Canad. Math. Bull. Vol. **46** (1), 2003 pp. 140–148

An Explicit Cell Decomposition of the Wonderful Compactification of a Semisimple Algebraic Group

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Abstract. We determine an explicit cell decomposition of the wonderful compactification of a semisimple algebraic group. To do this we first identify the $B \times B$-orbits using the generalized Bruhat decomposition of a reductive monoid. From there we show how each cell is made up from $B \times B$-orbits.

1 Introduction

The most commonly studied cell decompositions in algebraic geometry are the ones obtained by the method of [1]. If $S = k^*$ acts on a smooth complete variety X with finite fixed point set $F \subseteq X$, then $X = \bigsqcup_{\alpha \in F} C_\alpha$ where $C_\alpha = \{x \in X \mid \lim_{t \to 0} tx = \alpha\}$. Furthermore, each C_α is isomorphic to an affine space. If further, a semisimple group G acts on X extending the action of S, we may assume that each C_α is stable under the action of some Borel subgroup B of G with $S \subseteq B$. Thus, each C_α is a union of B-orbits.

In case X is the (two-sided) wonderful compactification of a semisimple group G, the above procedure has been carried out in [4]. In fact, they obtain results for a more general class of wonderful compactifications. Let G be a semisimple algebraic group, and suppose $\sigma\colon G \to G$ is an involution (so that $\sigma \circ \sigma = \mathrm{id}_G$) with $H = \{x \in G \mid \sigma(x) = x\}$. The *wonderful compactification* of G/H (according to [4]) is the unique normal G-equivariant compactification X of G/H obtained by considering an irreducible representation $\rho\colon G \to G\ell(V)$ of G with $\dim(V^H) = 1$ and with highest weight in general position. Then let $h \in V^H$ be nonzero and define

$$X = \overline{\rho(G)[h]} \subseteq \mathbb{P}(V),$$

the Zariski closure of the orbit of $[h]$. (See [4, Section 2] for details.) In this paper we restrict our attention to the special case where the group is $G \times G$ and $\sigma\colon G \times G \to G \times G$ is given by $\sigma(g, h) = (h, g)$. It is easy to see that, in this case, the $G \times G$-variety $(G \times G)/H$ can be canonically identified with G with its two-sided G-action.

Much important work has been accomplished since [4] appeared. See [13], [12], [2], [5]. In particular Brion [2] obtains much information about the structure of X. Among other things, he finds a BB-decomposition $X = \bigsqcup_{x \in F} C_x$ from which he then

Received by the editors March 5, 2001; revised October 14, 2001.
AMS subject classification: 14L30, 14M17, 20M17.

https://doi.org/10.4153/CMB-2003-014-1 Published online by Cambridge University Press

obtains a basis of the Chow ring of X of the form $\{\overline{By(x)B}\}_{x\in F}$. He also identifies explicitly how each cell C_x is made up from $B \times B$-orbits.

In this paper, we identify an explicit cell decomposition of X. Our approach is somewhat different than [2]. While Brion uses the combinatorics of spherical embeddings, we use the results of [9] to construct our cells directly from $B \times B$-orbits on a reductive monoid. We first identify the $B \times B$-orbits using the results of [9] on generalized Bruhat decomposition for reductive monoids. Indeed, there is a precise monoid analogue R of the Weyl group W, so that $X = \bigsqcup_{x\in R} BxB$. We then define for each "rank-one" element $r \in R$, a cell

$$C_r = \bigsqcup_{x\in \mathcal{C}_r} BxB \cong K^{n_r}$$

so that $X = \bigsqcup_{r\in R_1} C_r$. In particular, each $\mathcal{C}_r$ is defined using standard Weyl group combinatorics. These cells are canonically indexed by the fixed points of $T \times T$ on X. Our method also yields an explicit cell decomposition for each $G \times G$-orbit.

It appears that my cell decomposition agrees with the one of [2] (although we have not actually verified this). However, the monoid approach is sufficiently important to warrant a separate direct treatment. The interested reader should consult Brion's excellent paper for more information on the $B \times B$-orbit closures and the Chow ring for X.

To complete our results we give an explicit formula for the Poincaré polynomial of X_I.

2 Orbit Structure of X

In this section we assemble the relevant background information about the wonderful compactification of G.

