COMPOSITIO MATHEMATICA

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Compositio Math. **148** (2012), 1717–1735.

doi:10.1112/S0010437X12000437

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

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Algebraic boundaries of Hilbert's SOS cones

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Abstract

We study the geometry underlying the difference between non-negative polynomials and sums of squares (SOS). The hypersurfaces that discriminate these two cones for ternary sextics and quaternary quartics are shown to be Noether–Lefschetz loci of K3 surfaces. The projective duals of these hypersurfaces are defined by rank constraints on Hankel matrices. We compute their degrees using numerical algebraic geometry, thereby verifying results due to Maulik and Pandharipande. The non-SOS extreme rays of the two cones of non-negative forms are parametrized, respectively, by the Severi variety of plane rational sextics and by the variety of quartic symmetroids.

1. Introduction

A fundamental object in convex algebraic geometry is the cone $\Sigma_{n,2d}$ of homogeneous polynomials of degree $2d$ in $\mathbb{R}[x_1, \ldots, x_n]$ that are sums of squares (SOS). Hilbert [Hil88] showed that the cones $\Sigma_{3,6}$ and $\Sigma_{4,4}$ are strictly contained in the corresponding cones $P_{3,6}$ and $P_{4,4}$ of non-negative polynomials. Blekherman [Ble12] furnished a geometric explanation for this fact. Despite recent progress, however, the geometry of the sets $P_{3,6} \backslash \Sigma_{3,6}$ and $P_{4,4} \backslash \Sigma_{4,4}$ remains mysterious.

Here we extend known results on Hilbert's SOS cones by characterizing their *algebraic boundaries*, that is, the hypersurfaces that arise as Zariski closures of their topological boundaries. The algebraic boundary of the cone $P_{n,2d}$ of non-negative polynomials is the *discriminant* [Nie12], and this is also always one component in the algebraic boundary of $\Sigma_{n,2d}$. The discriminant has degree $n(2d-1)^{n-1}$, which equals 75 for $\Sigma_{3,6}$ and 108 for $\Sigma_{4,4}$. What we are interested in are the other components in the algebraic boundary of the SOS cones.

Theorem 1. *The algebraic boundary of $\Sigma_{3,6}$ has a unique non-discriminant component; it has degree 83 200 and is the Zariski closure of the sextics that are sums of three squares of cubics. Similarly, the algebraic boundary of $\Sigma_{4,4}$ has a unique non-discriminant component; it has degree 38 475 and is the Zariski closure of the quartics that are sums of four squares of quadrics. Both hypersurfaces define Noether–Lefschetz divisors in moduli spaces of K3 surfaces.*

Our characterization of these algebraic boundaries in terms of sums of few squares is a consequence of [Ble12, Corollaries 1.3 and 1.4]. What is new here is the connection to K3 surfaces, which elucidates the hypersurface of ternary sextics that are rank-3 quadrics in cubic forms, as well as the hypersurface of quartic forms in four variables that are rank-4 quadrics in quadratic forms. Their degrees are coefficients in the modular forms derived by Maulik and Pandharipande in their paper [MP07] on Gromov–Witten and Noether–Lefschetz theory. In § 2 we explain these concepts and present the proof of Theorem 1.

Received 11 July 2011, accepted in final form 7 May 2012, published online 15 October 2012.
2010 Mathematics Subject Classification 13J30 (primary), 14J28, 14P05, 65H10, 68W30 (secondary).
Keywords: positive polynomials, K3 surfaces, numerical algebraic geometry.

Section 3 concerns the cone dual to $\Sigma_{n,2d}$ and the varieties dual to our Noether–Lefschetz hypersurfaces in Theorem 1. Each of these is a determinantal variety, defined by rank constraints on a 10×10 Hankel matrix, and is parametrized by a Grassmannian via the global residue map of [CD05, § 1.6]. Hankel matrices are also known as *moment matrices* in the optimization literature (e.g. [Las10, § 3.2.1]) or as *catalecticants* in the commutative algebra literature (e.g. [IK99]).

Section 4 features another appearance of a Gromov–Witten number [KM94] in convex algebraic geometry. Building on work of Reznick [Rez07], we shall prove the following result.

Theorem 2. *The Zariski closure of the set of extreme rays of $P_{3,6}\backslash\Sigma_{3,6}$ is the Severi variety of rational sextic curves in the projective plane $\mathbb{P}^2$. This Severi variety has dimension* 17 *and degree* 26 312 976 *in the $\mathbb{P}^{27}$ of all sextic curves.*

We also determine the analogous variety of extreme rays for quartics in $\mathbb{P}^3$.

