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CONWAY POTENTIAL FUNCTIONS FOR LINKS IN $\mathbb{Q}$-HOMOLOGY 3-SPHERES

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We obtain a formula relating the Conway potential functions of links in S^3 which are connected by a framed surgery operation. Using this formula we extend the theory of Conway potential functions to links in all oriented $\mathbb{Q}$-homology 3-spheres.

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1. Introduction

Conway [2] discovered that the Alexander polynomial of an oriented link in S^3 may be normalized so as to enjoy many important properties. Hartley's article [3] is the basic reference. Using a refined version of Reidemeister Torsion, Turaev [8, Section 4] shows that this normalization can be extended to links in arbitrary $\mathbb{Z}$-homology 3-spheres.

Our study of surgery formulae for Casson's invariant [1] led us naturally to the problem of determining a normalization of the Alexander polynomials of oriented links in $\mathbb{Q}$-homology 3-spheres.

The construction given by Hartley [3] for the Conway potential function of a link in S^3 cannot be applied to links in other 3-manifolds since it depends crucially on the properties of knot projections on a plane. In the present approach, we assume the existence and properties of Conway potential functions for links in S^3 as stated in [3] and prove a surgery formula for these functions. This enables us to extend the theory to potential functions of links in $\mathbb{Q}$-homology 3-spheres where the surgery formula plays a central role. In particular, we give the relation between potential functions of links in possibly distinct manifolds having homeomorphic complements.

We list below the properties of the Conway potential functions; all of them, except the surgery formula III are straightforward generalizations of known properties of the Conway potential functions for links in S^3. Proofs and further explanations are given in the body of the text.

Let L be an oriented link of n components in an oriented $\mathbb{Q}$-homology 3-sphere M. Let Δ_L be the Alexander polynomial of L and let ∇_L be its Conway potential function.

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(I) Relation to the Alexander polynomial

$$\nabla_L(s_1,\dots,s_n)=\begin{cases}\Delta_L(s_1^2)/(s_1^d-s_1^{-d}) & \text{if } n=1\\ \Delta_L(s_1^2,s_2^2,\dots,s_n^2) & \text{if } n>1\end{cases}$$

where $d=|\text{torsion subgroup of } H_1(M\backslash L)|/|H_1(M)|$.

(II) Value at 1

$$\Delta_L(1,1,\dots,1)=\begin{cases} d & \text{if } n=1\\ lk_M(K_1,K_2) & \text{if } n=2 \text{ and } L=K_1\cup K_2\\ 0 & \text{if } n\geqq 3.\end{cases}$$

(III) Variance under surgery

Let $\mathbb{L}$ be an oriented framed link in a $\mathbb{Q}$-homology 3-sphere presenting an oriented link $\hat{L}$ in the surgered manifold $\chi(\mathbb{L})$. We suppose that $\chi(\mathbb{L})$ is again a $\mathbb{Q}$-homology 3-sphere and denote by B the framing matrix associated to $\mathbb{L}$. Then

$$\nabla_{\hat{L}}(s_1,s_2,\dots,s_n)=|B|^{-1}\nabla_L((s_1,s_2,\dots,s_n)\cdot B^{-1}).$$

(IV) Restriction

Suppose that $L=L_0\cup K_{m+1}\cup\cdots\cup K_n$, then

$$\nabla_L(s_1,s_2,\dots,s_m,1,1,\dots,1)=f_{L,L_0}(s_1,s_2,\dots,s_m)\nabla_{L_0}(s_1,s_2,\dots,s_m)$$

where

$$f_{L,L_0}(s_1,s_2,\dots,s_m)=\prod_{i=m+1}^{n}(s_1^{l_{1i}}s_2^{l_{2i}}\dots s_m^{-l_{mi}}-s_1^{l_{1i}}s_2^{-l_{2i}}\dots s_m^{-l_{mi}}).$$

(V) Symmetry

$$\nabla_L(s_1^{-1},s_2^{-1},\dots,s_n^{-1})=(-1)^n\nabla_L(s_1,s_2,\dots,s_n).$$

(VI) Orientation change

Let L' be the link resulting from a reversal of the orientation of the first component of L, then

$$\nabla_{L'}(s_1,s_2,\dots,s_n)=-\nabla_L(s_1^{-1},s_2,\dots,s_n).$$

(VII) Ambient orientation change

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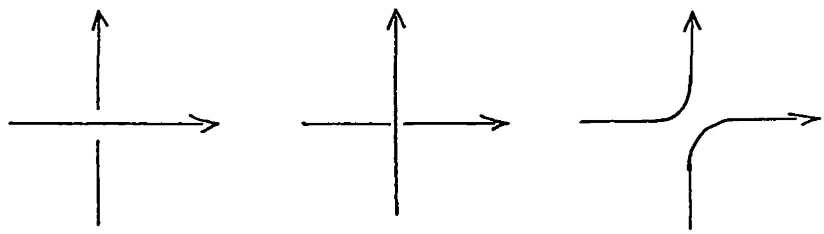
Let $\sim L$ denote the link L considered in the manifold $-M$, then

$$\nabla_{\sim L}(s_1, s_2, \ldots, s_n) = (-1)^{n-1}\nabla_L(s_1, s_2, \ldots, s_n).$$

(VIII) Skein relation

Let L_+, L_- and L_0 be oriented links in M differing only in a 3-ball as pictured below:



L_+ $\quad$ L_- $\quad$ L_0

Setting all the variables corresponding to the components appearing in the diagram equal to s (and leaving those that remain unchanged) we have

$$\nabla_{L_+} - \nabla_{L_-} = (s - s^{-1})\nabla_{L_0}.$$

In Section 1 we define the notions of oriented framed link and surgery presentation of a link; we recall the definition of Alexander polynomials of links in $\mathbb{Q}$-homology 3-spheres. In Section 2 we state the surgery formula for links in S^3 (Theorem 2.1) and define Conway potential functions for links in $\mathbb{Q}$-homology 3-spheres. The properties (I)–(VIII) listed above are established in Section 3. Finally Section 4 is devoted to the proof of Theorem 2.1 concerning the surgery formula for links in S^3.

We would like to thank Vladimir Turaev for informing us about his treatment of Conway potential functions for links in $\mathbb{Z}$-homology 3-spheres.

1. Definitions and preliminary notions

We shall work in the smooth, oriented category throughout this paper. Thus all manifolds and submanifolds will be smooth and oriented and diffeomorphisms between manifolds will preserve orientations. M will denote a $\mathbb{Q}$-homology 3-sphere, L a link in M and $T(L)$ a closed tubular neighbourhood of L in M.

If C_1 and C_2 are disjoint 1-cycles in M, a rational valued linking number is defined: $lk_M(C_1, C_2) \in \mathbb{Q}$ [7, §77]. The torsion pairing [7, §77] l: $H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$ is the (mod $\mathbb{Z}$) reduction of lk_M. If K is a knot in M, $lk_M(K, -)$: $H_1(M \setminus K; \mathbb{Q}) \to \mathbb{Q}$ provides a canonical isomorphism sending a meridian μ of K to 1.

