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An index of P. Hall for varieties of groups

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P. Hall defined the k-index of a variety $\underline{\underline{V}}$ of groups to be the least cardinal number r such that if a group G is generated by a set S and every subset of S of cardinality at most r generates a group in $\underline{\underline{V}}$ then $G \in \underline{\underline{V}}$. We show that the only variety which has finite k-index and contains a product of two non-trivial varieties is the variety of all groups. As a consequence of this and P. Hall's result that nilpotent varieties have finite k-index we show that a soluble variety or a variety generated by a finite group has finite k-index if and only if it is nilpotent.

1.

In a lecture at Oxford in August 1966, Hall defined the *k-index* of a variety $\underline{\underline{V}}$ of groups to be the least cardinal number r such that if a group G is generated by a set S and every subset of S of cardinality at most r generates a group in $\underline{\underline{V}}$ then $G \in \underline{\underline{V}}$. At the same time he showed that nilpotent varieties have finite k-index and asked which other varieties have this property. Here we shall introduce a class N of varieties which is large enough to contain every soluble variety and every Cross variety (that is, a variety generated by a finite group) and shall show that the nilpotent varieties are the only varieties in N which have finite k-index. As a proof of Hall's result that nilpotent varieties have finite k-index has never been published and the result is used in

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this paper we give a proof in Section 3.

Our main result is

THEOREM 1. *If the variety* $\underline{V}$ *has finite k-index and contains the product of two non-trivial varieties then* $\underline{V}$ *is the variety of all groups.*

The reader is referred to Hanna Neumann's book [7] for all undefined notation and terminology.

Before we introduce the class N we make some preliminary remarks. The argument of Theorem 1 of Kovács and Newman [4], together with the fact that nilpotent varieties are finitely based (34.14 of [7]), shows that every non-nilpotent variety contains a non-nilpotent subvariety $\underline{V}$ all of whose proper subvarieties are nilpotent. We call such a variety $\underline{V}$ *just non-nilpotent*. A variety $\underline{V}$ is said to be *reducible* if $\underline{V}$ is contained in the product of two proper subvarieties, that is, $\underline{V} \subseteq \underline{UW}$ where $\underline{U} \subset \underline{V}$ and $\underline{W} \subset \underline{V}$. Otherwise $\underline{V}$ is said to be *irreducible*. We write $\underline{A}$ for the variety of all abelian groups and, for any positive integer m, $\underline{A}_m$ for the variety of all abelian groups of exponent dividing m. If $\underline{V}$ is a reducible just non-nilpotent variety then $\underline{V} \subseteq \underline{UW}$ with $\underline{U} \subset \underline{V}$ and $\underline{W} \subset \underline{V}$. As $\underline{V}$ is just non-nilpotent, $\underline{U}$ and $\underline{W}$ are nilpotent, and $\underline{V}$ is soluble. It now follows, by Proposition 2 of Groves [2], that $\underline{V} = \underline{A}_p\underline{A}_q$ for some not necessarily distinct primes p and q.

We define N to be the class of all varieties which do not contain an irreducible just non-nilpotent subvariety. Clearly N contains all nilpotent varieties. Also N contains all Cross varieties: Kovács and Newman [3] point out that every just non-nilpotent variety contained in a Cross variety has the form $\underline{A}_p\underline{A}_q$ for distinct primes p and q. If $\underline{V} \in N$ and $\underline{U} \subseteq \underline{V}$ then clearly $\underline{U} \in N$. Also if $\underline{U}, \underline{V} \in N$ then $\underline{UV} \in N$: for if $\underline{W}$ is a subvariety of $\underline{UV}$ then

$$\underline{W} \subseteq (\underline{U} \cap \underline{W})(\underline{V} \cap \underline{W})$$

so that, if $\underline{W}$ is irreducible, $\underline{W} \subseteq \underline{U}$ or $\underline{W} \subseteq \underline{V}$. It now follows that N contains any subvariety of any product variety $\underline{V}_1\underline{V}_2 \ldots \underline{V}_k$ whenever each of the $\underline{V}_i$ is either abelian or Cross; that is, N contains every SC-variety, in the sense of Groves [2].