Theorem 3. *The Zariski closure of the set of extreme rays of $P_{4,4}\backslash\Sigma_{4,4}$ is the variety of* quartic symmetroids *in $\mathbb{P}^3$, that is, surfaces whose defining polynomial is the determinant of a symmetric 4×4 matrix of linear forms. This variety has dimension* 24 *in the $\mathbb{P}^{34}$ of all quartic surfaces.*

Section 5 presents an experimental study of the objects in this paper using numerical algebraic geometry. We demonstrate that the degrees 83 200 and 38 475 in Theorem 1 can be found from scratch using the software `Bertini` [BHSW]. This provides computational validation of the cited results of Maulik and Pandharipande [MP07]. Motivated by Theorem 3, we also show how to compute a symmetric determinantal representation (10) for a given quartic symmetroid.

A question one might ask is: *What is the point of integers such as* 38 475*?*

One answer is that the exact determination of such degrees signifies an understanding of deep geometric structures that can be applied to a wider range of subsequent problems. A famous example is the number 3264 of plane conics that are tangent to five given conics. The finding of that particular integer in the 19th century led to the development of intersection theory in the 20th century, and ultimately to numerical algebraic geometry in the 21st century. To be more specific, our theorems above furnish novel geometric representations of boundary sums of squares that are strictly positive, and of extremal non-negative polynomials that are not sums of squares. Apart from its intrinsic appeal within algebraic geometry, we expect that our approach, with its focus on explicit degrees, will be useful for applications in optimization and beyond.

2. Noether–Lefschetz loci of K3 surfaces

Every smooth quartic surface in $\mathbb{P}^3$ is a K3 surface. In our study of Hilbert's cone $\Sigma_{4,4}$ we care about quartic surfaces containing an elliptic curve of degree 4. Their defining quartic form is a sum of four squares. K3 surfaces also arise as double covers of $\mathbb{P}^2$ ramified along a smooth sextic curve. In our study of $\Sigma_{3,6}$ we care about K3 surfaces whose associated ternary sextic is a sum of three squares. This constraint on K3 surfaces also appeared in the proof by Colliot-Thélène [Col93] that a general sextic in $\Sigma_{3,6}$ is a sum of four but not three squares of rational functions.

Our point of departure is the Noether–Lefschetz theorem [GH85], which states that a general quartic surface S in $\mathbb{P}^3$ has Picard number 1. In particular, the classical result by Noether [Noe82] and Lefschetz [Lef21] states that every irreducible curve on S is the intersection of S with another surface in $\mathbb{P}^3$. This has been extended [PS71] to general quasi-polarized K3 surfaces (S, A),

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i.e. K3 surfaces S with a divisor A such that $A^2>0$ and $A\cdot C\geqslant 0$ for every curve C on S. For each even $l\geqslant 2$, the moduli space M_l of primitively quasi-polarized K3 surfaces (S,A) of degree $A^2=l$ is a quasi-projective variety of dimension 19, and for the general surface (S,A) in M_l, the Picard group is generated by A. The Picard group $\mathrm{Pic}(S)$ of a K3 surface S is an integral lattice of finite rank r with an even symmetric bilinear form defined by the intersection product. The moduli space of K3 surfaces with a given Picard lattice has dimension $20-r$ (for a detailed discussion see [DK07]). In particular, quasi-polarized surfaces (S,A) whose Picard group has rank at least 2 form a countable union of divisors, called *Noether–Lefschetz divisors* in M_l. Here we are interested in one Noether–Lefschetz divisor in M_2 and one in M_4.

Maulik and Pandharipande [MP07] described two types of Noether–Lefschetz divisors. We briefly recall these from their paper. The first type of divisor is defined via the Picard lattice. If $\mathrm{Pic}(S)$ is generated by the classes A and B with intersection matrix

$$\begin{pmatrix} A^2 & A\cdot B \\ A\cdot B & B^2 \end{pmatrix} = \begin{pmatrix} l & d \\ d & 2h-2 \end{pmatrix}, \tag{1}$$

then the Picard lattice $(\mathrm{Pic}(S),A)$, with its distinguished element A, has *discriminant*

$$\Delta=\Delta_l(h,d)=d^2-2lh+2l$$

and *coset*

$$\delta = d \bmod l \in \left(\frac{\mathbb{Z}}{l\mathbb{Z}}\right)\Big/ \pm .$$

The Noether–Lefschetz divisor

$$P_{\Delta,\delta}\subset M_l$$

is defined to be the closure of the locus of quasi-polarized K3 surfaces (S,A) whose Picard lattice $(\mathrm{Pic}(S),A)$ has rank 2 with discriminant Δ and coset δ.