Let G be a connected, semisimple group of adjoint type defined over the algebraically closed field k. The *wonderful compactification* X of G can be obtained as follows:

Let $G_1 = G\times k^*$, and consider normal reductive monoids M with 0 and unit group G_1[6, 7]. By the results of [11], this yields a systematic procedure for constructing two-sided compactifications of G. We can easily identify X as follows:

Let $\rho\colon G_1 \to G\ell(V)$ be an irreducible, faithful representation such that $\rho(1,t)(v) = tv$ for all $t \in k^*, v \in V$. Define

$$M_1 = \overline{\rho(G_1)} \subseteq \mathrm{End}(V) \quad \text{and let}$$
$$M = \text{the normalization of } M_1.$$

Assume also that the highest weight λ of $\rho|G$ is regular (*i.e.*: if $\lambda = \Sigma m_i\lambda_i$, where $\{\lambda_i\}$ is the set of fundamental dominant weights, then $m_i > 0$ for each i). Define

$$X = X_\lambda = (M \setminus \{0\})/k^*.$$

We assume G is of adjoint type to ensure that X is smooth. See Proposition 3.4.

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Proposition 2.1 *X is independent of λ.*

Proof Let $T \subseteq G$ be a maximal torus. By [10, Theorem 7.1.5], M is the unique, normal monoid with polyhedral root system (Z, Φ, C_λ) where $C_\lambda \subseteq X(T) \oplus \mathbb{Z} = Z$ is the smallest normal cone containing $\{(w^*(\lambda), 1) \mid w \in W\}$. Here $w^*(\lambda)(t) = \lambda(wtw^{-1})$ for $t \in T \times k^*$. If λ' is another regular, dominant weight, one checks that (Z, ϕ, C_λ) and $(Z, \phi, C_{\lambda'})$ are projectively equivalent polyhedral root systems in the sense of [11, Definition 3.5]. Thus, by [11, Theorem 3.7], $X_\lambda \cong X_{\lambda'}$ as $G \times G$-varieties. ■

Proposition 2.2 *X is the wonderful compactification of [4].*

Proof By our definition, $X = \overline{G \cdot \tilde{h}}$ where $\tilde{h} \in \mathbb{P}\big(\mathrm{End}(V)\big)$ is the class of $1_V \in \mathrm{End}(V) \cong V \otimes V^*$. But this corresponds to the construction of [4] via their Proposition 1.7 and Section 2.1. Indeed $H = \{(g, g)\} \subseteq G \times G$ (so my $\tilde{h}$ above is fixed by H under the action $G \times G \times \mathrm{End}(V) \to \mathrm{End}(V)$, $(g, y)(x) = gxy^{-1}$). Thus, my $\tilde{h}$ corresponds to their $\tilde{h}$. Notice also that their λ corresponds to my $\lambda \otimes \lambda^*$. ■

We now describe the orbit structure of X. We use the results of [8], [9], [10], along with our description above of X as $(M \setminus 0)/k^*$.

Let T be a maximal torus of G contained in the Borel subgroup B of G. Let Δ be the set of positive simple roots and let $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of simple reflections. For $I \subseteq \Delta$ or S we let P_I be the corresponding standard parabolic.

Proposition 2.3 *$X = \bigsqcup_{I \subseteq S} Ge_I G$ where $e_I \in \bar{T}$ is a representative of the unique T-orbit with $P_I = \{g \in G \mid e_I g \in Ge_I\}$.*

Proof By [9, Section 3.3.1], $M = \bigsqcup_{e \in \Lambda} GeG$ where $\Lambda = \{e \in E(\bar{T}) \mid eB \subseteq Be\}$ and $E(\bar{T}) = \{e \in \bar{T} \mid e^2 = e\}$. But from [8, Theorem 4.16], $\Lambda \cong \mathcal{P}(S) = \{I \subseteq S\}$ via $e \leftrightarrow I$ where $e = e_I$, since for this M, $J_0 = \phi$ because λ is a regular weight. The statement about P_I follows from [8, Corollary 4.12]. ■

Let $\Lambda = \{e_I \mid I \subseteq S\}$. Define

$$R = \overline{N_G(T)}/\sim$$

where $x \sim y$ if $xT = yT$. Notice that for $x \in \overline{N_G(T)}$, $Tx = xT$.

Proposition 2.4 *$X = \bigsqcup_{x \in R} BxB$.*

Proof See [9, Corollary 5.8]. R above corresponds to $\mathcal{R} \setminus \{0\}$ of [9]. ■

Notice that

$$R = \bigsqcup_{I \subseteq S} We_I W$$

so that also

$$Ge_IG = \bigsqcup_{x \in We_IW} BxB.$$

Here W is the Weyl group of G relative to T. It turns out that there is a normal form for the elements of R.

