Canad. Math. Bull. Vol. **27** (3), 1984

TOPOLOGICAL COMPLETENESS OF ORDER INTERVALS IN RIESZ SPACES

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ABSTRACT. It is shown that if L is a Dedekind complete Riesz space equipped with a locally solid topology T defined by strongly $(A, 0)$ Riesz pseudonorms, then order intervals of L are T-complete. This is an extension of a well known theorem of Nakano. The second part of the paper gives a necessary and sufficient condition for topological completeness of order intervals in a Dedekind σ-complete Riesz space which has a weak order unit and which is equipped with a locally solid σ-Fatou topology.

1. **Introduction.** A basic and well-known theorem which gives a sufficient condition for topological completeness of order intervals in Riesz spaces is due to H. Nakano and may be stated as follows.

THEOREM 1.1 (H. Nakano). *If L is a Dedekind complete Riesz space equipped with a linear space locally solid topology T defined by continuous Fatou Riesz pseudonorms, then order intervals of L are T-complete.*

On the other hand, W. A. J. Luxemburg [5] has shown that if (L, ρ) is a normed Riesz space and if L is Dedekind σ-complete, then order intervals of L are ρ-complete if and only if the Riesz norm ρ has property $(A, 0)$ (see below). Our intention in the first part of this paper is to show that a direct extension of the sufficiency part of the Luxemburg characterization yields a proper extension of the Nakano theorem.

In the final section of this paper, we consider Riesz spaces which are Dedekind σ-complete and which are equipped with locally solid topologies defined by σ-Fatou Riesz pseudonorms. Such spaces, in general, do not have topologically complete order intervals and arise naturally in problems concerning vector-valued measures [3]. We show that the question of topological completeness of order intervals in such spaces may be reduced to the corresponding question for much smaller subsets. This places in the context of Riesz spaces some results of Kluvánek for the case of certain function spaces [3].

For basic terminology and results in the theory of locally solid Riesz spaces, we refer to the monographs [1], [2]. Throughout this paper, we shall work

Received by the editors February 22, 1983 and, in revised form, June 15, 1983.
AMS Subject Classification (1980): Primary 46A40 Secondary 46A05.

directly with Riesz pseudonorms rather than with topologies with various properties. However, to facilitate comparison with the results of [1], we make some preliminary remarks. A Riesz pseudonorm ρ on the Riesz space L is called Fatou if and only if $0 \le f_\tau \uparrow_\tau f$ in L implies $\rho(f_\tau) \uparrow_\tau \rho(f)$. Similarly, replacing directed systems by increasing sequences yields the notion of σ-Fatou Riesz pseudonorm. A locally solid topology T on the Riesz space L is said to be a Fatou (σ-Fatou) topology if and only if T has a basis at zero consisting of solid and order closed (σ-order closed) sets. It is shown in 23B of [2] that a Fatou topology is defined by its continuous Fatou Riesz pseudonorms. An inspection of the proof of 23B of [2] shows also that each σ-Fatou topology is defined by its continuous σ-Fatou Riesz pseudonorms.

2. **Topologies defined by** $(A, 0)$ **Riesz pseudonorms.** The Riesz pseudonorm ρ on the Riesz space L is called an $(A, 0)$ Riesz pseudonorm if and only if whenever $\{f_n\}_{n=1}^\infty \subseteq L^+$ satisfies $f_n \downarrow_n 0$ and $\{f_n\}_{n=1}^\infty$ is a ρ-Cauchy sequence, it follows that $\rho(f_n) \to 0$. The relevance of this notion lies in a result of Luxemburg [5] that if L is Dedekind σ-complete and ρ is a Riesz norm on L, then order intervals of L are ρ-complete if and only if ρ is an $(A, 0)$ Riesz norm. It is not difficult to see that if ρ is a σ-Fatou Riesz pseudonorm, then also ρ is an $(A, 0)$ Riesz pseudonorm but the converse is false (see Example 3.1 below).

The following Arzelà-type characterization of $(A, 0)$ Riesz pseudonorms is due to W. A. J. Luxemburg for the case of Riesz norms [5]. The proof remains valid for Riesz pseudonorms, and is accordingly omitted.