The second type of Noether–Lefschetz divisor is defined by specifying a class $B\in\mathrm{Pic}(S)$ and intersection numbers $A\cdot B=d$ and $B^2=2h-2$. The divisor

$$D_{h,d}\subset M_l$$

is defined as the weighted sum

$$D_{h,d}=\sum_{\Delta\mid\Delta_l(h,d)}\mu(h,d,\Delta,\delta)[P_{\Delta,\delta}]$$

where $\mu(h,d,\Delta,\delta)\in\{0,1,2\}$ counts the number of classes $B\in\mathrm{Pic}(S)$ such that $A\cdot B=d$ and $B^2=2h-2$. A family $\pi:X\to C$ of K3 surfaces over a non-singular complete curve C with a divisor L on X that defines a quasi-polarization of degree $\int_S L^2=l$ for every K3 surface $S\subset X$ yields a morphism $\iota_\pi: C\to M_l$. The degree $NL^\pi_{h,d}$ of the divisor $\iota^*_\pi(D_{h,d})$ is the *Noether–Lefschetz number* for the family. The relevant enumerative geometry for these numbers was developed by Maulik and Pandharipande [MP07], and our result rests on theirs.

Proof of Theorem 1. It was shown in [Ble12] that $\partial\Sigma_{3,6}\backslash\partial P_{3,6}$ consists of ternary sextics F that are sums of three squares over $\mathbb{R}$. Over the complex numbers $\mathbb{C}$, such a sextic F is a rank-3 quadric in cubic forms, so it can be written as

$$F=fh-g^2 \quad \text{where } f,g,h\in\mathbb{C}[x_1,x_2,x_3]_3.$$

Let S be the surface of bidegree $(2,3)$ in $\mathbb{P}^1\times\mathbb{P}^2$ defined by the polynomial

$$G=fs^2+2gst+ht^2.$$

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If f, g and h are general, then the surface S is smooth. The canonical divisor on $\mathbb{P}^1 \times \mathbb{P}^2$ has bidegree $(-2, -3)$, so, by the adjunction formula, S is a K3 surface. The projection $S \to \mathbb{P}^2$ is two-to-one, ramified along the curve $\{F = 0\} \subset \mathbb{P}^2$. Up to the actions of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(3, \mathbb{C})$, there is an 18-dimensional family of surfaces of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. The pullback of a line under the projection $S \to \mathbb{P}^2$ determines a primitive polarization A on S with $A^2 = 2$, so these surfaces determine a divisor $D(2, 3)$ in the moduli space M_2. In particular, the general surface in the family has Picard group of rank 2. The fiber of the projection $S \to \mathbb{P}^1$ is a divisor B in $\mathrm{Pic}(S)$. The divisors A and B have the intersection numbers

$$\begin{pmatrix} A^2 & A \cdot B \\ A \cdot B & B^2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}. \tag{2}$$

Therefore A and B are independent in $\mathrm{Pic}(S)$. The class of B is clearly primitive, so if the Picard group has rank 2, it is generated by A and B. Computing the discriminant and the coset of the Picard lattice, we therefore get that $D(2, 3) \subset P_{9,1}$.

Conversely, any K3 surface S with Picard group generated by divisors A and B having the intersection matrix (2) has a natural embedding in $\mathbb{P}^1 \times \mathbb{P}^2$ as a divisor of bidegree $(2, 3)$. Indeed, the linear system $|A + B|$ defines an embedding of S into the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$; see [Sai74, Proposition 7.15 and Example 7.19]. So $P_{9,1} \subset D(2, 3)$, and therefore $P_{9,1} = D(2, 3)$.

A smooth surface S with a double cover $S \to \mathbb{P}^2$ of the plane ramified along a smooth sextic curve C is a polarized K3 surface $(S, A) \in M_2$ where A is the pullback of a line. Let $R \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a surface of bidegree $(2, 6)$, and consider the double cover of $X \to \mathbb{P}^1 \times \mathbb{P}^2$ ramified along R. Composing the double cover with the projection to the first factor, we obtain on the one hand a morphism $\Pi : X \to \mathbb{P}^1$ whose general fibers are K3 surfaces S. The pullback to X of a line in $\mathbb{P}^2$ restricts to a divisor A on S with $A^2 = 2$, so the family Π yields a morphism $\iota_\Pi : \mathbb{P}^1 \to M_2$. On the other hand, the projection $\pi : R \to \mathbb{P}^1$ of the ramification locus to the first factor yields a morphism $\sigma_\pi : \mathbb{P}^1 \to \mathbb{P}^{27}$ to the space of ternary sextics. The equality $P_{9,1} = D(2, 3)$ may now be interpreted, for a general form R, as

$$\iota_\Pi^{-1}(P_{9,1}) = \sigma_\pi^{-1}(\Sigma_3), \tag{3}$$

where Σ_3 is the Zariski closure of $\partial\Sigma_{3,6} \backslash \partial P_{3,6}$. The right-hand side of (3) is the intersection of the conic $\sigma_\pi(\mathbb{P}^1) \subset \mathbb{P}^{27}$ and the hypersurface Σ_3, so

