Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Agnes Bokanyi-Toth, School of Science Reception, Schofield Building, Loughborough University, Loughborough, LE11 3TU. Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most elegant solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by 13th September 2024 and will be published in the November 2024 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the Mathematical Association website

https:www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2024.3 (Paul Stephenson)

General Lucas sequences are defined by the recursion $l_{n+2} = l_{n+1} + l_n$. Two particular ones are the Fibonacci sequence, starting 0, 1, 1, 2, ... and the Lucas sequence, starting 2, 1, 3, 4, ... Show that, after the third term, the two sequences have no terms in common.

Problem 2024.4 (Paul Stephenson)

Consider a convex hexagon whose vertices are grid points on a square lattice of unit edge and which has no boundary points other than vertices. By Pick's theorem, its area is given by $A = i + \frac{1}{2}b - 1$, where i is the number of interior points and b is the number of boundary points. Show that its area must be at least 4 units.

Solutions to 2024.1 and 2024.2

Both problems were solved by Aarar Gupta. Problem 2024.1 was solved by Ryan Bhaskar. Problem 2024.2 was solved by Nikta Vodolazkyi.

Problem 2024.1 (Nick Lord)

For positive real numbers α, β, γ, consider the inequality

$$\alpha p^2 + \beta q^2 > \gamma r^2 \qquad (*)$$

(a) Show that, if $\frac{1}{\gamma} \leqslant \frac{1}{\alpha} + \frac{1}{\beta}$, then (*) holds whenever p, q, r are the sides of a non-degenerate triangle.

(b) Is the converse of (a) true?

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Solution (Ryan Bhaskar)

$$\frac{1}{\gamma} \geqslant \frac{1}{\alpha} + \frac{1}{\beta}. \tag{1}$$

By reciprocating both sides

$$\gamma \leqslant \frac{1}{\frac{1}{\alpha} + \frac{1}{\beta}}. \tag{2}$$

If p, q and r are the sides of a non-degenerate triangle, then $r < p + q$. By squaring both sides

$$r^2 < (p + q)^2. \tag{3}$$

Multiplying (2) and (3) gives

$$\gamma r^2 < \frac{(p + q)^2}{\frac{1}{\alpha} + \frac{1}{\beta}}. \tag{4}$$

So it suffices to show that the right-hand side of (4) is less that $\alpha p^2 + \beta q^2$.

By the AM-GM inequality

$$\frac{\alpha}{\beta}p^2 + \frac{\beta}{\alpha}q^2 \geqslant 2pq.$$

By adding $p^2 + q^2$ to both sides

$$p^2 + \frac{\alpha}{\beta}p^2 + \frac{\beta}{\alpha}q^2 + q^2 \geqslant p^2 + q^2 + 2pq,$$

$$\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\left(\alpha p^2 + \beta q^2\right) \geqslant (p + q)^2,$$

$$\alpha p^2 + \beta q^2 \geqslant \frac{(p + q)^2}{\frac{1}{\alpha} + \frac{1}{\beta}}. \tag{5}$$

From (4) and (5), it follows that $\alpha p^2 + \beta q^2 > \gamma r^2$.

(b) The converse is not true. A counter-example is provided by $\alpha = \beta = 2$, $\gamma = 1, p = 1, q = 2$ and $r = 3$.

To justify that this is a valid counter-example, consider the following:

$$\frac{1}{\gamma} \geqslant \frac{1}{\alpha} + \frac{1}{\beta}, \tag{6}$$

$$\alpha p^2 + \beta q^2 > \gamma r^2, \tag{7}$$

$$p + q > r. \tag{8}$$

Expressions (6) and (7) are satisfied and (8) is not (so p, q and r are not the sides of a non-degenerate triangle).

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Problem 2024.2 (Himadri Lal Das)

For a positive integer n, $d(n)$ denotes the number of digits in n. Prove that

$$\frac{20}{19} + \frac{30}{29} + \dots + \frac{100}{99} \leqslant \sum_{n=1}^{\infty} \frac{10^{d(n)^2 - 4d(n) + 3}}{n^{d(n)-1}} \leqslant \frac{10}{9} + \frac{20}{19} + \dots + \frac{90}{89}.$$

Solution (Himadri Lal Das)

$$\sum_{n=1}^{\infty} \frac{10^{d(n)^2 - 4d(n) + 3}}{n^{d(n)-1}} = \sum_{n=1}^{\infty} \frac{10^{(d(n)-2)^2 - 1}}{n^{d(n)-1}}$$

$$= 1 + 1 + \dots + \frac{10^{-1}}{10} + \frac{10^{-1}}{11} + \dots + \frac{1}{100^2} + \frac{1}{101^2} + \dots .$$

The series is convergent. Now we do the grouping for the above series where the n th term of the grouping contains the elements $\frac{10^{(d(k)-2)^2-1}}{n^{d(k)-1}}$ such that $d(k) = n$. So we have

$$\sum_{n=1}^{\infty} \frac{10^{(d(n)-2)^2-1}}{n^{d(n)-1}} = (1 + 1 + \dots + 1) + \frac{1}{10}\left(\frac{1}{10} + \frac{1}{11} + \dots + \frac{1}{99}\right) + \dots$$

$$\dots + 10^{(n-2)^2-1}\left(\frac{1}{10^{(n-1)^2}} + \frac{1}{(10^{(n-1)} + 1)^{n-1}} + \dots + \frac{1}{(10^n - 1)^{n-1}}\right) + \dots$$

$$\leqslant (1 + 1 + \dots + 1) + \frac{1}{10}\left(1 + \frac{1}{2} + \dots + \frac{1}{9}\right) + \dots + \frac{1}{10^{n-1}}\left(1 + \frac{1}{2^{n-1}} + \dots + \frac{1}{9^{n-1}}\right) + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{10^k} + \sum_{k=0}^{\infty} \frac{1}{20^k} + \dots + \sum_{k=0}^{\infty} \frac{1}{90^k} = \frac{10}{9} + \frac{20}{19} + \dots + \frac{90}{89}.$$

Also,

$$\sum_{n=1}^{\infty} \frac{10^{(d(n)-2)^2-1}}{n^{(d(n)-1)}} = (1 + 1 + \dots + 1) + \frac{1}{10}\left(\frac{1}{10} + \frac{1}{11} + \dots + \frac{1}{99}\right) + \dots$$

$$+ 10^{(n-2)^2-1}\left(\frac{1}{10^{(n-1)^2}} + \frac{1}{(10^{n-1} + 1)^{n-1}} + \dots + \frac{1}{(10^n - 1)^{n-1}}\right) + \dots$$

$$\geqslant (1 + 1 + \dots + 1) + \frac{1}{10}\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{10}\right) + \dots + \frac{1}{10^{n-1}}\left(\frac{1}{2^{n-1}} + \dots + \frac{1}{10^{n-1}}\right) + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{20^k} + \sum_{k=0}^{\infty} \frac{1}{30^k} + \dots + \sum_{k=0}^{\infty} \frac{1}{100^k} = \frac{20}{19} + \frac{30}{29} + \dots + \frac{100}{99}.$$

Prize Winners

The first prize of £25 is awarded to Aarar Gupta. The second prize of £20 is awarded to Ryan Bhaskar.

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