Proposition 2.1. *Let ρ be a Riesz pseudonorm on the Riesz space L. The pseudonorm ρ is an $(A, 0)$ Riesz pseudonorm if and only if $0 \le f_n \in L$, $n = 1, 2, \ldots, \{f_n\}$ is order convergent to zero and $\{f_n\}$ is ρ-Cauchy imply $\rho(f_n) \to 0$.*

Lemma 2.2. *Let the Riesz space L be Dedekind σ-complete and let ρ be an $(A, 0)$ Riesz pseudonorm on L. If $\{f_n\}_{n=1}^\infty \subseteq L^+$ is an order bounded sequence with $\rho(f_n - f_{n+1}) < 2^{-n}$, for $n = 1, 2, \ldots,$ then*

$$\rho\left(f_n - \sup_m \inf_{i \ge m} f_i\right) \le 2^{-n+1}, \qquad n = 1, 2, \ldots$$

Proof. Let the natural number n be given and suppose that k, m are natural numbers with $n < m \le k$. From

$$\left| f_n - \inf_{m \le i \le k} f_i \right| \le \sum_{j=n}^{m-1} |f_j - f_{j+1}| + f_m - f_m \wedge \cdots \wedge f_k$$
$$\le \sum_{j=n}^{k-1} |f_j - f_{j+1}|$$

it follows that

$$\rho\left(\left| f_n - \inf_{m \le i \le k} f_i \right|\right) < 2^{-n+1}, \qquad n < m \le k,$$

https://doi.org/10.4153/CMB-1984-048-1 Published online by Cambridge University Press

If now $k' > k$, then from

$$(1) \qquad \left| \inf_{m \le i \le k} f_i - \inf_{m \le i \le k'} f_i \right| \le \sum_{j=k}^{k'-1} |f_j - f_{j+1}|$$

it follows that the sequence $\{\inf_{k \ge i \ge m} f_i\}_{k \ge m}$ is ρ-Cauchy for each fixed $m > n$. Consequently, it follows from Proposition 2.1, that

$$(2) \qquad \rho\left(f_n - \inf_{i \ge m} f_i\right) \le 2^{-n+1}, \qquad m > n.$$

An inequality similar to (1) above shows that the sequence $\{\inf_{i \ge m} f_i\}_{m=1}^{\infty}$ is ρ-Cauchy and a further appeal to Proposition 2.1 and (2) above concludes the proof of the lemma.

We remark that, for the case of σ-Fatou Riesz pseudonorms, the preceding lemma is proved in [2] Lemma 23D while a topological version for σ-Fatou topologies is given in [1] Lemma 12.5. As an immediate consequence of the lemma, we have the following result, proved otherwise for the case of Riesz norms in [5], Theorem 61.7.

COROLLARY 2.3. *Let the Riesz space L be Dedekind σ-complete. If ρ is an $(A, 0)$ Riesz pseudonorm on L, then order intervals of L are ρ-complete.*

A further application of the lemma concerns sequential completeness.

THEOREM 2.4. *If the Riesz space L is Dedekind complete and if T is a linear space Hausdorff locally solid topology on L defined by a family $\{\rho\}$ of continuous $(A, 0)$ Riesz pseudonorms, then order intervals of L are sequentially T-complete.*

For the case of σ-Fatou topologies, the preceding theorem is proved in [1], Theorem 13.2. By working directly with pseudonorms and appealing to Lemma 2.2 above, it is easily seen that the proof given in [1] carries over to the present setting and, accordingly, the details are omitted.

3. **An extension of Nakano's theorem**. The Riesz pseudonorm ρ on the Riesz space L is said to be strongly $(A, 0)$ if and only if $\{f_\tau\} \subseteq L^+$, $0 \le f_\tau \downarrow_\tau 0$ and $\{f_\tau\}$ is a ρ-Cauchy net implies that $\rho(f_\tau) \downarrow_\tau 0$. It is shown in [5], Theorem 61.3 that if ρ is a Riesz norm, then ρ is an $(A, 0)$ Riesz norm if and only if ρ is strongly $(A, 0)$. Further, it is not difficult to see that if ρ is a Fatou pseudonorm on L, then ρ is strongly $(A, 0)$. The converse to this statement, however, is false as is shown by the following example which may be found in [4].