$$\deg \Sigma_3 = \tfrac{1}{2} \deg \iota_\Pi^*(P_{9,1}).$$

We shall derive this number from the Noether–Lefschetz number $NL_{9,1}^\Pi$, the degree of the divisor $\iota_\Pi^*(D_{1,3})$, computed in [MP07, § 6]. The divisor $D_{1,3}$ is a weighted sum

$$D_{1,3} = \sum_{\Delta | \Delta_2(1,3)} \mu(1, 3, \Delta, 1)[P_{\Delta,1}],$$

so we evaluate each summand of $\iota_\Pi^*(D_{1,3})$. If the Picard group of $S \subset X$ has rank 2 and contains divisors A and B as above, then these divisors generate $\mathrm{Pic}(S)$ and the discriminant equals $\Delta = \Delta_2(1, 3) = 9$. In particular, if $\Delta \neq 9$, then the pullback $\iota_\Pi^*(P_{\Delta,1})$ is trivial. Furthermore, when $\Delta = 9$, there are exactly two divisor classes on S, namely B and $3A - B$, that have self-intersection $B^2 = (3A - B)^2 = 0$ and intersection number $A \cdot B = A \cdot (3A - B) = 3$. Therefore the coefficient $\mu(1, 3, 9, 1)$ equals 2 and

$$\deg \Sigma_3 = \tfrac{1}{2} \deg \iota_\Pi^*(P_{9,1}) = \tfrac{1}{4} \deg \iota_\Pi^*(D_{1,3}) = \tfrac{1}{4} NL_{1,3}^\Pi.$$

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In [MP07, Corollary 3], the Noether–Lefschetz number $NL^{\Pi}_{h,d}$ is expressed as the coefficient of a monomial in the expansion of a modular form Θ^{Π}_l of weight 21/2 as a power series in $q^{1/(2l)}$, where $l = A^2$ is the degree of the polarization. The exponent of the relevant monomial is $\eta = \Delta_l(h,d)/(2l)$, where $\Delta_l(h,d)$ is the discriminant of the intersection matrix $\left(\begin{smallmatrix} l & A\cdot B \\ A\cdot B & B^2 \end{smallmatrix}\right)$.

For the family Π, the modular form Θ^{Π}_2 has the expansion

$$\Theta^{\Pi}_2 = -1 + 150q + 1248q^{5/4} + 108600q^2 + 332800q^{9/4} + 5113200q^3 + \cdots .$$

In our case, we have $\eta = 9/4$, since $l = 2$ and the intersection matrix (2) has discriminant 9. We conclude that the number

$$\tfrac{1}{4} NL^{\Pi}_{1,3} = \tfrac{1}{4} \cdot 332800 = 83200$$

equals the degree of the hypersurface of sextics that are sums of three squares.

We now come to the case of quartic surfaces in $\mathbb{P}^3$. It was shown in [Ble12] that $\partial\Sigma_{4,4} \backslash \partial P_{4,4}$ consists of quartic forms F that are sums of four squares over $\mathbb{R}$. Over the complex numbers $\mathbb{C}$, such a quartic F is a rank-4 quadric in quadrics:

$$F = fg - hk = \det\begin{pmatrix} f & h \\ k & g \end{pmatrix} \quad \text{for some } f, g, h, k \in \mathbb{C}[x_1, x_2, x_3, x_4]_2. \tag{4}$$

The K3 surface S defined by F contains two distinct pencils of elliptic curves on S, one defined by the rows and one by the columns of the 2×2 matrix. Up to the action of $\mathrm{SL}(4,\mathbb{C})$, the determinantal quartics (4) form an 18-dimensional family and hence determine a divisor $D(2,2)$ in the moduli space M_4. The Picard group of a general surface S in this family therefore has rank 2, and the class A of a plane section and the class B of an elliptic curve in one of the two elliptic pencils are independent. The classes of A and B are clearly primitive, so if the Picard group has rank 2, it is generated by A and B. The discriminant Δ and the coset δ of the Picard lattice can therefore be computed from the intersection matrix

$$\begin{pmatrix} A^2 & A\cdot B \\ A\cdot B & B^2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix}. \tag{5}$$

So $\Delta = \Delta_4(1,4) = 16$ and $\delta = 0$, which implies that $D(2,2) \subset P_{16,0}$.

Conversely, let $(S, A) \in P_{16,0}$ be a quasi-polarized surface with Picard group generated by A and B and intersection matrix (5). Then the linear system $|A|$ defines a embedding of S as a smooth quartic surface in $\mathbb{P}^3$ (see [Sai74]). The general curve in the linear system $|B|$ is embedded as an elliptic quartic curve on the quartic surface S. But any elliptic quartic curve in $\mathbb{P}^3$ is a complete intersection of two quadric surfaces, say $\{f = 0\}$ and $\{h = 0\}$. A quartic polynomial that defines S must therefore have the form $fg - hk$ for suitable quadratic polynomials g and k. We conclude that $P_{16,0} \subset D(2,2)$, and hence $D(2,2) = P_{16,0}$.

