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FINDING COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS BY ITERATION

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In this paper it is shown that a particular iteration scheme converges weakly to a common fixed point of a finite set of nonexpansive mappings. This result is an improvement of two related theorems in the literature.

Let X be a Banach space, C a convex subset of X. Let $T_1, T_2, \dots, T_k$ be a family of nonexpansive selfmaps of C. Kuhfittig [4] defined the following iteration scheme. Let $U_0 = I$, I the identity map, $0 < \alpha < 1$,

$$
\begin{aligned}
&U_1 = (1-\alpha)I + \alpha T_1 U_0, \\
&U_2 = (1-\alpha)I + \alpha T_2 U_1, \\
&\qquad \dots \\
&U_k = (1-\alpha)I + \alpha T_k U_{k-1}, \\
&x_0 \in C,\ x_{n+1} = (1-\alpha)x_n + \alpha T_k U_{k-1} x_n,\ n \geqslant 0.
\end{aligned}
\tag{1}
$$

Define $F = \bigcap_{i=1}^{k} F(T_i)$, where $F(T_i)$ denotes the fixed point set of T_i. Then, if each T_i is a nonexpansive selfmap of C, with $F \neq \emptyset$, C compact, X strictly convex, Kuhfittig [4] showed that (1) converges strongly to a common fixed point of the family. His second result is that, if X is uniformly convex and satisfies Opial's condition and C is closed and convex, then (1) converges weakly to a fixed point in F.

A Banach space is said to satisfy Opial's condition if, whenever $\{x_n\}$ is a convergent sequence in X with limit x_0, then, for any $x \neq x_0$

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\|.$$

The purpose of this note is to improve [4, Theorem 2] by removing the hypothesis of Opial's condition.

The result to be proved is the following.

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Theorem. *Let X be a uniformly convex Banach space, C a closed convex subset of X, $T_1, T_2, \ldots, T_k$ a family of nonexpansive selfmaps of C with $F \neq \emptyset$. Then $\{x_n\}$, defined by (1), converges weakly to a common fixed point of the family.*

The proof of the Theorem will require the following lemmas.

Let C be a subset of a Banach space X, $T : C \to X$, $x_0 \in C$. Then the Mann iterative scheme $M(x_0, t_n, T)$ is the sequence $\{x_n\}$ defined by $x_{n+1} = (1 - t_n)x_n + t_n T x_n$. If the sequence $\{t_n\}$ satisfies $0 \leqslant t_n \leqslant b < 1$, $\sum t_n = \infty$, and $x_n \in C$ for each positive integer n, then $M(x_0, t_n, T)$ is said to satisfy condition A.

Lemma 1. [**3**, Lemma 2] *Let C be a subset of a Banach space X, T a nonexpansive map from C into X. If $M(x_0, t_n, T)$ satisfies condition A and is bounded, then* $\lim \|x_n - Tx_n\| = 0$.

Let C be a closed bounded convex subset of a uniformly convex space X. A map T is said to be semicontractive if there exists a map $V : C \times C \to X$ such that $T(u) = V(u, u)$ for each $u \in C$ while, (a) for each fixed v in C, $V(\cdot, v)$ is nonexpansive from C to X, and (b) for each fixed $u \in C$, $V(u, \cdot)$ is completely continuous from C to X uniformly for u in bounded subsets of C.

Lemma 2. [**2**, Theorem 3] *Let X be uniformly convex, C a closed bounded convex subset of X, T a semicontractive mapping of C into X. Then:*

(a) *$(I - T)$ is demiclosed, and*
(b) *$(I - T)(C)$ is closed in X.*

Proof of the Theorem: From [4], the U_i and $T_i U_{i-1}$ are nonexpansive and $\{T_1, T_2, \ldots, T_k\}$ and $\{U_1, U_2, \ldots, U_k\}$ have the same fixed point set. Let $p \in F$, set $S = T_k U_{k-1}$.

For any $x \in C$, $p \in F$, define $E = \{u \in X : \|u - p\| \leqslant r\} \cap C$, where $r := \|x - p\|$. Then E is a nonempty bounded convex subset of C which is invariant under the U_i and T_i and contains $x_0 = x$. Thus, without loss of generality we may assume that C is bounded.