Definition 1.1. The *longitude* of a knot K in M is the unique class $\lambda \in H_1(\partial T(K); \mathbb{Q})$ satisfying

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(i) $\mu \cdot \lambda = 1$ in H_1 $(\partial T(K); \mathbb{Q})$ and

(ii) the image of λ in $H_1(M \backslash K; \mathbb{Q})$ is zero.

Note that the first condition is equivalent to λ being rationally homologous to K in $T(K)$ while the second is equivalent to $lk_M(K, \lambda) = 0$.

Suppose now that $l(K, K) \equiv -a/b \pmod{\mathbb{Z}}$ where $gcd(a, b) = 1$. It can be shown that $b\lambda$ is represented by an essential simple closed curve on $\partial T(K)$. Indeed we have the following more general result.

Lemma 1.2. *A class $p\mu + q\lambda \in H_1(\partial T(K); \mathbb{Q})$ is represented by an essential simple closed curve on $\partial T(K)$ if and only if $q \in \mathbb{Z}$ and there is a $c \in \mathbb{Z}$ coprime with q such that $p = c - (qa)/b$.*

Proof. Let π be a parallel curve to K on $\partial T(K)$, that is π is a simple closed curve on $\partial T(K)$ which is isotopic to K in $T(K)$. Now we necessarily have $\mu \cdot \pi = 1$ in $H_1(\partial T(K))$ and, after possibly altering π by an integral number of copies of μ, we may suppose $lk_M(K, \pi) = -a/b$. It follows that $\lambda = (a/b)\mu + \pi$. As μ and π form a basis for the integral homology of $\partial T(K)$, the lemma follows from the fact that an integral class is represented by an essential simple closed curve on $\partial T(K)$ if and only if it is a primitive class. □

We shall call such a pair (p, q) a *framing* of K.

Definition 1.3. By a *framed link* $\mathbb{L}$ in M we mean an underlying link $L = K_1 \cup K_2 \cup \cdots \cup K_n \subseteq M$ and a sequence $(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)$ of framings for the components of L.

Denote by $\chi(\mathbb{L})$ the manifold obtained by performing surgery on M along L as indicated by $\mathbb{L}$, the framing of K_j giving the surgery meridian.

Set $lk_M(K_i, K_j) = l_{ij}$, $1 \leqq i \neq j \leqq n$. We associate to $\mathbb{L}$ the *framing matrix*

$$B = \begin{bmatrix} p_1 & & & \\ & p_2 & q_j l_{ij} & \\ & q_j l_{ij} & & \\ & & & p_n \end{bmatrix}.$$

When M is a $\mathbb{Z}$-homology 3-sphere, B is a presentation matrix of $H_1(\chi(\mathbb{L}))$ and thus $|B| = \pm |H_1(\chi(\mathbb{L}))|$. In general $|B| = \pm |H_1(\chi(\mathbb{L}))| / |H_1(M)|$. We shall assume henceforth that $\chi(\mathbb{L})$ is a $\mathbb{Q}$-homology 3-sphere. This is equivalent to requiring $|B| \neq 0$.

The union of the cores of the surgery tori determine a link

$$\hat{L} = \hat{K}_1 \cup \hat{K}_2 \cup \cdots \cup \hat{K}_n \subseteq \chi(\mathbb{L})$$

oriented so that the meridian of $\hat{K}_j$ is

$$\hat{\mu}_j = p_j \mu_j + q_j \lambda_j.$$

Here μ_j and λ_j are the meridian and longitude of K_j.

Definition 1.4. We say $\mathbb{L}$ is a *surgery presentation* of $\hat{L}$.

Either collection of meridians $\{\mu_j\}_{j=1,\dots,n}$ or $\{\hat{\mu}_j\}_{j=1,\dots,n}$ forms a basis for $H_1(M\backslash L;\mathbb{Q})$ and B is the transition matrix from the former to the latter. It follows that if L_2 is presented by $\mathbb{L}_1$ with matrix B_2 and L_1 is presented by $\mathbb{L}$ with matrix B_1 then L_2 is presented by $\mathbb{L}'$ with matrix B_1B_2. Here $\mathbb{L}'$ has the same underlying link as $\mathbb{L}$ and framings corresponding to the meridians of L_2. In particular if $\mathbb{L}$ presents L with matrix B then the meridians of L define a framed link $\mathbb{L}^{-1}$ presenting L with matrix B^{-1}.

Lemma 1.5. *Let* $1\leqq i,j\leqq n$ and $c(i,j)$ *be the* (i,j) *cofactor of* B. *Set* $\hat{M}=\chi(\mathbb{L})$, *then*

$$lk_{\hat{M}}(\hat{K}_i,\hat{K}_j)=\begin{cases} -\dfrac{1}{q_i|B|}c(i,j) & \text{if } q_i\neq 0\\ \dfrac{1}{p_i|B|}\displaystyle\sum_{k\neq i} l_{ki}c(k,j) & \text{if } q_i=0.\end{cases}$$

Proof. Let $a_j=c(j,j)$ and $b_j=\bar{L}_{i\neq j}\sum_{i\neq j}l_{ij}c(i,j)$. Using the identities $\lambda_k=\sum_{i\neq k}l_{ik}\mu_i$ $(1\leqq k\leqq n)$ in $H_1(X;\mathbb{Q})$ (where X is the exterior of L), it can be shown that in this group,

$$b_j\mu_j-a_j\lambda_j=\sum_{i\neq j}\left\{c(i,j)\lambda_i-\left(\sum_{k\neq i}l_{ki}c(k,j)\right)\mu_i\right\}\qquad 1\leqq j\leqq n. \tag{1.I}$$

When $q_i\neq 0$,

$$c(i,j)\lambda_i-\left(\sum_{k\neq i}l_{ki}c(k,j)\right)\mu_i=(p_i/q_i)c(i,j)\mu_i+c(i,j)\lambda_i=(1/q_i)c(i,j)\hat{\mu}_i \tag{1.II}$$

in $H_1(X;\mathbb{Q})$. When $q_i=0$, $c(i,j)=0$ and $\hat{\mu}_i=p_i\mu_i$ so that

$$c(i,j)\lambda_i-\left(\sum_{k\neq i}l_{ki}c(k,j)\right)\mu_i=-(1/p_i)\left(\sum_{k\neq i}l_{ki}c(k,j)\right)\hat{\mu}_i \tag{1.III}$$

in $H_1(X;\mathbb{Q})$. Thus $b_j\mu_j-a_j\lambda_j$ is null homologous in the exterior of $\hat{K}_j$. As $\hat{\mu}_j\cdot(b_j\mu_j-a_j\lambda_j)=-|B|$, it follows $\hat{\lambda}_j=-(1/|B|)(b_j\mu_j-a_j\lambda_j)$. Now $\hat{\lambda}_j$ and $\hat{K}_j$ are equal as rational classes in the exterior of $\hat{K}_i$, hence when $q_i\neq 0$ reference to (1.I) and (1.II) shows $lk_{\hat{M}}(\hat{K}_i,\hat{K}_j)=-(1/q_i|B|)c(i,j)$. When $q_i=0$ referring to (1.I) and (1.III) shows $lk_{\hat{M}}(\hat{K}_i,\hat{K}_j)=(1/p_i|B|)(\sum_{k\neq i}l_{ki}c(k,j))$. □

