}) .$$

Then S^* is an open neighbourhood of x . But $S^* \subseteq V_{kL}[x]$ as required. Hence V does generate the topology on X .

Note 2. If V is a totally ordered base for a uniformity on X ,

then $\{A_V : V \in \mathcal{V}\}$ is an Abian structure for X , where

$$A_V = \{\text{int } V[x] : x \in X\}.$$

This establishes the equivalence of possessing an Abian structure, satisfying Condition 1, and possessing a compatible uniformity with a totally ordered base.

REMARK 1. By Theorem 3 and the result of Hayes [3], if X satisfies Condition 1, then it is paracompact. Furthermore if $\kappa \neq \aleph_0$, then every open cover has a clopen partition refinement.

REMARK 2. If X has Abian structure $\{A_\lambda : \lambda \in \Lambda\}$, where $\text{ord}(\Lambda)$ is either 1 or infinite regular, then every point has a neighbourhood base well-ordered by \supseteq , for instance $\{S_\lambda(x) : \lambda \in \Lambda\}$. For each point x let $\text{on}(x)$ be the smallest ordinal of such a neighbourhood base. Then $\text{on}(x)$ is either 1 or $\text{ord}(\Lambda)$.

For clearly $\text{on}(x) \leq \text{ord}(\Lambda)$ as the example $\{S_\lambda(x) : \lambda \in \Lambda\}$ shows. But if N is a neighbourhood base well-ordered by \supseteq , and $\text{ord}(N) < \text{ord}(\Lambda)$, then as N is a P_α -space, where $\text{ord}(\Lambda) = \omega_\alpha$, $\cap N$ is a neighbourhood of x , and hence the smallest neighbourhood of x . Hence $\text{on}(x) = 1$.

This gives a stronger analogue to first countability than might be expected.

REMARK 3. Connected topological spaces which possess Abian structures are pseudometrizable. For if X is not pseudometrizable, then $\text{ord}(\Lambda) > \omega_0$ and X is not indiscrete. By regularity and Sikorski [7, §2 (iv)], X is disconnected.

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