Since S is nonexpansive, using (1),

$$\begin{aligned} \|x_{n+1} - p\| = \big\|(1 - \alpha)x_n + \alpha S x_n\big\| &\leqslant (1 - \alpha)\|x_n - p\| + \alpha\|Sx_n - p\| \\ &\leqslant (1 - \alpha)\|x_n - p\| + \alpha\|x_n - p\| = \|x_n - p\|. \end{aligned}$$

Therefore $\lim \|x_n - p\|$ exists, which implies that $\{x_n\}$ is bounded.

From Lemma 1, $\lim_n \|x_n - Sx_n\| = 0$.

The assumption that X is uniformly convex implies that it is reflexive. The boundedness of $\{x_n\}$ implies that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point $q \in C$. Since S is nonexpansive, if one defines V by $V(u, v) = Su + v$,

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then V is semicontractive, and, from Lemma 2, S is demiclosed. This means that, if $\{x_{n_i}\}$ converges weakly to a point q, then, since $\lim_i \|(I-S)x_{n_i}\| = 0$, $(I-S)q = q$; that is, q is a fixed point of $S = T_k U_{k-1}$.

A uniformly convex space is strictly convex, so one can use the argument of [**4**], which we now do, to show that $q \in F$.

Suppose that q is not a common fixed point of T_{k-1} and U_{k-2}. Then the closed line segment $[q, T_{k-1}U_{k-2}q]$ has positive length. Define

$$z = U_{k-1}q = (1-\alpha)q + \alpha T_{k-1}U_{k-2}q.$$

By hypothesis there exists a point w such that $T_1 w = T_2 w = \cdots = T_k w = w$. Since $\{T_i\}$ and $\{U_i\}$ have the same fixed point set, it follows that $T_{k-1}U_{k-2}w = w$. Since T_{k-1} and U_{k-2} are nonexpansive,

$$\|T_{k-1}U_{k-2}q - w\| \leqslant \|q - w\| \tag{2}$$

and

$$\|T_k z - w\| \leqslant \|z - w\|.$$

Therefore w is at least as close to $T_k z$ as to z. But $T_k z = T_k U_{k-1} q = q$, so that w is at least as close to q as to $z = (1-\alpha)q + \alpha T_{k-1}U_{k-2}q$. Since X is strictly convex, it follows that

$$\|q - w\| < \|T_{k-1}U_{k-2}q - w\|,$$

contradicting (2), so that $T_{k-1}U_{k-2}q = q$. It then follows from $U_{k-1} = (1-\alpha)I + \alpha T_{k-1}U_{k-2}$ that $U_{k-1}q = (1-\alpha)q + \alpha q = q$ and $q = T_k U_{k-1} q = T_k q$. Therefore q is a common fixed point of T_k and U_{k-1}.

Since $T_{k-1}U_{k-2}q = q$, we may repeat the above argument to obtain the result that $T_{k-2}U_{k-2}q = q$ and that q is therefore a common fixed point of T_{k-1} and U_{k-2}. Continuing in the same manner, it then follows that $T_1 U_0 q = q$ and q is a common fixed point of T_2 and T_1. Thus q is a common fixed point of $\{T_i : i = 1, 2, \dots, k\}$. ▯

Corollary. [**4**, Theorem 2] *If X is a uniformly convex Banach space satisfying Opial's condition and C is a closed convex subset of X, and if the family of mappings $\{T_i : i = 1, 2, \dots, k\}$ satisfies* (1), *then, for any $x \in C$, the sequence $\{x_n\}$ converges weakly to a common fixed point.*

In a recent paper, Atsushiba and Takahashi [**1**] proved the following.

Theorem AT. *Let X be a uniformly convex Banach space which satisfies Opial's condition or whose norm if Fréchet differentiable. Let C be a nonempty closed convex*

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subset of X, S *and* T *a pair of commuting nonexpansive selfmaps of* C, *with* $F(S) \cap F(T) \neq \emptyset$. *Let* $x_1 \in C$ *and define* $\{x_n\}$ *by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\frac{1}{n^2}\sum_{i,j=0}^{n} S^i T^j x_n \quad \text{for } n \in \mathbb{N}, \tag{3}$$

where $\{\alpha_n\}$ *satisfies* $0 \leqslant \alpha_n \leqslant a < 1$. *Then* $\{x_n\}$ *converges weakly to a common fixed point of* T *and* S.

Iteration scheme (1) is much simpler than (3). In addition, the theorem of this paper shows that (1) is a much more general iteration scheme than (3), since the hypotheses of Opial's condition and commutativity of the maps are not required.

Finally we note that, in (1) one can replace α, in the formula for x_{n+1}, with a sequence $\{t_n\}$, satisfying Condition A.

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