Definition 1.6. Let L_1,L_2 be two links in M with $L_2=L_1\cup K_{n+1}\cup\cdots\cup K_m$. Define

$$f_{L_2,L_1}(s_1,s_2,\dots,s_n)=\prod_{i=n+1}^{m}(s_1^{l_{1i}}s_2^{l_{2i}}\dots s_n^{l_{ni}}-s_1^{-l_{1i}}s_2^{-l_{2i}}\dots s_n^{-l_{ni}}).$$

Let $\mathbf{s}=(s_1,s_2,\ldots,s_n)$ be an n-tuple of indeterminates and A an $n\times m$ rational matrix. Define $\mathbf{s}\cdot A$ to be the m-tuple

$$\mathbf{s}\cdot A=(s_1^{a_{11}}s_2^{a_{21}}\ldots s_n^{a_{n1}},\ldots,s_1^{a_{1m}}s_2^{a_{2m}}\ldots s_n^{a_{nm}}).$$

If B is an $m\times r$ matrix then $(\mathbf{s}\cdot A)\cdot B=\mathbf{s}\cdot(AB)$, thus we have a right-action of $GL(n,\mathbb{Q})$ on $\mathbb{Z}[\mathbb{Q}^n]$ via $f(\mathbf{s})\cdot A=f(\mathbf{s}\cdot A)$. Here we have written $\mathbb{Q}^n$ exponentially as

$$\{s_1^{x_1}s_2^{x_2}\ldots s_n^{x_n}\,|\,x_1,x_2,\ldots,x_n\in\mathbb{Q}\}.$$

Lemma 1.7. (i) *If* $L_1\subseteq L_2=L_1\cup K_{n+1}\cup\cdots\cup K_m\subseteq L_3=L_2\cup K_{m+1}\cup\cdots\cup K_t$ *are links in* M, $\mathbf{s}=(s_1,s_2,..,s_n)$ *and* $\mathbf{1}$ *denotes a string of ones then*

$$f_{L_3,L_1}(\mathbf{s})=f_{L_3,L_2}(\mathbf{s},\mathbf{1})f_{L_2,L_1}(\mathbf{s}).$$

(ii) *If* $\mathbb{L}_1$ *presents* $\hat{L}_1$ *with framing matrix* B_1 *and* $\mathbb{L}_2=\mathbb{L}_1\cup K_{n+1}^{(\varepsilon_{n+1},0)}\cup\cdots\cup K_m^{(\varepsilon_m,0)}$ *presents* $\hat{L}_2$ *with framing matrix* B_2*, then* $\hat{L}_1$ *is a sublink of* $\hat{L}_2$ *and*

$$f_{\hat{L}_2,\hat{L}_1}(\mathbf{s})=\varepsilon f_{L_2,L_1}(\mathbf{s}\cdot B_1^{-1})$$

where $\varepsilon=\varepsilon_{n+1}\varepsilon_{n+2}\ldots\varepsilon_m$.

Proof. Part (i) is straightforward. To prove (ii) we write

$$f_{L_2,L_1}(\mathbf{s})=\prod_{i=n+1}^{m}(g_i(\mathbf{s})-g_i(\mathbf{s})^{-1})\quad\text{where}\quad g_i(\mathbf{s})=s_i^{l_{1i}}s_2^{l_{2i}}\ldots s_n^{l_{ni}}.$$

Let $\hat{g}_i$ be the corresponding function for the pair $(\hat{L}_2,\hat{L}_1)$. For $1\leqq j\leqq n$, the exponent of s_j in $g_i(\mathbf{s}\cdot B_1^{-1})$ is $(1/|B_1|)\sum_{1\leqq k\leqq n}l_{ki}c_1(k,j)$ where $c_u(k,j)$ denotes the (k,j) cofactor of B_u for $u=1,2$. For $(n+1)\leqq i\leqq m$, $q_i=0$, so that Lemma 1.5 shows if $\hat{M}=\chi(\mathbb{L})$ then $lk_{\hat{M}}(\hat{K}_i,\hat{K}_j)=(\varepsilon/\varepsilon_i|B_1|)\sum_{1\leqq k\neq i\leqq m}l_{ki}c_2(k,j)$. It is readily verified that $c_2(k,j)=\varepsilon c_1(k,j)$ when $1\leqq k\leqq n$ and is zero otherwise. Thus the exponent of s_j in $g_i(\mathbf{s}\cdot B_1^{-1})$ equals $\varepsilon_i lk_{\hat{M}}(\hat{K}_i,\hat{K}_j)$. Hence

$$(g_i(\mathbf{s}\cdot B_1^{-1})-g_i(\mathbf{s}\cdot B_1^{-1})^{-1})=\varepsilon_i(\hat{g}_i(\mathbf{s})-\hat{g}_i(\mathbf{s})^{-1}).$$

The result follows. □

Next we recall the definition of the Alexander polynomial of a link in M. Our basic reference will be Hillman's book [4].

If X is a space let $T_1(X)$ be the torsion subgroup of $H_1(X)$ and $F_1(X)=H_1(X)/T_1(X)$. For a link $L=K_1\cup K_2\cup\cdots\cup K_n\subseteq M$ with exterior X, $F_1(X)\cong\mathbb{Z}^n$ and $F_1(X)\otimes\mathbb{Q}$ is canonically isomorphic to $\mathbb{Q}^n$, the ith generator, s_i say, corresponding to the meridian μ_i

of K_i. Let $p:\tilde{X}\to X$ be the cover associated to the surjection $\pi_1(X)\to F_1(X)$ and x a base point in X. Set $R=\mathbb{Z}[F_1(X)]$ and define $A(L)$ to be the R-module $H_1(\tilde{X},p^{-1}(x))$. Let $E_1(L)$ be the first elementary ideal of $A(L)$. This is an ideal in R defined as in Chapter III of [4]. The first Alexander element of L is any generator Δ_L of the smallest principal ideal of R containing $E_1(L)$. The following are straightforward generalisations of classical results. They are proven using arguments identical to those found in the theorems cited.

Theorem 1.8 (Theorem (IV.3(i)), [4]). *Let I denote the augmentation ideal of $\mathbb{Z}[F_1(X)]$. Then*

$$E_1(L)=\begin{cases}(\Delta_L) & \text{if } n=1\\ (\Delta_L)I & \text{if } n\geqq 2.\end{cases} \qquad \square$$

If L has $(n-1)\geqq 0$ components and $L_1=L\cup K_n$ then the inclusion of X_1 into X induces a surjection $\Phi:\mathbb{Z}[F_1(X_1)]\to\mathbb{Z}[F_1(X)]$. Let $[K_n]$ be the class of K_n in $F_1(X)$.

