ISOMORPHISMS INDUCED BY AUTOMORPHISMS

Dedicated to the memory of Hanna Neumann

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The object of this note is to record a property of finite, perfect, centrally closed groups, where, by definition, G is centrally closed if and only if whenever $E/Z \cong G$ and $Z \subseteq E' \cap Z(E)$, then $Z = 1$.

THEOREM. *Suppose G is a finite, perfect, centrally closed group and Z_1, Z_2 are central subgroups of G such that $G/Z_1 \cong G/Z_2$. Then $\alpha(Z_1) = Z_2$ for some $\alpha \in \mathrm{Aut}(G)$.*

PROOF. We may assume that $G = F/A$, where F is a free group. Let $Z(G) = R/A$ be the center of G. Then [1, p. 628 ff]

$$R/[R,F] = A/[R,F] \times F' \cap R/[R,F].$$

Since G is perfect, we have

$$F = F'A, \qquad F' \cap A = (F' \cap R) \cap A = [F,R],$$

and so there is an isomorphism

$$\rho: F/A \cong F'/[F,R].$$

Let $Z_i = L_i/A$, $i = 1,2$, so that $L_i \subseteq R$. Then

$$F/L_1 \cong F/A/L_1/A = G/Z_1 \cong G/Z_2 = F/A/L_2/A \cong F/L_2.$$

Thus, as F is free, there is an endomorphism π of F which induces an isomorphism of F/L_1 onto F/L_2, that is,

$$\pi(L_1) \subseteq L_2, \qquad F = \pi(F)\cdot L_2.$$

Hence, $F' = \pi(F)'[\pi(F), L_2]\cdot L_2' \subseteq \pi(F')[F,R] \subseteq F'$, and so

$$F' = \pi(F')[F,R]. \tag{1}$$

Also, $Z(F/L_i) = R/L_i$, since G is perfect so that its center and second center coincide. Hence, $\pi(R) \subseteq R$, whence $\pi([F,R]) \subseteq [F,R]$, and π induces an endomorphism π^* of $F'/[F,R]$. By (1), π^* is onto, and since $F'/[F,R]$ is finite, π^* is an automorphism. Then the composition

$$F/A \xrightarrow{\rho} F'/[F,R] \xrightarrow{\pi^*} F'/[F,R] \xrightarrow{\rho^{-1}} F/A$$

is an automorphism of G which carries Z_1 to Z_2.

REMARK. If G is a covering group of a simple group S, we can use the theorem to help determine all the isomorphism classes of perfect groups $\tilde{G}$ such that $\tilde{G}/Z(\tilde{G}) \cong S$; we are led to the study of the orbits of $\mathrm{Aut}(S)$ on the subgroups of the Schur multiplicator of S.

Bibliography

[1] B. Huppert, *Endliche Gruppen*, (Springer, 1967).

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