Proposition 2.5 *Let $x \in R$. Then there exist unique $u, v \in W$, and $e_I \in \Lambda$ such that*

i) $x = ue_Iv$, *and*
ii) $I \subseteq \{s \in S \mid \ell(us) > \ell(u)\} = I_u$.

Proof This follows from the results of [6], but we indicate a direct proof.

We can write $x = we_Iy$ for some $I \subseteq S$ and some $w, y \in W$. By Proposition 2.3, I is unique. From well-known results about Coxeter groups we can write $w = uc$ where $c \in W_I$ and $\ell(us) > \ell(u)$ for $s \in I$. Furthermore, u and c are unique. But $ce_I = e_Ic$, so we can write $x = ue_Iv$ where $v = cy$. ■

We refer to the decomposition $x = ue_Iv$ as the *normal form* of x.

To obtain our cell decomposition of X, we first "solve" the corresponding problem in R. Define

$$\begin{aligned} R_1 &= \{x \in R \mid Tx = xT = x\} \\ &= We_\phi W \cong W \times W \end{aligned}$$

the "rank-one" elements of R. Notice that $R_1 = X^{T\times T}$. Not surprisingly, our cells are indexed by the elements of R_1. Define

$$\varphi \colon R \to R_1$$

by $\varphi(x) = ue_\phi v$ if $x = uev$ is the normal form of x.

Definition 2.6 For $r = ue_\phi v \in R_1$, define

$$\mathcal{C}_r = \varphi^{-1}(r) = \{x \in R \mid x = ue_Iv, \text{ normal form } I \subseteq I_u\}.$$

So $R = \bigsqcup_{r \in R_1} \mathcal{C}_r$.

3 The Cells of X and X_I

Definition 3.1 Let $r \in R_1$. Define

$$C_r = \bigsqcup_{x \in \mathcal{C}_r} BxB.$$

For $I \subseteq S$ define

$$C_{I,r} = C_r \cap X_I$$

https://doi.org/10.4153/CMB-2003-014-1 Published online by Cambridge University Press

where $X_I = \overline{Ge_IG}$.

We refer to C_r and $C_{I,r}$ as *cells*. In this section we determine the structure of C_r and $C_{I,r}$. In particular, we find that each of these cells is isomorphic to an affine space.

Let U be the unipotent radical of B, and U^- the unipotent radical of B^-.

Proposition 3.2 *Let $x \in R$ and write $x = ue_Iv$ in normal form. Then*

$$\begin{aligned} BxB &= (Uu \cap uU)(e_IT)(vU \cap U^-v) \\ &\cong_0 (Uu \cap uU) \times e_IT \times (vU \cap U^-v) \end{aligned}$$

where $\cong_0$ means isomorphism if char $(k) = 0$, *and bijection if* char $(k) > 0$.

Proof Let $e = e_I$ and notice that $eB = eC_B(e) = C_B(e)e$, since $eB \subseteq Be$. Here $C_B(e) = \{y \in B \mid ye = ey\}$. Notice also that $vBv^{-1} = (vBv^{-1} \cap B)(vBv^{-1} \cap U^-)$ (direct product of varieties) where B^- is the Borel subgroup opposite to B relative to T. So

$$\begin{aligned} evBv^{-1} &= e(vBv^{-1} \cap B)(vBv^{-1} \cap U^-) \\ &= eV(vBv^{-1} \cap U^-) \\ &= Ve(vBv^{-1} \cap U^-) \end{aligned}$$

where $V \subseteqq C_B(e)$ is some connected subgroup with $T \subseteq V$. Indeed, $e(vBv^{-1} \cap B) \subseteq eB = eC_G(e)$. So by [7, Theorem 6.1(iii)] we can take $V = vBv^{-1} \cap C_G(e)$. So we get

$$(*) \qquad evB = Ve(vB \cap U^-v).$$

We now look at Bue. Recall first that $\ell(us) > \ell(u)$ for any $s \in I$. By [1, Proposition 2.3.3] this is the same as saying $uC_B(e_I)u^{-1} \subseteq B$, or equivalently,

$$(**) \qquad C_B(e_I) \subseteq u^{-1}Bu \cap B.$$

Thus,

$$\begin{aligned} u^{-1}Bue &= (u^{-1}Bu)(u^{-1}Bu \cap U^-)e \\ &= (u^{-1}Bu \cap B)e \end{aligned}$$