Theorem 1.9 (Theorem (VII.2(i)), [4]).

$$\Phi(E_1(L_1))=\begin{cases}(|T_1(X)|) & \text{if } n=1\\ ([K_n]-1)E_1(L) & \text{if } n\geqq 2.\end{cases} \qquad \square$$

Under our identification of $F_1(X)\otimes\mathbb{Q}$ with $\mathbb{Q}^n$, there is a natural inclusion $\Psi:R=\mathbb{Z}[F_1(X)]\to\mathbb{Z}[\mathbb{Q}^n]$.

Definition 1.10. The *Alexander polynomial* of L is the fractional Laurent polynomial

$$\Delta_L(s_1,s_2,\dots,s_n)=(1/|H_1(M)|)\Psi(\Delta_L).$$

As usual $\Delta_L(s_1,\dots,s_n)$ is defined only up to multiplication by units of R. This is denoted by the "$\doteq$" sign.

One may also prove the analogues of the Torres symmetry properties for Alexander polynomials (see Theorem (VII.1) of [4]). For the moment we shall only assume this property for the polynomials of knots. If K is a knot in M then after multiplying $\Delta_K(s)$ by $\pm$ an appropriate rational power of s, $\Delta_K(s)$ satisfies

$$\begin{cases}\Delta_K(1)=d\\ \Delta_K(s^{-1})=\Delta_K(s).\end{cases} \tag{1.10}$$

Here $d=|T_1(X)|/|H_1(M)|$. We shall assume that $\Delta_K(s)$ has been so normalized in what follows. The symmetry property for links in general manifolds actually follows from the results of Section 3.

Combining Theorems (1.8) and (1.9) with the expression for $[K_n]$ in terms of the canonical basis of $F_1(X) \otimes \mathbb{Q}$ gives:

Theorem 1.11 (Torres restriction formula). *If* $L_1 = L \cup K_n (n \geqq 1)$ *then*

$$\Delta_{L_1}(s_1, s_2, \dots, s_{n-1}, 1) = \begin{cases} d & \text{if } n=1 \\ [(s_1^{l_{12}} - 1)/(s_1^d - 1)]\,\Delta_L(s_1) & \text{if } n=2 \\ (s_1^{l_{1n}} s_2^{l_{2n}} \dots s_{n-1}^{l_{n-1\,n}} - 1)\,\Delta_L(s_1, s_2, \dots, s_{n-1}) & \text{if } n \geqq 3. \end{cases} \qquad \square$$

Finally note that if $\mathbb{L}$ presents $\hat{L}$, then L and $\hat{L}$ have the same exteriors. From the construction of the Alexander polynomials of L and $\hat{L}$ it is then clear that these polynomials differ only by a reparametrisation of $\mathbb{Q}^n$. Indeed, if B is the framing matrix of $\mathbb{L}$ we have

$$\Delta_L(\mathbf{s}) \doteq (|H_1(M)|/|H_1(\chi(\mathbb{L}))|)\,\Delta_L(\mathbf{s} \cdot B^{-1}) \doteq |B|^{-1}\,\Delta_L(\mathbf{s} \cdot B^{-1}). \tag{1.12}$$

If L is a knot this equation is exact.

2. Definition of Conway potential function for links in $\mathbb{Q}$-homology 3-spheres

In this section we define Conway potential functions for links in $\mathbb{Q}$-homology 3-spheres using the properties of these functions for links in S^3. Our main tool is the following theorem giving a surgery formula for Conway potential functions of links in S^3. This theorem will be proved in Section 4.

Theorem 2.1. *Let $\mathbb{L}$ be a framed link in S^3 such that $\chi(\mathbb{L})$ is homeomorphic to S^3. Let $\hat{L}$ be the link presented by $\mathbb{L}$ and B be the associated framing matrix. Then*

$$\nabla_{\hat{L}}(\mathbf{s}) = |B|^{-1} \nabla_L(\mathbf{s} \cdot B^{-1}). \tag{2.2}$$

We shall need the following lemmas:

Lemma 2.3. *Let L be a link in a $\mathbb{Q}$-homology 3-sphere M, then there exists a link L^* in M such that:*

(i) *L is a sublink of L^*.*
(ii) *$M \backslash L^*$ is homeomorphic to $S^3 \backslash L^0$ for some link L^0 in S^3.*
(iii) $f_{L^*, L}(\mathbf{s}) \neq 0$.

Proof. Let $\mathbb{E}$ be a framed link in S^3 such that there is a homeomorphism $H: \chi(\mathbb{E}) \to M$. We may isotope L in M so that $H(E) \cap L = \varnothing$ and that for each component C_i of E there is a component K_j of L such that $lk_M(H(C_i), K_j) \neq 0$. The links $L^* = L \cup H(E)$ and $L^0 = E \cup H^{-1}(L)$ satisfy the conditions above. $\square$

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Let M, L, L^* and L^0 be as in Lemma 2.3 and let $h: M\backslash L^* \to S^3\backslash L^0$ be a homeomorphism. Let $\mathbb{L}^0$ be the framing of L^0 associated to the curves $h(\mu_i^*)$ in $\partial T(L^0)$ where the μ_i^* are the meridians of L^*. Let B be the framing matrix of $\mathbb{L}^0$. We say that B is the framing matrix associated to h. Set

$$\nabla_{L^*}(\mathbf{s}) = |B|^{-1}\nabla_{L^0}(\mathbf{s}\cdot B^{-1}).$$

Lemma 2.4. *∇_{L^*} is well-defined.*

Proof. If $h_1: M\backslash L^* \to S^3\backslash L_1^0$ and $h_2: M\backslash L^* \to S^3\backslash L_2^0$ are two such homeomorphisms with associated framing matrices B_1 and B_2, we must show that

$$|B_1|^{-1}\nabla_{L_1^0}(\mathbf{s}\cdot B_1^{-1}) = |B_2|^{-1}\nabla_{L_2^0}(\mathbf{s}\cdot B_2^{-1}).$$

Now $h_2h_1^{-1}: S^3\backslash L_1^0 \to S^3\backslash L_2^0$ is a homeomorphism with framing matrix $B_2B_1^{-1}$ (see the remarks following Definition 1.4). Theorem 2.1 shows that

$$\nabla_{L_1^0}(\mathbf{t}) = |B_1B_2^{-1}|\nabla_{L_2^0}(\mathbf{t}\cdot B_1B_2^{-1}).$$

Setting $\mathbf{s} = \mathbf{t}\cdot B_1$ finishes the proof. □

Definition 2.5. Let M, L, L^* and L^0 be as in Lemma 2.3, we define the Conway potential function ∇_L of the link L in M to be

$$\nabla_L(\mathbf{s}) = \frac{1}{f_{L^*,L}(\mathbf{s})}\nabla_{L^*}(\mathbf{s}, \mathbf{1}).$$

To see that ∇_L is well-defined we need the following lemma.