EXAMPLE 3.1. If $f = \{f(n, m) : n, m = 1, 2, \ldots\}$ is a double sequence of real numbers, define

$$\rho(f) = \sum_{n=1}^{\infty} 2^{-n}\left\{\sup_m |f(n, m)| + n \varlimsup_m f(n, m)\right\}$$

and let $L=\{f:\rho(f)<\infty\}$. L is a Dedekind complete Riesz space and the Riesz norm ρ on L is strongly $(A, 0)$. The largest σ-Fatou Riesz pseudonorm dominated by ρ is the Lorentz seminorm

$$\sigma(f)=\sum_{n=1}^{\infty} 2^{-n}\left\{\sup_m |f(n, m)|\right\}.$$

It is easily checked that σ is not equivalent to ρ. In fact, if k is a natural number and if $f_k(n, m)$ is 1 if $n=k$ and 0 otherwise, then $\rho(f_k)=(1+k)\sigma(f_k)$.

In view of the preceding example, the following theorem, which is the main result of this section, is a proper extension of Nakano's theorem.

THEOREM 3.2. *If the Riesz space L is Dedekind complete and if T is a linear space locally solid topology on L defined by a family $\{\rho\}$ of continuous strongly $(A, 0)$ Riesz pseudonorms, then order intervals of L are T-complete.*

Proof. Let $0\leq w\in L$ and let $\{f_\alpha\}\subseteq[0, w]$ be a T-Cauchy net. For each $\rho\in\{\rho\}$, set $N(\rho)=\{f\in L:\rho(f)=0\}$ and observe that the strong $(A, 0)$ property implies that each $N(\rho)$ is a band in L. If $C(\rho)=\{f\in L: f\perp N(\rho)\}$, then $L=N(\rho)\oplus C(\rho)$ and let $P_\rho: L\to C(\rho)$ be the band projection of L onto $C(\rho)$. By Corollary 2.3 above, there exists $0\leq u_\rho\in L$ with $P_\rho u_\rho=u_\rho$ such that $\rho(f_\alpha-u_\rho)\to 0$. It follows that $P_\rho(u_\rho\wedge w)=u_\rho$ and so $0\leq u_\rho\leq P_\rho w\leq w$. If $\rho, \rho'\in\{\rho\}$ and if $\rho\leq\rho'$, it is easily seen that $u_\rho=P_\rho u_{\rho'}\leq u_{\rho'}$. Since it may be assumed that $\rho+\rho'\in\{\rho\}$ whenever $\rho, \rho'\in\{\rho\}$, it follows that

$$P_\rho P_{\rho'} u_{\rho'}=P_\rho P_{\rho'} u_{\rho+\rho'}=P_{\rho'} P_\rho u_\rho$$

Consequently, if $u_{\rho'}\geq u_\rho$, then

$$u_\rho\leq P_\rho u_{\rho'}=P_\rho P_{\rho'} u_{\rho'}=P_{\rho'} P_\rho u_\rho\leq u_\rho$$

so that $u_\rho=P_\rho u_{\rho'}$. It follows that the system $\{u_\rho\}\subseteq[0, w]$ is upwards directed and T-Cauchy. Set $u=\sup_\rho u_\rho$ and observe that $P_\rho u=u_\rho$ follows from the strong $(A, 0)$ property of ρ.

It now follows that $\rho(f_\alpha-u)=\rho(f_\alpha-u_\rho)\to 0$ for all $\rho\in\{\rho\}$ and the proof of the theorem is complete.

4. **Topological completeness in Dedekind σ-complete Riesz spaces**. The theorem of Nakano in general fails if the basic assumptions are weakened to the corresponding σ-properties. A simple example is the Riesz space L of all bounded Borel functions on the closed interval $[0, 1]$, equipped with the locally convex solid topology T of pointwise convergence. In this example, the Riesz space L is Dedekind σ-complete and has a strong order unit. The topology T is defined by σ-Fatou Riesz pseudonorms and yet it is clear that order intervals in L are not T-complete. In this setting, we now show that the question of topological completeness of order intervals may be reduced to the correspond-

ing question concerning much smaller subsets. We make first some preliminary remarks.