since $(u^{-1}Bu \cap U^-)e = e$. Indeed, if $U_\alpha \subseteq u^{-1}Bu \cap U^-$ then $U_\alpha \nsubseteq C_G(e_I)$ since if $U_\alpha \subseteq C_G(e_I)$ then by $(**)$ above $C_B(e_I) \subsetneqq C_{u^{-1}Bu}(e_I)$ which is impossible for dimension reasons. Therefore,

$$(***) \qquad Bue = (Bu \cap uB)e.$$

So using $(*)$ and $(***)$ we obtain

$$u^{-1}BuevB = (u^{-1}Bu \cap B)eV(vB \cap U^-v) = (u^{-1}Bu \cap B)e(vB \cap U^-v)$$

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since by $(**)$ $V \subseteq C_B(e) \subseteq u^{-1}Bu \cap B$. Thus, $BuevB = (Bu \cap uB)e(vB \cap U^{-}v)$ and so

$$BuevB = (Uu \cap uU)(eT)(vU \cap U^{-}v).$$

Now suppose that $aetb = cesd$ where $a, c \in u^{-1}Uu \cap U$ and $b, d \in vUv^{-1} \cap U^{-}$ and $s, t \in T$. So $c^{-1}aet = esdb^{-1}$. But $c^{-1}aet \in UeT \subseteq \bar{B}$ and $esdb^{-1} \in eTU^{-} \subseteq \overline{B^{-}}$, while $\bar{B} \cap \overline{B^{-}} = \bar{T}$. Thus $c^{-1}aet \in \bar{T}$ and so $c^{-1}ae \in \bar{T}$. But then $c^{-1}ae = e$ because $c^{-1}a \in U$.

However, if $ge = e$ then $g \in B^{-}$ and hence $c^{-1}a \in U \cap U^{-} = \{1\}$. Similarly $b = d$, and thus $es = et$ as well. This proves that

$$m\colon (Uu \cap uU) \times eT \times (vU \cap U^{-}v) \to BuevB,$$

$m(y, et, z) = yetz$, is bijective. If char $(k) = 0$ then by Zariski's Main Theorem, m is an isomorphism. ■

Proposition 3.3 *If $r = ue_\phi v$ then*

$$C_r = (Un \cap uU)(Z_u)(vU \cap U^{-}v)$$
$$\cong_0 (Uu \cap uU) \times Z_u \times (vU \cap U^{-}v)$$

where $Z_u = \bigsqcup_{I \subseteq I_u} e_I T \subseteq \bar{T}$.

Proof

$$C_r = \bigsqcup_{ue_I v \in \mathcal{C}_r} (Uu \cap uU)(e_I T)(vU \cap U^{-}v)$$
$$= (Uu \cap uU)\Big(\bigsqcup_{I \subseteq I_u} e_I T\Big)(vU \cap U^{-}v)$$

So apply Proposition 3.2. ■

Proposition 3.4 *$Z_u \subseteq \bar{T}$ is a subvariety isomorphic to $k^{i(u)}$ where $i(u) = |I_u|$.*

Proof We may assume here that $u = 1$, so that $Z_1 = \bigsqcup_{I \subseteq S} e_I T$. Any other Z_u is a T-orbit closure of Z_1 of the required dimension.

By the proof of Proposition 2.1 X can be obtained from the reductive monoid M whose polyhedral root system is (Z, Φ, C_λ) where $Z = X(T) \oplus \mathbb{Z}$ and C_λ is the smallest normal cone of Z containing $\big\{ \big(u^*(\lambda), 1\big) \mid w \in W \big\}$. We may also assume that (Z, Φ, C_λ) is *integral* in the sense of [11, Definition 2.1]. If $T_1 = T \times k^* \subseteq G(M) = G \times k^*$ is the maximal torus of $G(M)$ corresponding to $T \subseteq G$, then $\pi^{-1}(Z_1) = \{x \in \overline{T_1} \mid xe_\phi \neq 0\}$, where $\pi\colon M \setminus 0 \to X$ is the canonical projection. One checks that