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a general 3-fold of bidegree $(1,4)$. The projection $\Pi : X \to \mathbb{P}^1$ defines a family of K3 surfaces, and the pullback of a plane from the second projection restricts to a polarization A on the fibers S of Π of degree $A^2 = 4$. So, on the one hand, the family Π yields a morphism $\iota_\Pi : \mathbb{P}^1 \to M_4$. On the other hand, the projection Π to the first factor also yields a morphism $\sigma_\Pi : \mathbb{P}^1 \to \mathbb{P}^{34}$ to the space of quartic forms in four variables.

The equality $P_{16,0} = D(2,2)$ may now be interpreted, for a general 3-fold X, as

$$\iota_\Pi^{-1}(P_{16,0}) = \sigma_\Pi^{-1}(\Sigma_4), \tag{6}$$

where Σ_4 is the Zariski closure of $\partial\Sigma_{4,4}\backslash\partial P_{4,4}$. The right-hand side of (6) is the intersection of the line $\sigma_\Pi(\mathbb{P}^1)\subset\mathbb{P}^{34}$ and the hypersurface Σ_4, so

$$\deg\,\Sigma_4=\deg\,\iota_\Pi^*(P_{16,0}).$$

We shall derive this number from the Noether–Lefschetz number $NL^\Pi_{16,0}$ computed in [MP07, § 6]. Recall that $NL^\Pi_{16,0}$ is the degree of the divisor $\iota_\Pi^*(D_{1,4})$ and that $D_{1,4}$ is a weighted sum

$$D_{1,4}=\sum_{\Delta|\Delta_4(1,4)}\mu(1,4,\Delta,0)[P_{\Delta,1}].$$

We evaluate each summand of $\iota_\Pi^*(D_{1,4})$. If the Picard group of $S\subset X$ has rank 2 and contains divisors A and B as above, then these divisors generate the Picard group, and so $\Delta=\Delta_4(1,4)=16$. In particular, if $\Delta\neq 16$, then the pullback $\iota_\Pi^*(P_{\Delta,0})$ is trivial. Furthermore, when $\Delta=16$, there are exactly two divisor classes on S, namely B and $2A-B$, that have self-intersection $B^2=(2A-B)^2=0$ and intersection number $A\cdot B=A\cdot(2A-B)=4$. Therefore the coefficient $\mu(1,4,16,0)$ equals 2 and

$$\deg\,\Sigma_4=\deg\,\iota_\Pi^*(P_{16,0})=\tfrac{1}{2}\deg\,\iota_\Pi^*(D_{1,4})=\tfrac{1}{2}NL^\Pi_{1,4}.$$

The number $NL^\Pi_{1,4}$ is the coefficient of a monomial in the expansion of a modular form Θ^Π_l of weight 21/2 as a power series in $q^{1/(2l)}$, where $l=A^2$ is the degree of the polarization. Here, the exponent of the relevant monomial is

$$\eta=\frac{\Delta_4(1,4)}{8}=\frac{16}{8}=2.$$

For the family Π, the modular form Θ^Π_4 has the expansion

$$\Theta^\Pi_4=-1+108q+320q^{9/8}+5016q^{3/2}+76950q^2+136512q^{17/8}+\cdots.$$

This was shown in [MP07, Theorem 2]. We conclude that the degree of the hypersurface of homogeneous quartics in four unknowns that are sums of four squares is

$$\tfrac{1}{2}NL^\Pi_{1,4}=\tfrac{1}{2}\cdot 76950=38475.$$

This completes the proof of Theorem 1. □

Remark 4. It was pointed out to us by Giorgio Ottaviani that the smooth ternary sextics that are rank-3 quadrics in cubic forms are known to coincide with the smooth sextics that have an effective even theta characteristic (cf. [Ott07, Proposition 8.4]).

3. Rank conditions on Hankel matrices

We now consider the convex cone $(\Sigma_{n,2d})^\vee$ dual to the cone $\Sigma_{n,2d}$. Its elements are the linear forms ℓ on $\mathbb{R}[x_1,\ldots,x_n]_{2d}$ that are non-negative on squares. Each such linear form ℓ is represented by its associated quadratic form on $\mathbb{R}[x_1,\ldots,x_n]_d$, which is defined by $f\mapsto\ell(f^2)$. The symmetric matrix which expresses this quadratic form with respect to the monomial basis of $\mathbb{R}[x_1,\ldots,x_n]_d$ is denoted by H_ℓ, and it is called the *Hankel matrix* of ℓ. It has format $\binom{n+d-1}{d}\times\binom{n+d-1}{d}$, and its rows and columns are indexed by elements of $\{(i_1,i_2,\ldots,i_n)\in\mathbb{Z}^n_{\geqslant 0}:i_1+i_2+\cdots+i_n=d\}$. We shall examine the two cases of interest.

