INVERSION THEOREMS FOR THE LAPLACE-STIELTJES TRANSFORM

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1. Introduction and main results. The Laplace–Stieltjes transform $f(x)$ of the function $\alpha(t)$ is defined by

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t) \equiv \lim_{R\uparrow\infty} \int_0^R e^{-xt}\, d\alpha(t), \tag{1}$$

where $\alpha(t)$ is a function of bounded variation in each closed interval $[0, R]$ $(R > 0)$ and the right-hand side of (1) is supposed to be convergent for some $x = x_0$. The Laplace transform $f(x)$ of the function $\phi(t)$ is defined by

$$f(x) = \int_0^\infty e^{-xt}\phi(t)\, dt \equiv \lim_{R\uparrow\infty} \int_0^R e^{-xt}\phi(t)\, dt, \tag{2}$$

where $\phi(t) \in L_1(0, R)$ for each $R > 0$ and the right-hand side of (2) is supposed to be convergent for some $x = x_0$.

Improving a result due to Phragmen and Doetsch **(3**, pp. 286–288**)**, D. Saltz **(6)** obtained the following inversion theorem for the Laplace–Stieltjes transform.

THEOREM A. *If $f(x)$ is the Laplace–Stieltjes transform of $\alpha(t)$, then*

$$\lim_{\uparrow\infty} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{nst}}{n!} f(ns) = \begin{cases} \left(1 - \dfrac{1}{e}\right)(\alpha(0+) - \alpha(0)) & \textit{for } t = 0, \\[2ex] \left(1 - \dfrac{1}{e}\right)\alpha(t+) + \dfrac{1}{e}\alpha(t-) - \alpha(0) & \textit{for } t > 0; \end{cases}$$

here $\alpha(t\pm)$ denotes, respectively,

$$\lim_{h\downarrow 0} (t \pm h).$$

In this paper we prove a general theorem for the Laplace–Stieltjes transform. Particular cases of this theorem generalize Theorem A and also an inversion theorem for the Laplace transform due to D. V. Widder **(8)**. Also, our theorem gives formulas for $\alpha^{(r)}(z)$ by means of the values of $f(x)$ for real x. The formal idea leading to our main result is the following. Let

$$K(s) \equiv \sum_{n=1}^{\infty} a_n\, e^{\lambda_n s}.$$

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Then we have formally for the Laplace transform $f(x)$ of $\phi(t)$

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n s x} f(\lambda_n s) = \int_0^{\infty} K(s(x-t))\phi(t)\,dt.$$

If, for some function $h(s)$, the function $\Phi_s(t, x) \equiv h(s)K(s(x-t))$ forms (as $s \uparrow \infty$) a kernel of a suitable singular integral (**4**, Chapter VII), then formally

$$\lim_{s\uparrow\infty} h(s) \sum_{n=1}^{\infty} a_n \exp(\lambda_n s) f(\lambda_n s) = \phi(x),$$

which is an inversion formula for the Laplace transform. The same idea is applicable to other integral transforms.

Let $K(s)$ denote a function such that $K(-s)$ is the Laplace–Stieltjes transform of a function $\beta(t)$ satisfying $\beta(t) = 0$ for $0 \leqslant t \leqslant c$ (for some $c > 0$). That is

$$K(s) \equiv \int_c^{\infty} e^{st} d\beta(t) \qquad (c > 0).$$

For two real numbers x, y the Kronecker symbol $\delta_{x,y}$ is defined by $\delta_{x,y} = 0$ for $x \neq y$ and $\delta_{x,y} = 1$ for $x = y$.

Our main result is given by the following theorem.

THEOREM 1. *Let $f(x)$ be the Laplace–Stieltjes transform of $\alpha(t)$. Suppose that for some $z \geqslant 0$ and a certain non-negative integer r there are $2r + 2$ numbers such that*

$$\alpha(t) = \begin{cases} \displaystyle\sum_{k=0}^{r} \frac{1}{k!} b_k (t-z)^k + o((t-z)^r) & \textit{as } t \downarrow z, \\[2ex] \displaystyle\sum_{k=0}^{r} \frac{1}{k!} c_k (t-z)^k + o((z-t)^r) & \textit{as } t \uparrow z \end{cases} \tag{3}$$