Suppose that L is a Dedekind σ-complete Riesz space. If ρ is a σ-Fatou Riesz pseudonorm on L, then the null ideal $N_\rho=\{f\in L: \rho(f)=0\}$ is a σ-ideal in L. If L_ρ denotes the quotient Riesz space L/N_ρ, then L_ρ is again a Dedekind σ-complete Riesz space and the quotient map is a Riesz σ-homomorphism of L onto L_ρ. If $f\in L$, then $[f]_\rho$ will denote the element of L_ρ determined by f. If ρ is σ-Fatou Riesz pseudonorm on L, and if the sequence $\{f_n\}\subseteq L$ is order convergent to an element $f\in L$, then $\rho(f)\le\liminf \rho(f_n)$. See, for example, the implication (i) $\Rightarrow$ (ii) of [1], Theorem 11.4.

The element f of the Riesz space L is called a component of the element $0\le e\in L$ if and only if $0\le f\le e$ and $f\wedge(e-f)=0$. If the Riesz space L is Dedekind σ-complete, then the set of all components of e is a Dedekind σ-complete Boolean algebra. Suppose now that $0\le e$ is a weak order unit for the Dedekind σ-complete Riesz space L. If u is any element of L and if t is any real number, then $e(t;u)$ will denote the component of e in the principal band generated by $(te-u)^+$. The system $\{e(t;u)\}$ is called the spectral system of u with respect to e. The basic properties of spectral systems are given in Chapter 6 of [6]. We recall specifically that

$$e(t;u)=\sup\{e\wedge n(te-u)^+: n=1,2,\ldots\}.$$

LEMMA 4.1. *Let L be a Dedekind σ-complete Riesz space, with weak order unit $e\ge 0$ and let ρ be a σ-Fatou Riesz pseudonorm on L. If the elements $f, g\in L$ satisfy $[f]_\rho=[g]_\rho$, then $[e(t;f)]_\rho=[e(t;g)]_\rho$ for every real number t.*

Proof. For each $n=1,2,3,\ldots$ and for each real number t, observe that

$$\begin{aligned}\rho(e\wedge n(te-f)^+-e\wedge n(te-g)^+)&\le\rho(n(te-f)^+-n(te-g)^+)\\&\le n\rho(g-f)=0.\end{aligned}$$

Now, the sequence $\{|e\wedge n(te-f)^+-e\wedge n(te-g)^+|\}$ is order convergent to $|e(t;f)-e(t;g)|$. Since ρ is σ-Fatou, it follows that $\rho(e(t;f)-e(t;g))=0$ and this proves the assertion of the lemma.

We may now state the principal result of this section, which finds its motivation in Theorem III 3.2 of [3].

THEOREM 4.2. *Let L be a Dedekind σ-complete Riesz space equipped with a locally solid topology T defined by a family $\{\rho\}$ of σ-Fatou Riesz pseudonorms. Assume that L has a weak order unit $e\ge 0$. The following statements are equivalent:*

(i) *The σ-complete Boolean algebra of components of e is T-complete.*

(ii) *Each order interval of L is T-complete.*

Proof. It clearly suffices to prove only the statement (i) $\Rightarrow$ (ii). Assume, then, that the Boolean algebra of principal components of e is T-complete.