The Hankel matrix for ternary sextics ($n = d = 3$) is the 10×10 matrix

$$H_\ell = \begin{bmatrix} a_{006} & a_{015} & a_{024} & a_{033} & a_{105} & a_{114} & a_{123} & a_{204} & a_{213} & a_{303} \\ a_{015} & a_{024} & a_{033} & a_{042} & a_{114} & a_{123} & a_{132} & a_{213} & a_{222} & a_{312} \\ a_{024} & a_{033} & a_{042} & a_{051} & a_{123} & a_{132} & a_{141} & a_{222} & a_{231} & a_{321} \\ a_{033} & a_{042} & a_{051} & a_{060} & a_{132} & a_{141} & a_{150} & a_{231} & a_{240} & a_{330} \\ a_{105} & a_{114} & a_{123} & a_{132} & a_{204} & a_{213} & a_{222} & a_{303} & a_{312} & a_{402} \\ a_{114} & a_{123} & a_{132} & a_{141} & a_{213} & a_{222} & a_{231} & a_{312} & a_{321} & a_{411} \\ a_{123} & a_{132} & a_{141} & a_{150} & a_{222} & a_{231} & a_{240} & a_{321} & a_{330} & a_{420} \\ a_{204} & a_{213} & a_{222} & a_{231} & a_{303} & a_{312} & a_{321} & a_{402} & a_{411} & a_{501} \\ a_{213} & a_{222} & a_{231} & a_{240} & a_{312} & a_{321} & a_{330} & a_{411} & a_{420} & a_{510} \\ a_{303} & a_{312} & a_{321} & a_{330} & a_{402} & a_{411} & a_{420} & a_{501} & a_{510} & a_{600} \end{bmatrix}. \tag{7}$$

The Hankel matrix for quaternary quartics ($n = 4, d = 2$) also has size 10×10, and is given by

$$H_\ell = \begin{bmatrix} a_{0004} & a_{0013} & a_{0022} & a_{0103} & a_{0112} & a_{0202} & a_{1003} & a_{1012} & a_{1102} & a_{2002} \\ a_{0013} & a_{0022} & a_{0031} & a_{0112} & a_{0121} & a_{0211} & a_{1012} & a_{1021} & a_{1111} & a_{2011} \\ a_{0022} & a_{0031} & a_{0040} & a_{0121} & a_{0130} & a_{0220} & a_{1021} & a_{1030} & a_{1120} & a_{2020} \\ a_{0103} & a_{0112} & a_{0121} & a_{0202} & a_{0211} & a_{0301} & a_{1102} & a_{1111} & a_{1201} & a_{2101} \\ a_{0112} & a_{0121} & a_{0130} & a_{0211} & a_{0220} & a_{0310} & a_{1111} & a_{1120} & a_{1210} & a_{2110} \\ a_{0202} & a_{0211} & a_{0220} & a_{0301} & a_{0310} & a_{0400} & a_{1201} & a_{1210} & a_{1300} & a_{2200} \\ a_{1003} & a_{1012} & a_{1021} & a_{1102} & a_{1111} & a_{1201} & a_{2002} & a_{2011} & a_{2101} & a_{3001} \\ a_{1012} & a_{1021} & a_{1030} & a_{1111} & a_{1120} & a_{1210} & a_{2011} & a_{2020} & a_{2110} & a_{3010} \\ a_{1102} & a_{1111} & a_{1120} & a_{1201} & a_{1210} & a_{1300} & a_{2101} & a_{2110} & a_{2200} & a_{3100} \\ a_{2002} & a_{2011} & a_{2020} & a_{2101} & a_{2110} & a_{2200} & a_{3001} & a_{3010} & a_{3100} & a_{4000} \end{bmatrix}. \tag{8}$$

We note that what we call a Hankel matrix is known as a *moment matrix* in the literature on optimization and functional analysis [Las10], and it is called a *(symmetric) catalecticant* in the literature on commutative algebra and algebraic geometry [IK99].

The dual cone $(\Sigma_{3,6})^\vee$ is the spectrahedron consisting of all positive semidefinite Hankel matrices (7). The dual cone $(\Sigma_{4,4})^\vee$ is the spectrahedron consisting of all positive semidefinite matrices (8). This convex duality offers a way of representing our Noether–Lefschetz loci via their projective dual varieties.