(for $z = 0$, the second assumption of $\alpha(t)$ is not needed. In this case we define $c_k = 0$ for $0 \leqslant k \leqslant r$). *If for a function $K(s)$ and some non-negative integer q*

$$\int_c^{\infty} e^{st} |d\beta(t)| < +\infty$$

for all real s,

$$\lim_{s\uparrow\infty} K(s) \equiv -a_0$$

exists, and

$$\int_{-\infty}^{+\infty} |u^q K^{(q+1)}(u)|\,du < +\infty,$$

then for $0 \leqslant p \leqslant \min(r, q)$

$$\lim_{s\uparrow\infty} \left\{ s^p \int_c^{\infty} e^{szt} t^p f(st)\,d\beta(t) - \sum_{n=0}^{p} (b_n - c_n) s^{p-n} K^{(p-n)}(0) + \delta_{0,z}\, s^p K^{(p)}(0)\alpha(0) \right\} = -a_0 c_p - (1 - \delta_{0,z})\delta_{0,p}\, a_0 \alpha(0). \tag{4}$$

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In particular if $p = 0$ *or if* $p > 0$ *and* $b_k = c_k$ *for* $0 \leqslant k < p$, *then for* $z > 0$

$$(5)\qquad \lim_{s\uparrow\infty} s^p \int_c^\infty e^{szt} t^p f(st)\, d\beta(t) = K(0)b_p - (K(0) + a_0)c_p + \delta_{0,p}\, a_0\, \alpha(0).$$

If the function $\alpha(t)$ satisfies (3) for some $z \geqslant 0$, then for $z > 0$ we have $b_0 = \alpha(z+)$ and $c_0 = \alpha(z-)$ and for $z = 0$ we have $b_0 = \alpha(0+)$ and $c_0 = 0$. For $p = 0$, conclusion (4) of Theorem 1 takes the form

$$\lim_{s\uparrow\infty} \int_c^\infty e^{szt} f(st)\, d\beta(t) = K(0)\alpha(z+) - (K(0) + a_0)\alpha(z-) + a_0\, \alpha(0).$$

If $a_0 = 0$, then the right-hand side is $K(0)(\alpha(z+) - \alpha(z-))$; and if $K(0) \neq 0$ we have a formula for the jump $\alpha(z+) - \alpha(z-)$ of the function $\alpha(t)$ at the point $t = z$. If $\alpha(t)$ is continuous at the point $t = z$, then $\alpha(z+) = \alpha(z-)$ and the right-hand side is equal to $-a_0(\alpha(z)$; if $a_0 \neq 0$, then we have an inversion formula for the Laplace–Stieltjes transform.

With the function $\alpha(t)$ and a fixed point $z \geqslant 0$, associate the two functions $\alpha_+(t) \equiv \alpha_+(z; t)$ and $\alpha_-(t) \equiv \alpha_-(z; t)$ defined by $\alpha_+(z; t) = \alpha(z+)$ for $t = z$, $\alpha_+(z; t) = \alpha(t)$ for $t > z$, and $\alpha_-(z; t) = \alpha(z-)$ for $t = z$,

$$\alpha_-(z; t) = \alpha(t) \text{ for } 0 \leqslant t < z.$$

When $z = 0$, we define $\alpha_+(z; t)$ only. It is known (**2**, p. 114 (36)) that if $\alpha_+^{(r)}(z)$ and $\alpha_-^{(r)}(z)$ exist, then assumption (3) on $\alpha(t)$ at the point $t = z$ is satisfied by the constants $b_k = \alpha_+^{(k)}(z)$, $c_k = \alpha_-^{(k)}(z)\,(0 \leqslant k \leqslant r)$. If $\alpha^{(r)}(z)$ exists for some positive integer r and for some $z > 0$, then conclusion (4) of Theorem 1 takes the form

$$\lim_{s\uparrow\infty} s^p \int_c^\infty e^{szt} t^p f(st)\, d\beta(t) = -a_0\, \alpha^{(p)}(z) + \delta_{0,p}\, a_0\, \alpha(0)$$

for $0 \leqslant p \leqslant \min(r, q)$. If $a_0 \neq 0$, then we have an inversion formula for the pth derivative of $\alpha(t)$ at the point $t = z$.

Suppose the function $\beta(t)$ defining $K(s)$ is a step function with jumps a_k at $t = k$ $(k = 1, 2, \ldots)$, that is $\beta(t) = 0$ for $0 \leqslant t \leqslant 1$ and $\beta(t) = \sum_{k<t} a_k$ for $t > 1$. Then

$$K(s) = \sum_{n=1}^\infty a_n\, e^{ns}.$$

Associate with $K(s)$ the power series

$$H(z) = \sum_{n=0}^\infty a_n\, z^n$$

if

$$\lim_{s\uparrow\infty} K(s) \equiv -a_0$$

exists. In this case $K(s) = H(e^s) - a_0$. It is easy to see that for such $K(s)$ Theorem 1 can be stated in the following form.