$$\mathcal{O}\big(\pi^{-1}(Z_1)\big) = k[C_\lambda][1/\chi]$$

where $\chi = \delta\lambda$ (multiplicative notation) corresponds to $(1, \lambda) \in Z = X(T) \oplus \mathbb{Z}$. (So $\delta\colon T \times k^* \to k^*$ is the projection). Now if $s_\alpha \in S$ then $s_\alpha(\chi) = \alpha^{-k}\chi$ for some $k > 0$, since λ is regular. So $s_\alpha(\chi)\chi^{-1} = \alpha^{-k} \in \mathcal{O}\big(\pi^{-1}(Z_1)\big)$. Thus $\alpha^{-1} \in \mathcal{O}\big(\pi^{-1}(Z_1)\big)$ since $\mathcal{O}\big(\pi^{-1}(Z_1)\big)$ is integrally closed in $\mathcal{O}(T \times k^*)$. It follows that $\mathcal{O}\big(\pi^{-1}(Z_1)\big) = k[\alpha_1^{-1}, \dots, \alpha_\ell^{-1}, \chi, \chi^{-1}]$ since G is of adjoint type and C_λ is integral. Thus $\mathcal{O}(Z_1) = k[\alpha_1^{-1}, \dots, \alpha_\ell^{-1}]$. Here ℓ is the rank of G. ■

Theorem 3.5 *Let $r = ue_\phi v \in R_1$. Then there is a bijective morphism*

$$m\colon k^{n_r} \to C_r$$

where $n_r = \ell(w_0) - \ell(u) + i(u) + \ell(v)$. Here $w_0 \in W$ is the longest element, so that $\ell(w_0) = |\Phi^+|$.

Proof We have from Propositions 3.3 and 3.4 that $\dim(C_r) = \dim(Uu \cap uU) + \dim(Z_u) + \dim(vU \cap U^- v)$. One checks easily that $\dim(Uu \cap uU) = \ell(w_0) - \ell(u)$ and that $\dim(vU \cap U^- v) = \ell(v)$. Also, from Proposition 3.4 $\dim(Z_u) = i(u)$. ■

We now consider the cell decomposition for $X_I = \overline{Ge_I G}$. Recall from [4] that X_I is a smooth, spherical $G \times G$-subvariety of X and therefore must have a $B \times B$-equivariant cell decomposition of its own. This is straightforward since $X^{T \times T} \subseteq X_I$. Given $r \in R_1$ and $I \subseteq S$ recall that

$$C_{I,r} = C_r \cap X_I.$$

Clearly, $X_I = \bigsqcup_{r \in R_1} C_{I,r}$. But we can say more.

Theorem 3.6 *Let $r = ue_\phi v \in R_1$. Then there is a bijective morphism*

$$m\colon k^{n_{I,r}} \to C_{I,r}$$

where $n_{I,r} = \ell(w_0) - \ell(u) + |I \cap I_u| + \ell(v)$.

Proof By inspection $C_{I,r} = \bigsqcup_{J \subseteq I \cap I_u} Bue_J vB$ and so $C_{I,r} = (Uu \cap uU)(Z_{I,u})(vU \cap U^- v)$ where $Z_{I,u} = \bigsqcup_{J \subseteq I \cap I_u} e_J T$. So the proof proceeds as in 3.4 and 3.5. ■

Remark 3.7 We have defined C_r via 3.1. However there is a direct definition in terms of the monoid M. Let $r \in R_1$. Then we can write $r = er$ for some unique, rank-one idempotent $e \in E_1(R) = E(\bar{T}) \cap R_1$. It turns out that if

$$\tilde{C}_r = \{y \in M \mid eBy = eBey \subseteq rB\}$$

then $C_r = \tilde{C}_r / k^*$.

4 Betti Numbers of X_I

If Y is a smooth projective algebraic variety, then it is of interest to calculate the Betti numbers of Y,

$$\beta_i(Y) = \dim_F\bigl(H^i(Y;F)\bigr),$$

for some appropriate (Weil) cohomology theory. It is well known that one can calculate $\beta_i(Y)$ using the Weil zeta function of any smooth reduction of Y to the algebraic closure of a finite field. If further, Y has a cell decomposition into affine spaces then we obtain

$$\beta_i(Y) = \begin{cases} \text{the number of cells of dimension } i/2, & i \text{ even} \\ 0, & i \text{ odd.} \end{cases}$$

In our situation we do not yet know that our cells are affine spaces (unless char $(k) = 0$), but the bijection of Theorem 3.6 will remain a bijection after suitable reduction to a finite field. Let

$$P(X,t) = \sum_{i\geq 0}(-1)^i\beta_i(X)t^i$$

be the Poincaré polynomial of X.

Theorem 4.1

$$P(X_I,t) = \Bigl(\sum_{u\in W} t^{2(\ell(w_0)-\ell(u)+|I_u\cap I|)}\Bigr)\Bigl(\sum_{v\in W} t^{2\ell(v)}\Bigr)$$

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