On the other hand we will show that the variety $\underline{\underline{O}}$ of all groups is not in N. Bachmuth, Mochizuki and Walkup [1] have shown that the variety $\underline{\underline{B}}_5$ of all groups of exponent dividing 5 is not soluble. Thus it contains a just non-nilpotent subvariety $\underline{\underline{W}}$. If $\underline{\underline{W}}$ were reducible then $\underline{\underline{W}}$ would equal $\underline{\underline{A}}_p\underline{\underline{A}}_q$ for some not necessarily distinct primes p and q, but clearly $\underline{\underline{A}}_p\underline{\underline{A}}_q \nsubseteq \underline{\underline{B}}_5$. Thus $\underline{\underline{O}}$ contains an irreducible just non-nilpotent variety $\underline{\underline{W}}$, and so $\underline{\underline{O}} \notin N$.

We show that the theorem above implies

COROLLARY. *A variety* $\underline{\underline{V}}$ *of* N *has finite k-index if and only if* $\underline{\underline{V}}$ *is nilpotent.*

Proof. If $\underline{\underline{V}} \in N$ is nilpotent, then $\underline{\underline{V}}$ has finite k-index, by Hall's result, proved in §3. Suppose conversely that $\underline{\underline{V}} \in N$ has finite k-index and, by way of contradiction, assume $\underline{\underline{V}}$ is non-nilpotent. Let $\underline{\underline{U}}$ be a just non-nilpotent subvariety of $\underline{\underline{V}}$. As $\underline{\underline{V}} \in N$, $\underline{\underline{U}}$ is reducible, so $\underline{\underline{U}} = \underline{\underline{A}}_p\underline{\underline{A}}_q$ for some not necessarily distinct primes p and q. Now by Theorem 1 $\underline{\underline{V}} = \underline{\underline{O}}$ as $\underline{\underline{V}}$ has finite k-index and contains a product of two non-trivial varieties. However $\underline{\underline{O}} \notin N$ and we have a contradiction.

2.

Before proving Theorem 1 we briefly outline a definition of the twisted wreath product, a concept due to B.H. Neumann [6]. Suppose A and B are groups, related as follows. B has a subgroup H with a right transversal T, $B = HT$, and there is a homomorphism $\theta : H \to \operatorname{Aut}A$. Let $A^{(T)}$ denote the group of functions of finite support from T to A with componentwise multiplication. Then the (restricted) twisted wreath product $A \operatorname{wr}_\theta B$ of A by B is the semidirect product of $A^{(T)}$ by B, such that for $\mu \in A^{(T)}$, $b \in B$ and $t \in T$,

$$\mu^b(t) = \mu(s)^{\theta(h)}$$

where $tb^{-1} = h^{-1}s$, $s \in T$ and $h \in H$. It can be shown that, up to isomorphism, $A \operatorname{wr}_\theta B$ is independent of the choice of T. If H and θ are trivial then $A \operatorname{wr}_\theta B$ is the (restricted) standard wreath product and

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is denoted by $A \text{ wr } B$. We regard $A^{(T)}$ and B as subgroups of $A \text{ wr}_\theta B$ in the usual way. We shall always take $1 \in T$ and identify the element $a \in A$ with the function $\mu \in A^{(T)}$ defined by $\mu(1) = a$ and $\mu(t) = 1$ for all $t \neq 1$. Thus A and B are both subgroups of $A \text{ wr}_\theta B$. The following lemma, whose proof is left to the reader, is used several times in proving Theorem 1.

(i). *Let* $G = A \text{ wr}_\theta B$ *where* $H \leq B$ *and* $\theta : H \to \text{Aut}A$. *Suppose* $A_1 \leq A$, $B_1 \leq B$ *and* $B_1 \cap H = \{1\}$. *Then, if* A *and* B *are identified with subgroups of* G , *as above, we have*

$$\text{gp}(A_1, B_1) \cong A_1 \text{ wr } B_1 .$$

We shall use (i) to prove

(ii). *Suppose* $\underline{\underline{V}}$ *is a variety of finite k-index* r *and* $\underline{\underline{V}}$ *contains* $C \text{ wr } D^r$, *where* C *and* D *are nontrivial cyclic groups and* D^r *denotes the direct product of* r *copies of* D . *Then* $\underline{\underline{V}}$ *contains* $(C*C) \text{ wr } D^r$, *where* $C*C$ *is the free product of* C *with itself.*