THEOREM 2. *Let $f(x)$ be the Laplace–Stieltjes transform of $\alpha(t)$. Suppose that* (3) *is satisfied for some $z \geqslant 0$ and a certain non-negative integer r. If for an entire function*

$$H(z) \equiv \sum_{n=0}^{\infty} a_n z^n$$

and some non-negative integer q we have

$$\lim_{x \uparrow \infty} H(x) = 0$$

and

$$\int_0^{\infty} \left| x^{-1} (\log x)^q \sum_{k=1}^{q+1} A_k{}^{(q+1)} x^k H^{(k)}(x) \right| dx < +\infty,$$

where by definition

$$x^m \equiv \sum_{k=1}^{m} A_k{}^{(m)} x(x-1)\ldots(x-k+1),$$

then we have for $0 \leqslant p \leqslant \min(r, q)$

$$\lim_{s \uparrow \infty} \left\{ s^p \sum_{n=1}^{\infty} n^p a_n e^{nzs} f(ns) - \sum_{n=0}^{p} (b_n - c_n) s^{p-n} \left[\frac{d^{p-n}}{dt^{p-n}} H(e^t) \right]_{t=0} + \delta_{0,z}\, s^p \left[\frac{d^p}{dt^p} H(e^t) \right]_{t=0} \alpha(t) \right\}$$
$$= -a_0 b_p + \delta_{0,p}\, a_0\, \alpha(0).$$

It is easy to see (**5**, p. 205 (4)) that

$$\frac{d^m}{dx^m} H(e^x) = \sum_{k=1}^{m} A_k^{(m)} e^{kx} H^{(k)}(e^x) \qquad \text{for } m > 0.$$

Hence for $m > 0$ we have

$$\left[\frac{d^m}{dx^m} H(e^x) \right]_{x=0} = \sum_{k=1}^{m} A_k^{(m)} H^{(k)}(1).$$

The numbers $A_k{}^{(m)}$ are the Stirling numbers of the second kind (**5**, p. 168).

We now give some methods for obtaining functions $H(z)$ satisfying the assumptions of Theorem 2. By using particular functions $H(z)$ suggested by these methods, we obtain from Theorem 2 inversion and jump formulas for the Laplace–Stieltjes transform and for the Laplace transform. These include Theorem A and another known result as special cases.

First method. Let $R(z)$ be an entire function of finite order $\rho \geqslant 0$. Then for each integer $b > \rho$ and each positive number c the function $H(z) \equiv e^{-cz^b} R(z)$ satisfies the assumptions of Theorem 2 on the function $H(z)$ for all non-negative integers q. This follows from the fact that the derivative of an entire function of finite order is of the same order (**1**, p. 13, Theorem 2.4.1). By choosing

$R(z) = (z-1)^k$, $b = 1$, and $c = 1$, we obtain the following result from Theorem 2.

THEOREM 3. *Let $f(x)$ be the Laplace–Stieltjes transform of $\alpha(t)$. Suppose that (3) is satisfied for some $z \geqslant 0$ and a certain non-negative integer r. Then for $k, r = 0, 1, 2, \ldots$, we have (if empty sums are defined as equal to zero)*

$$(6)\quad \lim_{s\uparrow\infty}\left\{s^r\sum_{n=1}^{\infty} n^r a_{k,n}\, e^{nzs} f(ns) - \sum_{n=0}^{r-k}(b_n - c_n)s^{r-n}\left[\frac{d^{r-n}}{dt^{r-n}}(e^t-1)^k e^{-e^t}\right]_{t=0} + \delta_{0,z}\, s^r\left[\frac{d^r}{dt^r}(e^t-1)^k e^{-e^t}\right]_{t=0}\alpha(0)\right\} = (-1)^{k+1}[b_r - \delta_{0,r}\,\alpha(0)],$$

where

$$a_{k,n} = (-1)^{k-n}\sum_{p=0}^{\min(k,n)}\binom{k}{p}\Big/(n-p)!.$$

Theorem 3 reduces to Theorem A when $k = r = 0$.

If we add the two consequences (6) of Theorem 3 for $r = 0$ and $k = 0, 1$ we obtain the following result:

$$\lim_{s\uparrow\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{nzs}}{(n-1)!}f(ns) = \frac{1}{e}(b_0 - c_0) + \delta_{0,z}\frac{1}{e}\alpha(0).$$

For $z > 0$, we have $b_0 - c_0 = \alpha(z+) - \alpha(z-)$; that is, we have here a formula for the jump of the function $\alpha(t)$ at the point $t = z > 0$. For $z = 0$, we have

$$\frac{1}{e}(b_0 - c_0) + \frac{1}{e}\delta_{0,z}\,\alpha(0) = \frac{1}{e}(\alpha(0+) + \alpha(0)).$$

Consequence (6) of Theorem 3 with $k = r + 1$ has the following form:

$$\lim_{s\uparrow\infty} s^r\sum_{n=1}^{\infty} n^r a_{r+1,n}\, e^{nzs} f(ns) = (-1)^r(b_r - \delta_{0,r}\,\alpha(0)).$$

This is a formula for b_r. In particular if $\alpha^{(r)}(z)$ exists for some $z \geqslant 0$, then $b_r = \alpha^{(r)}(z)$ and we have an inversion formula for the rth derivative of $\alpha(t)$. For $r = 0$, we have $b_0 = \alpha(z+)$. In this case, we have an inversion formula for $\alpha(z+)$.