We show first that the order interval $[0, e]$ is T-complete. Let $\{f_\alpha\} \subseteq [0, e]$ be a T-Cauchy net. From Corollary 2.3, it follows that for each $\rho \in \{\rho\}$, there exists an element $f_\rho \in [0, e]$ such that $\rho(f_\alpha - f_\rho) \xrightarrow[\alpha]{} 0$. If now $\rho, \rho' \in \{\rho\}$ and $\rho \leq \rho'$ (i.e. $\rho(f) \leq \rho'(f)$ for all $0 \leq f \in L$) it follows that $\rho(f_\rho - f_{\rho'}) = 0$. Consequently, for each real number t and for each $\rho' \in \{\rho\}$ with $\rho \leq \rho'$, it follows from Lemma 4.1 that $\rho(e(t; f_\rho) - e(t; f_{\rho'})) = 0$. Since the sum of continuous σ-Fatou Riesz pseudonorms is again a continuous σ-Fatou Riesz pseudonorm, it may clearly be assumed that whenever $\rho, \rho' \in \{\rho\}$, there exists $\rho'' \in \{\rho\}$ with $\rho, \rho' \leq \rho''$. Thus for each real number t, the system $\{e(t; f_\rho)\}$ is a T-Cauchy net of principal components of e, and so, by hypothesis, there exists a principal component of e, e_t say, such that $e(t; f_\rho) \xrightarrow[\rho]{} e_t$. Observe that $\rho(e(t; f_\rho) - e_t) = 0$ for each t and each $\rho \in \{\rho\}$. It is clear that we may take $e_t = 0$ if $t < 0$ and $e_t = e$ if $t > 1$. For each real t, we now set

$$e(t) = \sup\{e_r : r < t, \ r \text{ rational}\}.$$

It is not difficult to see that the system $\{e(t)\}$ of principal components of e has the property that $e(t') \uparrow e(t)$ as $t' \uparrow t$. Moreover, $e(t) = 0$ if $t \leq 0$ and $e(t) = e$ if $t > 1$. We now observe that

$$\rho(e(t; f_\rho) - e(t)) = 0 \tag{1}$$

for each $\rho \in \{\rho\}$ and each real t. In fact,

$$\begin{aligned} [e(t)]_\rho &= \sup\{[e_r]_\rho : r < t, \ r \text{ rational}\} \\ &= \sup\{[e(r; f_\rho)]_\rho : r < t, \ r \text{ rational}\} \\ &= [e(t; f_\rho)]_\rho, \end{aligned}$$

using the fact that the system $\{e(t; f_\rho)\}$ is the spectral system of f_ρ with respect to e and the fact that the quotient map of L onto L_ρ is a Riesz σ-homomorphism. For each natural number $n = 1, 2, \ldots,$ set

$$f_n = \sum_{k=1}^{2^{n+1}} k/2^n [e(k/2^n) - e(k - 1/2^n)]$$

Observe that $0 \leq f_n \uparrow_n \leq e$. Let $f = \sup_n f_n$. It is not difficult to see, by using Theorem 40.8 of [6], that the system $\{e(t)\}$ is the spectral system of f with respect to e. Since $\{e(t; f_\rho)\}$ is the spectral system of f_ρ with respect to e, it follows from (1) that $\rho(f - f_\rho) = 0$ for every $\rho \in \{\rho\}$. Thus the net $\{f_\alpha\}$ is T-convergent to f and so the order interval $[0, e]$ is T-complete.

Let now $0 \leq h \in L$ be given and suppose that $\{f_\alpha\} \subseteq [0, h]$ is a T-Cauchy net. Since

$$\rho(f_\alpha \wedge ne - f_{\alpha'} \wedge ne) \leq \rho(f_\alpha - f_{\alpha'})$$

for each $\rho \in \{\rho\}$, each natural number n and indices α, α' and since each order interval $[0, ne]$ is T-complete, it follows that there exists a sequence $\{h_n\} \subseteq [0, h]$ such that

$$\sup_n \rho(h_n - f_\alpha \wedge ne) \underset{\alpha}{\rightarrow} 0.$$

By passing to the quotient space, it may be assumed that T is Hausdorff (see [1], Theorem 14.3). Consequently, it follows that $h_n \uparrow_n$. Let $h_0 = \sup_n h_n$. For each α, the sequence $\{|h_n - f_\alpha \wedge ne|\}$ is order convergent to $|h_0 - f_\alpha|$ and so

$$\begin{aligned}\rho(h_0 - f_\alpha) &\leq \liminf \rho(h_n - f_\alpha \wedge ne) \\ &\leq \sup_n \rho(h_n - f_\alpha \wedge ne) \underset{\alpha}{\rightarrow} 0\end{aligned}$$

for each $\rho \in \{\rho\}$. It follows that the order interval $[0, h]$ is T-complete and this completes the proof of the theorem.