Lemma 2.6. *Let L be a link in a $\mathbb{Q}$-homology 3-sphere M. Let L_1^* and L_2^* be links in M such that:*

(i) $L \subseteq L_1^* \subseteq L_2^*$

(ii) *$M\backslash L^*$ is homeomorphic to $S^3\backslash L_1^0$ for some link L_1^0 in S^3*

(iii) *$f_{L_1^*,L}(\mathbf{s}) \neq 0$ and $f_{L_2^*,L}(\mathbf{s}) \neq 0$,*

then

$$\frac{1}{f_{L_1^*,L}(\mathbf{s})}\nabla_{L_1^*}(\mathbf{s}, \mathbf{1}) = \frac{1}{f_{L_2^*,L}(\mathbf{s})}\nabla_{L_2^*}(\mathbf{s}, \mathbf{1}).$$

Proof. Let $h: M\backslash L_1^* \to S^3\backslash L_1^0$ be a homeomorphism and let L^0 be the sublink of L_1^0 corresponding to L. Consider the link $h(L_2^*\backslash L_1^*)$ in S^3 and set $L_2^0 = L_1^0 \cup h(L_2^*\backslash L_1^*)$. Then the restriction of h to $M\backslash L_2^*$ is a homeomorphism between $M\backslash L_2^*$ and $S^3\backslash L_2^0$. Let B_1 and B_2 be the framing matrices associated to L_1^* and L_2^*, then B_2 is of the form

$$\begin{pmatrix} B_1 & 0 \\ X & I \end{pmatrix}$$

since the image under h of a meridian of a component of $L_2^*\backslash L_1^*$ is a meridian of the corresponding component of L_2^0. We must show that:

$$f_{L_2^*,L}(\mathbf{s})\nabla_{L_1^0}((\mathbf{s},\mathbf{1})\cdot B_1^{-1})=f_{L_1^*,L}(\mathbf{s})\,\nabla_{L_2^0}((\mathbf{s},\mathbf{1})\cdot B_2^{-1}).$$

Since

$$\nabla_{L_2^0}((\mathbf{s},\mathbf{1})\cdot B_2^{-1})=\nabla_{L_2^0}((\mathbf{s},\mathbf{1})\cdot B_1^{-1},\mathbf{1})=f_{L_2^0,L_1^0}((\mathbf{s},\mathbf{1})\cdot B_1^{-1})\nabla_{L_1^0}((\mathbf{s},\mathbf{1})\cdot B_1^{-1}),$$

Lemma 1.7 gives the result. □

Theorem 2.7. *The function ∇_L is well-defined for any link L in a $\mathbb{Q}$-homology 3-sphere.*

Proof. Let L be a link in a $\mathbb{Q}$-homology 3-sphere M. For $i=1,2$ let L_i^* be a link in M such that $L\subseteq L_i^*$, $f_{L_i^*,L}\neq 0$ and $M\backslash L_i^*$ is homeomorphic to $S^3\backslash L_i^0$ for some link L_i^0 in S^3. We must show:

$$\frac{1}{f_{L_1^*,L}(\mathbf{s})}\nabla_{L_1^*}(\mathbf{s},\mathbf{1})=\frac{1}{f_{L_2^*,L}(\mathbf{s})}\nabla_{L_2^*}(\mathbf{s},\mathbf{1}). \tag{2.8}$$

We may suppose that $L=L_1^*\cap L_2^*$. We can isotope if necessary $L_2^*\backslash L$ in M so that $L_3^*=L_1^*\cup L_2^*$ satisfies

$$f_{L_3^*,L_1^*}(\mathbf{s})\neq 0 \quad \text{and} \quad f_{L_3^*,L_2^*}(\mathbf{s})\neq 0.$$

Note that

$$f_{L_3^*,L_i^*}(\mathbf{s},\mathbf{1})=f_{L_3^*,L}(\mathbf{s}).$$

Lemma 1.7(i) shows that $f_{L_3^*,L}(\mathbf{s})\neq 0$ and Lemma 2.6 that both sides of the equality (2.8) are equal to

$$\frac{1}{f_{L_3^*,L}(\mathbf{s})}\nabla_{L_3^*}(\mathbf{s},\mathbf{1}).$$ □

3. Properties of the Conway potential functions

In this section we prove the properties of the Conway potential functions listed in the introduction using the analogous properties known to hold for links in S^3 (see [3]). We fix the following notation: Let L be a link in a $\mathbb{Q}$-homology 3-sphere M and let L^* be a link in M such that $L \subseteq L^*$, $f_{L^*,L} \neq 0$ and $M \backslash L^*$ is homeomorphic to $S^2 \backslash L^0$ with framing matrix B_0.

(III) Variance under surgery. Let $\mathbb{L}$ be a framing on L with framing matrix B. Let $\hat{L}^*$ be the image of the link L^* in $\chi(\mathbb{L})$. Then $\hat{L} \subseteq \hat{L}^*$ and $\chi(\mathbb{L}) \backslash \hat{L}^*$ is homeomorphic to $S^3 \backslash L^0$ with associated framing matrix

$$B^* = B_0 \begin{pmatrix} B & 0 \\ X & I \end{pmatrix}$$

By definition

$$\nabla_L(\mathbf{s}) = \frac{|B_0|^{-1}}{f_{L^*,L}(\mathbf{s})} \nabla_{L^0}((\mathbf{s}, \mathbf{1}) \cdot B_0^{-1}).$$

Lemma 1.7(ii) shows that $f_{\hat{L}^*,\hat{L}}(\mathbf{s}) = \mathrm{f}_{\mathrm{L}^*,\mathrm{L}}(\mathbf{s} \cdot B^{-1}) \neq 0$, hence

$$\nabla_{\hat{L}}(\mathbf{s}) = \frac{|B^*|^{-1}}{f_{\hat{L}^*,\hat{L}}(\mathbf{s})} \nabla_{L^0}((\mathbf{s}, \mathbf{1}) \cdot (B^*)^{-1})$$

$$= \frac{|B_0|^{-1}|B|^{-1}}{f_{L^*,L}(\mathbf{s} \cdot B^{-1})} \nabla_{L^0}(((\mathbf{s} \cdot B^{-1}), \mathbf{1}) \cdot B_0^{-1})$$

$$= |B|^{-1} \nabla_L(\mathbf{s} \cdot B^{-1}).$$

(IV) Restriction. We may suppose that the link L^* contains L'. By Lemma 1.7(i)

$$f_{L^*,L}(\mathbf{s}) = f_{L^*,L'}(\mathbf{s}, \mathbf{1}) f_{L',L}(\mathbf{s})$$

so $f_{L^*,L'} \neq 0$ and

$$\nabla_{L'}(\mathbf{s}, \mathbf{1}) = (1/f_{L^*,L'}(\mathbf{s}, \mathbf{1})) \nabla_{L^*}(\mathbf{s}, \mathbf{1})$$

$$= (f_{L',L}(\mathbf{s})/f_{L^*,L}(\mathbf{s})) \nabla_{L^*}(\mathbf{s}, \mathbf{1})$$

$$= f_{L',L}(\mathbf{s}) \nabla_L(\mathbf{s}).$$

(V) Symmetry. If L has n components, L^* and L^0 have m components, then ∇_{L^0} is $(-1)^m$ symmetric ([3]) and $f_{L^*,L}$ ia $(-1)^{m-n}$ symmetric. Thus ∇_L is $(-1)^n$ symmetric.