By Theorem 3 for $r = 1$ and $k = 0$ and the fact that

$$\int_0^\infty e^{-xt}\phi(t)\,dt = \int_0^\infty e^{-xt}\,d\alpha(t)$$

for

$$\alpha(t) = \int_0^t \phi(u)\,du \qquad (t \geqslant 0)$$

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if the left-hand side exists, we obtain the following result.

THEOREM 4. *Let $f(x)$ be the Laplace transform of $\phi(t)$. Suppose that for some $t = z > 0$ the function $\phi(t)$ is the derivative of its indefinite integral; then*

$$\lim_{s\uparrow\infty} s\sum_{n=1}^{\infty} (-1)^{n-1}\frac{e^{nzs}}{(n-1)!}f(ns) = \phi(z).$$

Theorem 4, with the additional restriction that $\phi(t)$ is continuous, was given by D. V. Widder **(8)**.

Second method. If $R(z)$ is an entire function of finite integral order $\rho > 0$ and of finite type τ, then, for each $b > \tau$, the function $H(z) = e^{-bz^\rho}R(z)$ satisfies the assumptions of Theorem 2 on $H(z)$ for all non-negative integral values of q. This follows from the fact that the derivative of an entire function of finite order and type is of the same order and type **(1**, p. 13, Theorem 2.4.1**)**.

Third method. Let $R(z)$ be an entire function satisfying $R(x) = O(x^\alpha)$ $(x\uparrow\infty)$ for some $\alpha \geqslant -1$. Suppose $R(z)$ has zeros $z_1, \ldots, z_m$ of orders $\alpha_1, \ldots, \alpha_m$, respectively. If for some non-negative integer q there are integers $q_1, \ldots, q_m$ satisfying $0 \leqslant q_j \leqslant \alpha_j$ $(0 \leqslant j \leqslant m)$ and

$$q_1 + \ldots + q_m > q + \alpha + 1,$$

then the function

$$H(z) = \left\{\int_z^0 + \int_0^{+\infty}\right\}(t-z)^q(t-z_1)^{-q_1}\ldots(t-z_m)^{-q_m}R(t)dt$$

satisfies the assumptions of Theorem 2 on $H(z)$ for this q. For two integers m, n satisfying $2 \leqslant n \leqslant m$, the function

$$H(z) = \int_z^\infty (t-z)^q t^{-n}\sin^m t\,dt$$

satisfies the assumptions of Theorem 2 on $H(z)$ for each q with $0 \leqslant q \leqslant n-2$. By means of the identity

$$H(x) = \sum_{k=0}^{q}(-1)^q\binom{q}{k}x^{q-k}\left\{\int_0^\infty t^{k-n}\sin^m t\,dt - \int_0^x t^{k-n}\sin^m t\,dt\right\}$$

and the theorem of residues, it is possible to obtain the coefficients of the power-series expansion of $H(x)$ explicitly.

Fourth method. It is known that for $\nu \geqslant 0$ the function

$$H_\nu(z) \equiv \Gamma(\nu+1)2^\nu z^{-\nu}J_\nu(z) = \sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(\nu+1)}{m!\,\Gamma(m+\nu+1)}(z/2)^{2m}$$

is an entire function, that $J_\nu(z) = O(z^{-\frac{1}{2}})$ as $z\uparrow\infty$, and that

$$\frac{d}{dz}[z^{-\nu}J_\nu(z)] = -z^\nu J_{\nu+1}(z)$$

(**9**, p. 368, §17.5 and p. 360, §17.211). By these properties of $J_\nu(z)$ it follows that for two non-negative integers q, n and any real number ν with $\nu > q + n + \frac{1}{2}$, the function $H_\nu(z)$ satisfies the assumptions of Theorem 2 on $H(z)$ for this q. By substituting the function $z^n H_\nu(z)$ for the pairs $p = 0$, $n = 0$ and $p = 0$, $n = 1$ in Theorem 2, we get the following result.