Theorem 5. *The Hankel matrices (7) of rank* 7 *or less constitute a rational projective variety of dimension* 21 *and degree* 2640. *Its dual is a hypersurface, the Zariski closure of sums of three squares of cubics. Likewise, the Hankel matrices (8) of rank* 6 *or less constitute a rational projective variety of dimension* 24 *and degree* 28 314. *Its dual is a hypersurface, the Zariski closure of sums of four squares of quadrics.*

Proof. The fact that these varieties are rational and irreducible of the asserted dimensions can be seen as follows. Consider the Grassmannian $\mathrm{Gr}(3, 10)$ which parametrizes 3-dimensional linear subspaces F of the 10-dimensional space $\mathbb{R}[x_1, x_2, x_3]_3$ of ternary cubics. This Grassmannian is rational and its dimension equals 21. The *global residue* in $\mathbb{P}^2$, as defined in [CD05, § 1.6], specifies a rational map $F \mapsto \mathrm{Res}_{\langle F \rangle}$ from $\mathrm{Gr}(3, 10)$ into $\mathbb{P}((\mathbb{R}[x_1, x_2, x_3]_6)^*) \simeq \mathbb{P}^{27}$. The base

https://doi.org/10.1112/S0010437X12000437 Published online by Cambridge University Press

locus of this map is the resultant of three ternary cubics, so $\mathrm{Res}_{\langle F\rangle}$ is well-defined whenever the ideal $\langle F\rangle$ is a complete intersection in $\mathbb{R}[x_1, x_2, x_3]$. The value $\mathrm{Res}_{\langle F\rangle}(P)$ of this linear form on a ternary sextic P is the image of P modulo the ideal $\langle F\rangle$, and it can be computed via any Gröbner basis normal form. Our map $F \mapsto \ell$ is birational because it has an explicit inverse: $F = \mathrm{kernel}(H_\ell)$. The inverse simply maps the rank-7 Hankel matrix representing ℓ to its kernel.

The situation is entirely analogous for the $n = 4, d = 2$ case. Here we consider the 24-dimensional Grassmannian $\mathrm{Gr}(4, 10)$ which parametrizes 4-dimensional linear subspaces F of $\mathbb{R}[x_1, x_2, x_3, x_4]_2$. The global residue in $\mathbb{P}^3$ specifies a rational map

$$\mathrm{Gr}(4, 10) \dashrightarrow \mathbb{P}((\mathbb{R}[x_1, x_2, x_3, x_4]_4)^*) \simeq \mathbb{P}^{34}, \quad F \mapsto \mathrm{Res}_{\langle F\rangle}.$$

This map is birational onto its image, the variety of rank-6 Hankel matrices (8), and the inverse of that map takes a rank-6 Hankel matrix (8) to its kernel.

To determine the degrees of our two Hankel determinantal varieties, we argue as follows. The variety S_r of all symmetric 10×10 matrices of rank r or less is known to be irreducible and arithmetically Cohen–Macaulay; it has codimension $\binom{11-r}{2}$, and its degree is given by the following formula due to Harris and Tu [HT84]:

$$\mathrm{degree}(S_r) = \prod_{j=0}^{9-r}\left(\binom{10+j}{10-r-j}\Big/\binom{2j+1}{j}\right). \tag{9}$$

Thus S_r has codimension 6 and degree 2640 for $r = 7$, and it has codimension 10 and degree 28 314 for $r = 6$. The projective linear subspace of Hankel matrices (7) has dimension 27. Its intersection with S_7 was seen to have dimension 21. Hence the intersection has the expected codimension 6 and is proper. That the intersection is proper ensures that the degree remains 2640. Likewise, the projective linear subspace of Hankel matrices (8) has dimension 34, and its intersection with S_6 has dimension 24. The intersection has the expected codimension 10, and we conclude as before that the degree equals 28 314.

It remains to be shown that the two Hankel determinantal varieties are projectively dual to the Noether–Lefschetz hypersurfaces in Theorem 1. This follows from [Ble12, Theorem 1.6] for sextic curves in $\mathbb{P}^2$ and from [Ble12, Theorem 1.7] for quartic surfaces in $\mathbb{P}^3$. These results characterize the relevant extreme rays of $\Sigma_{3,6}^*$ and $\Sigma_{4,4}^*$, respectively. These rays are dual to the hyperplanes that support $\partial\Sigma_{3,6}$ and $\partial\Sigma_{4,4}$ at smooth points representing strictly positive polynomials. By passing to the Zariski closures, we conclude that the Zariski closures of $\partial\Sigma_{3,6}\backslash\partial P_{3,6}$ and $\partial\Sigma_{4,4}\backslash\partial P_{4,4}$ are dual to the Hankel determinantal varieties above. For a general introduction to the relationship between projective duality and cone duality in convex algebraic geometry, we refer to [RS10]. $\square$