Proof. Let $B = D_1 \times D_2 \times \ldots \times D_{r+1}$ where for each i , $D_i \cong D$ and D_i is generated by an element d_i . Let $h = d_1 d_2 \ldots d_{r+1}$ and let H be the cyclic subgroup of B generated by h . Let A be the normal closure of C in the free product $C*H$. Then there is a homomorphism $\theta : H \to \text{Aut}A$, such that $\theta(h)$ is the restriction to A of the inner automorphism of $C*H$ induced by h . Let c be a generator of C . Then it is easy to see that $A \text{ wr}_\theta B$ has a generating set

$$S = \{c, d_1, d_2, \ldots, d_{r+1}\} .$$

By (i), every r element subset of S generates a group isomorphic to a subgroup of $C \text{ wr } D^r$. Thus $A \text{ wr}_\theta B \in \underline{\underline{V}}$ since $C \text{ wr } D^r \in \underline{\underline{V}}$ and $\underline{\underline{V}}$ has k-index r . Let A_1 be the subgroup of A generated by c and c^h , and let $B_1 = D_1 \times D_2 \times \ldots \times D_r$. Then $A_1 \cong C*C$, $B_1 \cong D^r$ and

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$B_1 \cap H = \{1\}$. Thus $(C*C)$ wr D^r is in $\underline{\underline{V}}$, by (i).

Proof of Theorem 1. Suppose $\underline{\underline{V}}$ is a variety of finite k-index r which contains the product of two nontrivial varieties. Then $\underline{\underline{V}}$ contains a product variety $(\text{var}C)(\text{var}D)$ where C and D are nontrivial cyclic groups. Thus $\underline{\underline{V}}$ contains C wr D^r and, by (ii), $\underline{\underline{V}}$ contains $(C*C)$ wr D^r . But $C*C$ has an infinite cyclic subgroup C_1 and $\underline{\underline{V}}$ contains C_1 wr D^r (which is isomorphic to a subgroup of $(C*C)$ wr D^r). Thus, by (ii) again, $\underline{\underline{V}}$ contains (C_1*C_1) wr D^r and therefore contains C_1*C_1 . But C_1*C_1 is an absolutely free group of rank 2 and so has a subgroup which is free of countable rank (see problem 2, p. 122 of [5]). Thus $\underline{\underline{V}} = \underline{\underline{O}}$.

3.

In this section we prove

THEOREM 2 (Hall). *If* $\underline{\underline{V}}$ *is a variety which is nilpotent of class* c *then the* k*-index of* $\underline{\underline{V}}$ *is at most* $c + 1$.

Proof. Suppose G is a group with a generating set S such that every subset of S of cardinality at most $c + 1$ generates a group in $\underline{\underline{V}}$. We shall show that G satisfies every law of $\underline{\underline{V}}$, so that $G \in \underline{\underline{V}}$. Let X be an absolutely free group freely generated by the "variables" $x_1, x_2, \ldots$. If $v = v(x_1, \ldots, x_r)$ is an element of X which is a law of $\underline{\underline{V}}$ and $\alpha : X \to G$ is a homomorphism we need to show that $v\alpha = 1$. For each i , $x_i\alpha$ is a word in the elements of S . Thus there are elements v_i of X and a homomorphism $\beta : X \to G$ such that $x_i\beta \in S$ for all i , and $v\alpha = v(v_1, \ldots, v_r)\beta$. Let $w = v(v_1, \ldots, v_r)$. Then w is a law of $\underline{\underline{V}}$ and it suffices to show that $w\beta = 1$.

By the proof of 33.45 of [7], w is equal to a product of words $w_1, \ldots, w_s$ such that each w_i is a consequence of w (hence a law of $\underline{\underline{V}}$) and w_i involves each variable it contains. It suffices to show that

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$w_i\beta = 1$ for each i, and so, changing the notation, we may assume that w itself involves each variable it contains.

If w involves no more than $c + 1$ variables then $w\beta = 1$ by the hypothesis on S. Thus we may assume that w involves more than $c + 1$ variables. Certainly w involves $c + 1$ distinct variables so, by 33.38 of [7], w is an element of the $(c+1)$st term of the lower central series of X. Therefore (see problem 3, p. 297 of [5]), w is in the normal closure of the set of left normed commutators of the form $\left[x_{i(1)}, \ldots, x_{i(c+1)}\right]$, and thus it suffices to show that

$$\left[x_{i(1)}, \ldots, x_{i(c+1)}\right]\beta = 1 .$$

But this follows since β is a homomorphism and $\mathrm{gp}\left(x_{i(1)}\beta, \ldots, x_{i(c+1)}\beta\right)$ is in $\underline{V}$ and thus is nilpotent of class at most c.

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