THEOREM 5. *Let $f(x)$ be the Laplace–Stieltjes transform of $\alpha(t)$. Then for $\nu > \frac{1}{2}$ we have*

$$\lim_{s\uparrow\infty}\sum_{m=1}^{\infty}(-1)^{m-1}\frac{\Gamma(\nu+1)}{m!\,\Gamma(m+\nu+1)}2^{-2m}e^{2mzs}f(2ms)$$
$$= \begin{cases}(1-H_\nu(1))(\alpha(0+)-\alpha(0)) & \textit{for } z = 0,\\ (1-H_\nu(1))\alpha(z+)+H_\nu(1)\alpha(z-)-\alpha(0) & \textit{for } z > 0;\end{cases}$$

and for each $\nu > \frac{3}{4}$ and $z > 0$ we have

$$\lim_{s\uparrow\infty}\frac{1}{J_\nu(1)}\sum_{m=0}^{\infty}(-1)^m\frac{1}{m!\,\Gamma(m+\nu+1)}2^{-(2m+\nu)}e^{(2m+1)zs}f((2m+1)s)$$
$$= \alpha(z+)-\alpha(z-).$$

2. Proof of Theorem 1. In the proof of Theorem 1 we use the following six lemmas.

LEMMA 1. *If*

$$K(s) \equiv \int_c^\infty e^{st}\,d\beta(t) \qquad (c > 0)$$

exists for all real s, then

$$\lim_{s\downarrow-\infty} e^{-cs}K^{(r)}(s) = c^r[\beta(c+)-\beta(c)]$$

for $r = 0, 1, 2, \ldots$; in particular

$$\lim_{s\uparrow\infty} s^q K^{(r)}(-s) = 0$$

for $q, r = 0, 1, 2, \ldots$.

Proof. We have (**7**, p. 57, Theorem 5*a* and p. 12, Theorem 6b)

$$e^{-cs}K^{(r)}(s) = \int_0^\infty e^{sv}\,d\left\{\int_c^{c+v} u^r\,d\beta(u)\right\}.$$

Applying (**7**, p. 181, Theorem **1**), with $\gamma = 0$ and $A = c^r[\beta(c+)-\beta(c)]$, we have

$$\limsup_{s\downarrow-\infty}|e^{-cs}K^{(r)}(s) - c^r[\beta(c+)-\beta(c)]|$$
$$\leqslant \limsup_{v\downarrow 0}\left|\int_c^{c+v}u^r d\beta(u) - c^r[\beta(c+)-\beta(c)]\right| = 0.$$

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LEMMA 2. *If for $x \geqslant 0$, $g(x)$ is the $(r+1)$th indefinite integral of a function $g^{(r+1)}(x)$ $(r \geqslant 1)$,*

$$\lim_{x \uparrow \infty} g(x) \equiv a$$

exists and

$$\int_0^\infty u^r |g^{(r+1)}(u)| du < +\infty,$$

then

$$\lim_{x \uparrow \infty} x^k g^{(k)}(x) = 0 \qquad \text{for } 0 < k \leqslant r, \tag{7}$$

and

$$\int_0^\infty u^k |g^{(k+1)}(u)|\, du < +\infty \qquad \text{for } 0 \leqslant k < r. \tag{8}$$

Proof. For $0 < x < y$, the inequality

$$\int_x^y |g^{(r+1)}(t)|\, dt \leqslant x^{-r} \int_x^y t^r |g^{(r+1)}(t)|\, dt$$

implies by Cauchy's theorem that

$$\int^\infty g^{(r+1)}(t)\, dt$$

exists. This and the relation

$$g^{(r)}(y) - g^{(r)}(x) = \int_x^y g^{(r+1)}(t)\, dt \to \int_x^\infty g^{(r+1)}(t)\, dt \qquad (y \uparrow \infty)$$

implies that

$$\lim_{y \uparrow \infty} g^{(r)}(y) = a_r$$

exists. If $a_r \neq 0$, then for $y > x > 0$ we have

$$g^{(r-1)}(y) - g^{(r-1)}(x) = \int_x^y g^{(r)}(t)\, dt \to (\operatorname{sgn} a_r)(+\infty) \qquad (y \uparrow \infty).$$

Hence

$$\lim_{y \uparrow \infty} g^{(r-1)}(y) = (\operatorname{sgn} a_r)\cdot(+\infty).$$

Repeating this argument r times, we get

$$\lim_{y \uparrow \infty} g(y) = (\operatorname{sgn} a_r)\cdot(+\infty).$$

But this contradicts the assumption

$$\lim_{y \uparrow \infty} g(y) = a.$$

Thus we have

$$\lim_{y\uparrow\infty} g^{(r)}(y) = 0.$$

Hence

$$|x^r g^{(r)}(x)| = \left|x^r \int_x^\infty g^{(r+1)}(t)\,dt\right| \leqslant \int_x^\infty t^r |g^{(r+1)}(t)|\,dt \to 0 \qquad (x \uparrow \infty).$$

This proves (7) for $k = r$.