(I) and (II) Relation to the Alexander polynomial and value at 1. The putative relationship between Δ_L and ∇_L holds for links in S^3 ([3]) and so in particular for L^0.

From the definition of ∇_{L^*} and equation (1.12) it also holds for L^*. Using the restriction formulae for Δ (Theorem 1.11) and ∇ we see that it holds for L up to units of R when $n \geqq 2$. When L is a knot these restriction formulae plus the symmetry of Δ_L (equation (1.10)) and that of ∇_L show that $\nabla_L(s) = \pm \Delta_L(s^2)/(s^d - s^{-d})$.

We show next that

(i) $\nabla_L(s) = \Delta_L(s^2)/(s^d - s^{-d})$ whenever L is a knot.

(ii) $\nabla_L(1,1) = l_{12}$ when L has two components.

(iii) $\nabla_L(1,1,\ldots,1) = 0$ when L has more than two components.

Equation (iii) is an easy consequence of the restriction formula (IV) for ∇.

When $M = S^3$ (i) and (ii) hold ([3]) and note that any $\mathbb{Q}$-homology 3-sphere may be realised by a sequence of surgeries on knots in $\mathbb{Q}$-homology 3-spheres starting with S^3. Thus it suffices to prove that if (i) and (ii) are true in M then they are true in $M' = \chi(\mathbb{J})$ where J is a knot in M, $\mathbb{J} = K^{(p,q)}$ and $p \neq 0$.

We consider (i) first. Let K' be a knot in M' and let J' denote the knot presented by $\mathbb{J}$. We may assume that J' and K' are disjoint and that $l' = lk_{M'}(K', J') \neq 0$. Set $L' = J' \cup K'$. Let K be the knot in M corresponding to K' and set $\mathbb{L} = J^{(p,q)} \cup K^{(1,0)}$. If $l = lk_M(K, J)$ then Lemma 1.5 shows $l' = (l/p)$ and that $\nabla_{L'}(s,t) = (1/p)\nabla_L(s^{-l/p}t^{-ql'}, t)$. Setting $s = 1$ gives

$$(1/p)\nabla_L(t^{-ql'}, t) = f_{L',K'}(t)\nabla_{K'}(t) = \pm f_{L',K'}(t)\Delta_{K'}(t^2)/(t^d - t^{-d}). \tag{3.1}$$

Now by hypothesis $\nabla_L(1,1) = l$ and so letting t tend to 1 in equation (3.1) gives $l' = (l/p) = \pm l'$. Thus $\nabla_{K'}(s) = \Delta_{K'}(s^2)/(s^d - s^{-d})$ and as K' was arbitrary, (i) holds in M'.

Now suppose that $L' = K_1 \cup K_2$ is an arbitrary 2-component link in M'. Then

$$\nabla_{L'}(1,1) = \lim_{s \to 1} [(s^{l_{12}} - s^{-l_{12}})/(s^d - s^{-d})]\Delta_{K_1}(s^2) = l_{12}.$$

Thus (ii) holds in M'. This completes the proof of (I) and (II).

(VI) Orientation change. $\mathbb{L} = K_1^{(-1,0)} \cup K_2^{(1,0)} \cup \cdots \cup K_n^{(1,0)}$ presents L' and has framing matrix J where J is the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}.$$

Hence $\nabla_{L'}(\mathbf{s}) = |J|^{-1}\nabla_L(\mathbf{s} \cdot J^{-1}) = -\nabla_L(s_1^{-1}, s_2, \ldots, s_n)$.

(VII) Ambient orientation change. Let $\sim L$, $\sim L^*$ and $\sim L^0$ denote the links L, L^*, and L^0 in $-M$ and $-S^3$ respectively. Suppose that L has n components, L^* and L^0 have m components. As $(-M)\backslash(\sim L^*)$ is homeomorphic to $(-S^3)\backslash(\sim L^0)$ with associated matrix $-B_0$,

$$\nabla_{\sim L^*}(\mathbf{s}) = |-B_0|^{-1}\nabla_{\sim L^0}(\mathbf{s}\cdot(-B_0)^{-1}) = (-1)^m|B_0|^{-1}((-1)^m\nabla_{\sim L^0}(\mathbf{s}\cdot B_0^{-1}))$$

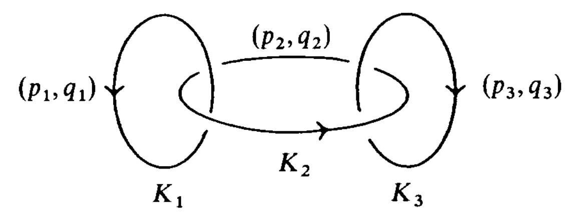
by (V) symmetry

$$= (-1)^{m-1}|B_0|^{-1}\nabla_{L^0}(\mathbf{s}\cdot B_0^{-1}).$$

Since $f_{\sim L^*,\sim L}(\mathbf{s}) = (-1)^{m-n}f_{L^*,L}(\mathbf{s})$ we have $\nabla_{\sim L}(\mathbf{s}) = (-1)^{n-1}\nabla_L(\mathbf{s})$.

(VIII) Skein relation. Let L_1 be a link in M such that $M\backslash L_1$ is homeomorphic to $S^3\backslash L_1^0$. Let $L_+^* = L_+ \cup L_1$, $L_-^* \cup L_1$ and $L_0^* = L_0 \cup L_1$. We may assume that L_1 was chosen so that $f_{L_+^*,L_+}(\mathbf{s})$, $f_{L_-^*,L_-}(\mathbf{s})$ and $f_{L_0^*,L_0}(\mathbf{s})$ are each nonzero. Denote by L_+^0, L_-^0 and L_0^0 the links in S^3 whose complements are homeomorphic to those of L_+^*, L_-^* and L_0^*. It may be verified that L_+^0, L_-^0 and L_0^0 are skein-related and as they lie in S^3 they satisfy the Conway's skein relation (for a proof, modify the argument in (4.2) in [3] appropriately). It is now a simple matter using the definition of ∇ to show the skein relation holds between the potential functions of L_+^*, L_-^* and L_0^*.

As an example, consider the following framed link $\mathbb{L}$ in S^3:



Using the free differential calculus of Fox, one can see that $\Delta_L(s_1,s_2,s_3) \doteq s_2 - 1$ so that $\nabla_L(s_1,s_2,s_3) = \varepsilon(s_2 - s_2^{-1})$ where $\varepsilon = \pm 1$ by property V. Using property IV, $\nabla_L(1,s,1) = (s-s^{-1})^2\nabla_{K_2}(s) = s - s^{-1}$, so that $\varepsilon = +1$. The framing matrix B of $\mathbb{L}$ is

$$B = \begin{pmatrix} p_1 & q_2 & 0 \\ q_1 & p_2 & q_3 \\ 0 & q_2 & p_3 \end{pmatrix}.$$

The manifold $\chi(\mathbb{L})$ is a Seifert fibre space over S^2 with at most three exceptional fibres and is a $\mathbb{Q}$-homology 3-sphere if $\beta = |B| \neq 0$. Lemma 1.5 shows that $lk(\hat{K}_1, \hat{K}_2) = p_3/\beta$ and $lk(\hat{K}_1, \hat{K}_3) = p_1/\beta$. Properties III and IV show that $\nabla_{\hat{L}}(1,s,1) = 1/\beta\,\nabla_L((1,s,1)\cdot B^{-1}) = 1/\beta(s^{p_1p_3/\beta} - s^{-p_1p_3/\beta}) = \nabla_{\hat{K}_2}(s)\,(s^{p_1/\beta} - s^{-p_1/\beta})\,(s^{p_3/\beta} - s^{-p_3/\beta})$ Set $s = u^\beta$ and $g_k(s) = (s^k - s^{-k})/(s - s^{-1})$, then $\Delta_{\hat{K}_2}(u^{2\beta}) = 1/\beta(g_{p_1p_3}(u)g_\beta(u))/(g_{p_1}(u)g_{p_3}(u))$.