The final result of the paper is a theorem which guarantees the topological completeness of a locally solid Riesz space L, given that L has topologically complete order intervals. A locally solid topology T on a Riesz space L is said to have the Levi property (σ-Levi property) if and only if every upwards directed T-bounded system (increasing T-bounded sequence) in L^+ has a supremum in L. It is shown by a simple argument in [1] Theorem 13.8, that if (L, T) is a locally solid Riesz space with T-complete order intervals, then L is T-complete if T has the Levi property. Under a mild restriction, we now show that a similar result holds for topologies which have the σ-Levi property. We recall that the Riesz space L is said to have finite or countable order basis if and only if there exists a sequence $\{v_n\} \subset L^+$, which may be assumed increasing, such that $f \in L$ and $|f| \wedge v_n = 0$ for all $n = 1, 2, \ldots$ implies $f = 0$.

PROPOSITION 4.3. *Let L be an Archimedean Riesz space with finite or countable order basis. Let T be a locally solid Hausdorff topology on L and suppose that each order interval of L is T-complete. If T is σ-Levi, then L is T-complete.*

Proof. Let $0 \leq v_n \uparrow_n \subseteq L$ be a countable order basis for L and let $\{f_\alpha\} \subseteq L$ be a Cauchy net. We may clearly assume that $f_\alpha \geq 0$ for each index α. Since order intervals in L are T-complete, it follows that the net $\{f_\alpha \wedge v_n\}$ is convergent, to h_n, say, for each natural number n. Since T is Hausdorff, it follows that $0 \leq h_n \leq h_{n+1}$ for $n = 1, 2, \ldots$. Since the net $\{f_\alpha\}$ is a Cauchy net, it follows that the sequence $\{h_n\}$ is T-bounded so that $h = \sup_n h_n$ exists in L since T has the σ-Levi property.

Observe now that $f_\alpha \wedge h \rightarrow h$. Indeed, if $f_\alpha \wedge h \rightarrow k$, then $f_\alpha \wedge h \wedge v_n \rightarrow k \wedge v_n$ so that $k \wedge v_n = h_n \wedge h = h_n$ for each $n = 1, 2, \ldots$. Since $\{v_n\}$ is an order basis for L, it follows that $h = k$. A similar argument now yields that $f_\alpha \wedge g \rightarrow h \wedge g$ for each $0 \leq g \in L$. Let ρ be any T-continuous Riesz pseudonorm on L. Let $\varepsilon > 0$ be given and choose α_0 such that $\rho(f_\alpha - f_{\alpha'}) < \varepsilon$ for $\alpha, \alpha' \geq \alpha_0$. Observe that

$$\rho(h - f_\alpha) \leq \rho(h - f_{\alpha'} \wedge (f_\alpha \vee h)) + \rho(f_\alpha - f_{\alpha'}).$$

Thus, given $\alpha \geq \alpha_0$, choose $\alpha' \geq \alpha_0$ such that

$$\rho(h - f_{\alpha'} \wedge (f_\alpha \vee h)) < \varepsilon$$

It follows that $\rho(h - f_\alpha) \leq 2\varepsilon$ for $\alpha \geq \alpha_0$ and so $f_\alpha \to h$. Thus L is T-complete.

The author would like to thank the referee for pointing out an error in an earlier version of this paper.

References

1. C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces* (Academic Press, New York, 1978).
2. D. H. Fremlin, *Topological Riesz spaces and measure theory* (Cambridge University Press, 1974).
3. Igor Kluvánek and Greg Knowles, *Vector Measures and Control Systems* (North Holland Mathematics Studies 20, Amsterdam, 1975).
4. W. A. J. Luxemburg and A. C. Zaanen, *Notes on Banach function spaces, Note III*, Nederl. Akad. Wetensch **A66** (1963) 239–250.
5. W. A. J. Luxemburg, *Notes on Banach function spaces, Note XVI A*, Nederl. Akad. Wetensch. **A68** (1965) 646–657.
6. W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, Amsterdam (North Holland, 1977).

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