If $r > 1$, then by the last limit relation we have

$$\int_x^y |g^{(r)}(t)|\,dt \leqslant \int_x^y |t^r g^{(r)}(t)| t^{-r}\,dt \to 0 \qquad \text{as } x, y \to \infty.$$

Hence

$$\int^\infty g^{(r-1)}(t)\,dt$$

exists. Now

$$|x^{r-1} g^{(r-1)}(x)| = \left|x^{r-1} \int_x^\infty (t^r g^{(r)}(t)) t^{-r}\,dt\right|$$
$$\leqslant (r-1)^{-1}\cdot \sup_{t\geqslant x} |t^r g^{(r)}(t)| \to 0 \qquad (x \uparrow \infty).$$

This proves (7) for $k = r - 1$. By repeating this argument, we prove (7) for $0 < k \leqslant r$. In order to prove (8) for $k = r - 1$ ($r > 0$), observe that for $0 < x < y$ both sets

$$A \equiv \{t | g^{(r)}(t) > 0, x < t < y\} \quad \text{and} \quad B \equiv \{t | g^{(r)}(t) < 0, x < t < y\}$$

are open. Hence A and B are the unions of a finite or denumerable number of pairwise disjoint open intervals $A = \bigcup_n (\alpha_n, \beta_n)$, $B = \bigcup_n (\gamma_n, \delta_n)$; and for $\alpha_n, \beta_n, \gamma_n, \delta_n$ different from x and y, we have

$$g^{(r)}(\alpha_n) = \ldots = g^{(r)}(\delta_n) = 0.$$

Now, since $g^{(r)}(u) = 0$ for $u \in (x, y) - A - B$,

$$\int_x^y u^{r-1} |g^{(r)}(u)|\,du = \sum_n \int_{\alpha_n}^{\beta_n} u^{r-1} g^{(r)}(u)\,du - \sum_n \int_{\gamma_n}^{\delta_n} u^{r-1} g^{(r)}(u)\,du.$$

Integrating by parts, we have

$$\int u^{r-1} g^{(r)}(u)\,du = \frac{1}{r} u^r g^{(r)}(u) - \frac{1}{r} \int u^r g^{(r+1)}(u)\,du.$$

Therefore

$$\int_x^y u^{r-1} |g^{(r)}(u)|\,du \leqslant \frac{1}{r} |x^r g^{(r)}(x)| + \frac{1}{r} |y^r g^{(r)}(y)| + \frac{1}{r} \int_x^\infty u^r |g^{(r+1)}(u)|\,du.$$

By (7) this implies, letting $x, y \to \infty$, that (8) is true for $k = r - 1$. Repeating this argument, we prove (8) for $0 \leqslant k < n$.

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An immediate consequence of Lemma 2 is the following result.

LEMMA 3. *If, for real* x, $g(x)$ *is the* $(r+1)$th *indefinite integral of a function* $g^{(r+1)}(x)$ $(r \geqslant 1)$,

$$\lim_{x\uparrow\infty} g(x) \quad \textit{and} \quad \lim_{x\downarrow-\infty} g(x)$$

exist, and

$$\int_{-\infty}^{\infty} |u^r g^{(r+1)}(u)|\, du < +\infty,$$

then

$$\lim_{x\uparrow\infty} x^k g^{(k)}(x) = \lim_{x\downarrow-\infty} x^k g^{(k)}(x) = 0 \qquad \textit{for } 0 < k \leqslant r$$

and

$$\int_{-\infty}^{+\infty} |u^k g^{(k+1)}(u)|\, du < +\infty \qquad \textit{for } 0 \leqslant k < r.$$

LEMMA 4. *Let* $f(x)$ *be the Laplace–Stieltjes transform of* $\alpha(t)$. *If for a function*

$$K(s) \equiv \int_0^\infty e^{st}\, d\beta(t) \qquad (\beta(t) = 0 \textit{ for } 0 \leqslant t \leqslant c > 0)$$

we have

$$\int_0^\infty e^{st}\, |d\beta(t)| < +\infty \qquad \textit{for all real } s,$$

then for $r = 0, 1, \ldots,$ *all real* z, *and all sufficiently large* s, *we have*

$$(9) \quad I \equiv \int_0^\infty e^{szt} t^r f(st)\, d\beta(t) = s\int_0^\infty K^{(r+1)}(sz - st)\alpha(t)\, dt - \alpha(0)K^{(r)}(sz).$$

Proof. The function $f(x)$ exists for some $x_0 > 0$. Hence by **(7**, p. 181, Theorem 1**)**,

$$\lim_{x\uparrow\infty} f(x) = \alpha(0+).$$

By **(7**, p. 57, Theorem 5a**)**, we have

$$\int_0^\infty e^{st} t^r\, |d\beta(t)| < +\infty \qquad \text{for all real } s.$$