4. The surgery formula for Conway potential functions of links in S^3

This section is devoted to the proof of Theorem 2.1. The surgery formula (2.2) is

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derived from the standard properties of Conway potential functions for links in S^3 as established in [3]. We shall use for the proof Rolfsen's version of the "calculus theorem" of Kirby [5]. It is worth mentioning that Theorem 2.1 has an elementary (though long) proof depending only on the basic properties of the potential function.

We mention briefly how Rolfsen moves are defined for oriented framed links (for more details, see [6]). Let $\mathbb{L}$ be an oriented framed link in S^3.

(i) **Trivial insertion:** Add to $\mathbb{L}$ another oriented component with framing $(\varepsilon, 0)$ with $\varepsilon = \pm 1$.

(ii) **Trivial deletion:** Delete such a component.

(iii) **Twist move:** Select a trivial component K_j of L, twist t times $(t \in \mathbb{Z})$ along a disc spanning K_j and replace the framings as follows:

$$\text{if } i \neq j \text{ change } (p_i, q_i) \text{ to } (p_i + tq_i lk(K_i, K_j)^2, q_i),$$

$$\text{if } i = j \text{ change } (p_j, q_j) \text{ to } (p_j, q_j + tp_j).$$

Let $\mathbb{L}$ be a framed link in S^3 such that $\chi(\mathbb{L})$ is homeomorphic to S^3. We know from (1.12), symmetry property (V) and the relation between Conway potential functions and Alexander polynomials that $\nabla_{\bar{L}}(\mathbf{s}) = \delta(\mathbb{L})|B|^{-1}\nabla_L(\mathbf{s} \cdot B^{-1})$ where $\delta(\mathbb{L}) = \pm 1$. To prove Theorem 2.1, we must show that $\delta(\mathbb{L}) = 1$.

Lemma 4.1. *Let $\mathbb{L}$ be a framed link in S^3 such that $\chi(\mathbb{L})$ is homeomorphic to S^3 and $\nabla_L \neq 0$, then there exists a sequence $\mathbb{L}_0, \mathbb{L}_1, \dots, \mathbb{L}_N = \mathbb{L}$ of framed links such that:*

(i) *$\mathbb{L}_0$ is the trivial knot with framing $(\pm 1, 0)$,*

(ii) $\nabla_{L_i} \neq 0$, $i = 0, \dots, N$,

(iii) *for $i = 0, \dots, N-1$, $\mathbb{L}_{i+1}$ is obtained from $\mathbb{L}_i$ by a Rolfsen move R_i.*

Proof. Since $\chi(\mathbb{L})$ is homeomorphic to S^3, the calculus theorem of Kirby [5, 6] shows that there is a sequence of framed links $\mathbb{L}_i$ connecting $\mathbb{L}_0$ to $\mathbb{L}_N = \mathbb{L}$ as in (iii). It may happen that for some i, $\nabla_{L_i} \neq 0$ while $\nabla_{L_{i+1}} = 0$. Denote by R_i the Rolfsen move changing $\mathbb{L}_i$ to $\mathbb{L}_{i+1}$ and note that R_i cannot be a twist move since in this case $\nabla_{L_{i+1}}(\mathbf{s}) = \pm|B|^{-1}\nabla_{L_i}(\mathbf{s} \cdot B^{-1})$ for some unimodular matrix B.

First case: R_i and R_{i+1} are insertions or deletions. In the set of links L_i, L_{i+1}, L_{i+2}, choose one with the greatest number of components and call it $\bar{L}$. Choose an oriented knot K disjoint from $\bar{L}$ such that K links each component of $\bar{L}$ once and consider the framed links $\mathbb{L}'_i = \mathbb{L}_i \cup K^{(1,0)}$, $\mathbb{L}'_{i+1} = \mathbb{L}_{i+1} \cup K^{(1,0)}$, $\mathbb{L}'_{i+2} \cup K^{(1,0)}$. Since K has linking number one with all other components, $\nabla_{L'_i}$, $\nabla_{L'_{i+1}}$ and $\nabla_{L'_{i+2}}$ are nonzero polynomials.

Second case: R_i is an insertion or deletion, R_{i+1} is a twist move on a component C of L_{i+1}. Amongst L_i, L_{i+1}, choose one with the greatest number of components and call it $\bar{L}$. Choose an oriented knot K disjoint from $\bar{L}$ such that:

(i) K links each component of $\bar{L}$ except C once,

(ii) $lk(K, C)=0$, $\nabla_{K\cup C}\neq 0$ and $\nabla_{\bar{K}\cup C}\neq 0$, where $\bar{K}$ denotes the image of K after the twist move R_{i+1}.

Set $\mathbb{L}'_i=\mathbb{L}_i\cup K^{(1,0)}$, $\mathbb{L}'_{i+1}=\mathbb{L}_{i+1}\cup K^{(1,0)}$, $\mathbb{L}'_{i+2}=\mathbb{L}_{i+2}\cup K^{(1,0)}$; then $\nabla_{L'_i}$, $\nabla_{L'_{i+1}}$ and $\nabla_{L'_{i+2}}$ are nonzero polynomials.

In both cases the sequence $\mathbb{L}_0,\dots,\mathbb{L}_i$, $\mathbb{L}'_i$, $\mathbb{L}'_{i+1}$, $\mathbb{L}'_{i+2}$, $\mathbb{L}_{i+2},\dots,\mathbb{L}_N$ has fewer links with trivial potential functions. Using this argument repeatedly proves Lemma 4.1. □

Lemma 4.2. *Let $\mathbb{L}$ be a framed link in S^3 such that $\chi(\mathbb{L})$ is homeomorphic to S^3. Let $\mathbb{L}'$ be obtained from $\mathbb{L}$ by a Rolfsen move R. Suppose that $\nabla_L\neq 0$ and $\nabla_{L'}\neq 0$, then $\delta(\mathbb{L})=\delta(\mathbb{L}')$.*

Proof. Let B and B' be the framing matrices of $\mathbb{L}$ and $\mathbb{L}'$. We know that

$$\nabla_{\mathcal{L}}(\mathbf{s})=\delta(\mathbb{L})|B|^{-1}\nabla_L(\mathbf{s}\cdot B^{-1}) \quad\text{and}\quad \nabla_{\mathcal{L}'}(\mathbf{s})=\delta(\mathbb{L}')|B'|^{-1}\nabla_L(\mathbf{s}\cdot(B')^{-1}).$$