Hence

$$\int_0^\infty e^{stz} t^r |f(st)|\, |d\beta(t)| < +\infty \qquad \text{for } s > x_0/c \text{ and all real } z.$$

Thus by **(7**, p. 41, Theorem 2.3a**)**, for $s > x_0/c$ and real z,

$$I = s\int_0^\infty e^{szt} t^{r+1}\left\{\int_0^\infty e^{-stu}\alpha(u)\, du\right\} d\beta(t) - K^{(r)}(sz)\alpha(0)$$

$$= s\int_0^\infty K^{(r+1)}(sz - su)\alpha(u)\, du - K^{(r)}(sz)\alpha(0).$$

The change of the order of integration is justified for $s > x_0/c$ by Fubini's theorem (7, p. 26, Theorem 15d) since by (7, p. 39, Theorem 2.2a),

$$|\alpha(t)| \leqslant Me^{x_0 t} \qquad (t \geqslant 0)$$

and

$$\int_0^\infty e^{szt}t^{r+1}\left\{\int_0^\infty e^{-stu}|\alpha(u)|\,du\right\}|d\beta(t)|$$
$$\leqslant M\int_c^\infty e^{szt}t^{r+1}\left\{\int_0^\infty e^{(x_0-st)}\,du\right\}|d\beta(t)|$$
$$\leqslant M\int_c^\infty e^{szt}t^{r+1}\left\{\int_0^\infty e^{(x_0-cs)u}\,du\right\}|d\beta(t)|$$
$$= M(cs-x_0)^{-1}\int_c^\infty e^{szt}t^{r+1}|d\beta(t)| < +\infty.$$

LEMMA 5. *Let $f(x)$ be the Laplace–Stieltjes transform of $\alpha(t)$. If a function*

$$K(s) \equiv \int_c^\infty e^{st}\,d\beta(t) \qquad (\beta(t) = 0 \text{ for } 0 \leqslant t \leqslant c > 0)$$

exists for all real s, then for $r = 0, 1, 2, \ldots$, each real z and each $\delta > 0$, we have

$$\lim_{s\uparrow\infty} s^{r+1}\int_{z+\delta}^\infty K^{(r+1)}(sz-st)\alpha(t)\,dt = 0.$$

Proof. The function $f(x)$ exists for some $x_0 > 0$. Therefore by (7, p. 39, Theorem 2.2a), $|\alpha(t)| \leqslant Me^{x_0 t}$ $(t \geqslant 0)$. Hence for $s > x_0/c$ we have

$$\left|s^{r+1}\int_{z+\delta}^\infty K^{(r+1)}(sz-st)\alpha(t)\,dt\right| \leqslant Ms^{r+1}\int_{z+\delta}^\infty |K^{(r+1)}(sz-st)|^{x_0 t}\,dt$$
$$= Me^{x_0 z}s^r\int_{-\infty}^{-\delta s}|K^{(r+1)}(u)|e^{-xu_0/s}\,du$$

(and by Lemma 1)

$$\leqslant M_1 e^{x_0 z}s^r\int_{-\infty}^{-\delta s}e^{(cs-x_0)u/s}\,du$$
$$= M_1 s^{r+1}(cs-x_0)^{-1}e^{x_0(z+\delta)}e^{-c\delta s}$$
$$\to 0 \qquad (s\uparrow\infty).$$

LEMMA 6. *Suppose that for some $z > 0$ the function $\alpha(t)$ is a bounded L-measurable function in $[0, z]$. If the function*

$$K(s) = \int_c^\infty e^{st}\,d\beta(t) \qquad (c > 0)$$

exists for all real s and if for a certain non-negative integer r we have

$$\int_0^\infty |u^r K^{(r+1)}(u)|\,du < +\infty,$$

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then

$$\lim_{s\uparrow\infty} s^{r+1} \int_0^{z-\delta} K^{(r+1)}(sz - st)\alpha(t)\, dt = 0.$$

for each $\delta(0 < \delta < z)$.

Proof. We have $|\alpha(t)| \leqslant M_1$ if $0 \leqslant t \leqslant z$. By substitution we obtain

$$\begin{aligned}\left| s^{r+1} \int_0^{z-\delta} K^{(r+1)}(sz - st)\alpha(t)\, dt\right| &\leqslant M_1 s^{r+1} \int_0^{z-\delta} |K^{(r+1)}(sz - st)|\, dt \\ &= M_1 s^r \int_{\delta s}^{zs} |K^{(r+1)}(u)|\, du \\ &\leqslant M_1 \delta^{-r} \int_{\delta s}^{\infty} |u^r K^{(r+1)}(u)|\, du \\ &\to 0 \qquad (s \uparrow \infty).\end{aligned}$$

Proof of Theorem 1. We have $0 \leqslant p \leqslant \min(r, q)$. It is obvious that (3) continues to be true if we write p in place of r. We have

$$\lim_{s\uparrow\infty} K(-s) = 0$$

by Lemma 1. Hence for $g(x) = K(x)$, all the assumptions of Lemma 3 are satisfied and this lemma yields

$$\int_{-\infty}^{+\infty} |u^p K^{(p)}(u)|\, du < +\infty.$$

Now in the conclusion (4) of our theorem, r and q do not appear explicitly (only through the restrictions on p). Hence it is enough to prove our theorem for the special case $p = q = r$.

Suppose $z > 0$ and $0 < \delta < z$ or $z = 0$ and $\delta > 0$. The function $f(x)$ exists for some $x_0 > 0$. Therefore, by Lemma 4, we have for $s > x_0/c$

$$\begin{aligned}(10)\quad & s^r \int_0^{\infty} e^{szt} t^r f(st)\, d\beta(t) \\ &= \left\{ \int_0^{z-\delta} + \int_{z-\delta}^{z} + \int_z^{z+\delta} + \int_{z+\delta}^{\infty} \right\} s^{r+1} K^{(r+1)}(sz - st)\alpha(t)\, dt \\ &\qquad\qquad - s^r K^{(r)}(sz)\alpha(0) \\ &\equiv I_1 + I_2 + I_3 + I_4 - s^r K^{(r)}(sz)\alpha(0)\end{aligned}$$

(for $z = 0$, we define $I_1 = I_2 = 0$). By Lemma 2,

$$(11)\quad \alpha(0)s^r K^{(r)}(sz) = \delta_{0,z} K^{(r)}(0)\alpha(0)s^r - [1 - \delta_{0,z}]\delta_{0,r}\, a_0\, \alpha(0) + o(1) \ (s \uparrow \infty).$$

By Lemmas 5 and 6, we have for each fixed δ

$$(12)\qquad \lim_{s\uparrow\infty} I_1 = \lim_{s\uparrow\infty} I_4 = 0.$$

Given $\epsilon > 0$, choose δ, $0 < \delta < z$, so small that we have both

$$\left|\alpha(t) - \sum_{k=0}^{r} b_k (t-z)^k/k!\right| < \epsilon(t-z)^r \qquad \text{for } z < t \leqslant z + \delta$$

and

$$\left|\alpha(t) - \sum_{k=0}^{r} c_k (t-z)^k/k!\right| < \epsilon(z-t)^r \qquad \text{for } z - \delta \leqslant t < z.$$

We now have

$$\left|I_3 - \sum_{k=0}^{r} b_k\, s^{r+1} \int_z^{z+\delta} K^{(r+1)}(sz - st)(t-z)^k (k!)^{-1}\, dt\right|$$

$$= \left|s^{r+1} \int_z^{z+\delta} K^{(r+1)}(sz - st)\left\{\alpha(t) - \sum_{k=0}^{r} b_k (t-z)^k/k!\right\} dt\right|$$

$$\leqslant \epsilon s^{r+1} \int_z^{z+\delta} (t-z)^r |K^{(r+1)}(sz - st)|\, dt$$

$$\leqslant \epsilon \int_{-\infty}^{+\infty} |u^r K^{(r+1)}(u)|\, du.$$

Hence

$$\left|I_3 - \sum_{k=0}^{r} (-1)^k b_k\, s^{r-k} \int_{-\delta s}^{0} K^{(r+1)}(u) u^k (k!)^{-1}\, du\right| \leqslant \epsilon \int_{-\infty}^{+\infty} |u^r K^{(r+1)}(u)|\, du.$$

Integrating the integral appearing in the kth term of the sum on the left-hand side of the last inequality by parts k times and applying Lemma 1 to each of the terms obtained, we obtain

$$\text{(13)} \qquad \left|I_3 - \sum_{k=0}^{r} b_k K^{(r-k)}(0) s^{r-k} + o(1)\right| \leqslant \epsilon \int_{-\infty}^{+\infty} |u^r K^{(r+1)}(u)|\, du \qquad (s \uparrow \infty).$$

In a similar way, but using Lemma 3 instead of Lemma 1, we have

$$\text{(14)} \qquad \left|I_2 + \sum_{k=0}^{r} c_k K^{(r-k)}(0) s^{r-k} + a_0 c_r + o(1)\right|$$

$$\leqslant \epsilon \int_{-\infty}^{+\infty} |u^r K^{(r+1)}(u)|\, du \qquad (s \uparrow \infty).$$

A combination of (10) to (14) completes the proof of the theorem.

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