First case. R is a trivial insertion or deletion. Let

$$\mathbb{L}=K_1^{(p_1,q_1)}\cup\cdots\cup K_n^{(p_n,q_n)} \quad\text{and}\quad \mathbb{L}'=\mathbb{L}\cup J_{n+1}^{(\varepsilon,0)}$$

with $\varepsilon=\pm 1$. Then $|B'|=\varepsilon|B|$ and

$$\nabla_{\mathcal{L}'}(\mathbf{s},1)=f_{\mathcal{L}',\mathcal{L}}(\mathbf{s})\nabla_{\mathcal{L}}(\mathbf{s})=\delta(\mathbb{L})|B|^{-1}f_{\mathcal{L}',\mathcal{L}}(\mathbf{s})\nabla_L(\mathbf{s}\cdot B^{-1}).$$

On the other hand:

$$\nabla_{\mathcal{L}'}(\mathbf{s},1)=\delta(\mathbb{L}')\varepsilon|B|^{-1}\nabla_{L'}(\mathbf{s}\cdot B^{-1},1)=\delta(\mathbb{L}')\varepsilon|B|^{-1}f_{L',L}(\mathbf{s}\cdot B^{-1})\nabla_L(\mathbf{s}\cdot B^{-1}).$$

Lemma 1.7(ii) shows that:

$$\delta(\mathbb{L})f_{L',L}(\mathbf{s}\cdot B^{-1})\nabla_L(\mathbf{s}\cdot B^{-1})=\delta(\mathbb{L}')f_{L',L}(\mathbf{s}\cdot B^{-1})\nabla_L(\mathbf{s}\cdot B^{-1}).$$

(a) Suppose first that there is an index i, $1\leqq i\leqq n$, such that $lk(K_i, J)\neq 0$, then $f_{L',L}\neq 0$ and $\nabla_L\neq 0$ so that $\delta(\mathbb{L}')=\delta(\mathbb{L})$.

(b) If $lk(K_i, J)=0$ for $1\leqq i\leqq n$, add a component K disjoint from L' such that $lk(J,K)=lk(K_i,K)=1$ for $1\leqq i\leqq n$, then using case (a): $\delta(\mathbb{L}\cup K^{(1,0)})=\delta(\mathbb{L})$, $\delta(\mathbb{L}'\cup K^{(1,0)})=\delta(\mathbb{L}')$ and $\delta(\mathbb{L}\cup K^{(1,0)})=\delta(\mathbb{L}'\cup K^{(1,0)})$. This shows $\delta(\mathbb{L}')=\delta(\mathbb{L})$.

Second case: R is a twist move.

(a) We first show that if K_1 is a trivial knot and $\mathbb{L}=K_1^{(1,m)}\cup K_2^{(1,0)}\cup\cdots\cup K_n^{(1,0)}$ is a framed link in S^3, then $\delta(\mathbb{L})=1$.

Let B be the framing matrix of $\mathbb{L}$. Insert a component K_0 such that $lk(K_0, K_i)=1$, $1\leqq i\leqq n$ and consider $\mathbb{L}^*=K_0^{(1,0)}\cup\mathbb{L}$. Then $\nabla_{\mathcal{L}^*}(s_0,s_1,\mathbf{1})=\delta(\mathbb{L}^*)\nabla_{L^*}(s_0,s_0^{-m}s_1,\mathbf{1})$. Let L_0 denote the link $K_0\cup K_1$ and $\hat{L}_0$ denote the corresponding link in $\chi(\mathbb{L}^*)$, then

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$$f_{\hat{L}^*,\hat{L}_0}(s_0,s_1)\,\nabla_{\hat{L}_0}(s_0,s_1)=\delta(\mathbb{L}^*)f_{L^*,L_0}(s_0,s_0^{-m}s_1)\,\nabla_{L_0}(s_0,s_0^{-m}s_1).$$

As $f_{L^*,L_0}\neq 0$, Lemma 1.7(ii) shows that

$$\nabla_{\hat{L}_0}(s_0,s_1)=\delta(\mathbb{L}^*)\,\nabla_{L_0}(s_0,s_0^{-m}s_1).$$

Setting $s_0=s_1=1$ and applying Lemma 1.5 gives $\delta(\mathbb{L}^*)=1$. The first case now implies $\delta(\mathbb{L})=\delta(\mathbb{L}^*)=1$.

(b) We consider now the general twist move: Let $\mathbb{L}$ be a framed link in S^3 and suppose that K_1 is a trivial knot. Perform a t-twist move along a disc spanning K_1 and let $\mathbb{L}'$ be the framed link obtained after the twist. Let B and B' be the framing matrices of $\mathbb{L}$ and $\mathbb{L}'$. They satisfy $B'=TB$ where

$$T=\begin{bmatrix} 1 & 0 & \cdots\cdots\cdots 0 \\ tlk(K_1,K_2) & 1 & 0\ \cdots\cdots 0 \\ \vdots & & \\ tlk(K_1,K_n) & 0 & \cdots\cdots\cdots 1 \end{bmatrix}.$$

Consider the framed link $\mathbb{L}_0=K_1^{(1,-t)}\cup K_2^{(1,0)}\cup\cdots\cup K_n^{(1,0)}$. Note that $\hat{L}$ is isotopic to $\hat{L}'$ and that L' is isotopic to $\hat{L}_0$. This shows that

$$\delta(\mathbb{L})\left|B\right|^{-1}\nabla_L(\mathbf{s}\cdot B^{-1})=\nabla_{\hat{L}}(\mathbf{s})=\nabla_{\hat{L}'}(\mathbf{s})=\delta(\mathbb{L}')\left|B'\right|^{-1}\nabla_{L'}(\mathbf{s}\cdot(B')^{-1}).$$

Setting $\mathbf{s}\cdot B^{-1}=\mathbf{u}\cdot T$ we get $\delta(\mathbb{L})\,\nabla_L(\mathbf{u}\cdot T)=\delta(\mathbb{L}')\,\nabla_{L'}(\mathbf{u})$. Using part (a) we see that $\delta(\mathbb{L}')=\delta(\mathbb{L})$. □

Proof of Theorem 2.1. We know that $\nabla_{\hat{L}}(\mathbf{s})=\delta(\mathbb{L})\left|B\right|^{-1}\nabla_{L'}(\mathbf{s}\cdot B^{-1})$ where $\delta(\mathbb{L})=\pm 1$. If $\nabla_L=0$, then (2.2) clearly holds. If $\nabla_L\neq 0$, let $\mathbb{L}_0,\mathbb{L}_1,\ldots,\mathbb{L}_N=\mathbb{L}$ be a sequence of framed links as in Lemma 4.1. Lemma 4.2 shows that $\delta(\mathbb{L})=\delta(\mathbb{L}_{N-1})=\cdots=\delta(\mathbb{L}_0)$. Obviously $\delta(\mathbb{L}_0)=1$ and (2.2) holds for $\mathbb{L}$. □

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