Remark 6. In the space $\mathbb{P}(\mathrm{Sym}^2 V)$ of quadratic forms on a 10-dimensional vector space V^*, the subvariety S_r of forms of rank r or less is the dual variety to $S_{10-r}^* \subset \mathbb{P}(\mathrm{Sym}^2 V^*)$. Identifying V with ternary cubics, the space of 10×10 Hankel matrices (7) form a 27-dimensional linear subspace $H \subset \mathbb{P}(\mathrm{Sym}^2 V^*)$. For $r \leqslant 3$, we have $\dim(S_r) < 27$, and the variety dual to $H_{10-r} = S_{10-r}^* \cap H$ equals the image Σ_r of the birational projection of S_r into H^*. That image is the variety of sextics that are quadrics of rank r or less in cubics. When $r \leqslant 2$, the projection from S_r^* to Σ_r is a morphism, so the degrees of these two varieties coincide. When $r = 3$, it is not a morphism and the degree drops to 83 200. A similar analysis works for $V = \mathbb{R}[x_1, x_2, x_3, x_4]_2$ with $r \leqslant 4$.

https://doi.org/10.1112/S0010437X12000437 Published online by Cambridge University Press

4. Extreme non-negative forms

For each of Hilbert's two critical cases, the hypersurface separating $\Sigma_{n,2d}$ and $P_{n,2d}\backslash\Sigma_{n,2d}$ was examined in § 2. We now take an alternative look at this separation; specifically, we focus on the extreme rays of the cone $P_{n,2d}$ of non-negative forms that do not lie in the SOS subcone $\Sigma_{n,2d}$. We begin with the following result on zeros of non-negative forms in the two Hilbert cases.

PROPOSITION 7. *Let F be a non-negative form in $P_{3,6}$ or $P_{4,4}$. If F has more than 10 real zeros, then F has infinitely many zeros and it is a sum of squares.*

Proof. The statement for $P_{3,6}$ was proved by Choi *et al.* in [CLR80]. They also showed the statement for the cone $P_{4,4}$ but with '11 zeros' instead of '10 zeros'. To reduce the number from 11 to 10, we use Kharlamov's theorem in [Kha72], which states that the number of connected components of any quartic surface in real projective 3-space is no greater than 10. See also Rohn's classical work on this subject in [Roh13]. □

Recall that a face of a closed convex set K in a finite-dimensional real vector space is *exposed* if it is the intersection of K with a supporting hyperplane. The extreme rays of K lie in the closure (and hence in the Zariski closure) of the set of exposed extreme rays [Sch93]. A polynomial $F \in P_{n,2d}\backslash\Sigma_{n,2d}$ that generates an exposed extreme ray of $P_{n,2d}$ is an *extreme non-negative form*.

Our first goal is to prove Theorem 2, which characterizes the Zariski closure of the semi-algebraic set of all extreme non-negative forms for $n = d = 3$.

Proof of Theorem 2. Suppose that $F \in P_{3,6}\backslash\Sigma_{3,6}$ is an extreme form. By [Rez07, Lemma 7.1], the polynomial F is irreducible. Moreover, we claim that $|\mathcal{V}_{\mathbb{R}}(F)| \geqslant 10$. It is not hard to show that F is an extreme non-negative form if and only if $\mathcal{V}_{\mathbb{R}}(F)$ is maximal among all forms in $P_{n,2d}$. This is due to the fact that the face dual to F in the dual cone of $P_{n,2d}$ is the conical hull of the linear functionals that are point evaluations on the real zeros of F. Since F generates an exposed extreme ray, it must be uniquely specified by its dual face, and thus it is the unique non-negative form vanishing on $\mathcal{V}_{\mathbb{R}}(F)$. In other words, if F is an extreme non-negative form and $\mathcal{V}_{\mathbb{R}}(F) \subseteq \mathcal{V}_{\mathbb{R}}(G)$ for some $G \in P_{n,2d}$, then $G = \lambda F$ for some $\lambda \in \mathbb{R}$. Now suppose that $|\mathcal{V}_{\mathbb{R}}(F)| \leqslant 9$. Then there is a ternary cubic g that vanishes on $\mathcal{V}_{\mathbb{R}}(F)$. We have $g^2 \in P_{3,6}$ and $\mathcal{V}_{\mathbb{R}}(F) \subseteq \mathcal{V}_{\mathbb{R}}(g)$. This contradicts maximality of $\mathcal{V}_{\mathbb{R}}(F)$. By Proposition 7 we conclude that $|\mathcal{V}_{\mathbb{R}}(F)| = 10$.

Let C be the sextic curve in the complex projective plane $\mathbb{P}^2$ defined by $F = 0$. Since C is irreducible, it must have non-negative genus. Each point in $\mathcal{V}_{\mathbb{R}}(F)$ is a singular point of the complex curve C. As this gives C ten singularities, it follows, by the genus formula, that C can have no more singularities and that all of the real zeros of F are ordinary singularities. The curve C has genus zero and is therefore an irreducible rational curve.

Let $\mathcal{S}_{6,0}$ denote the Severi variety of rational sextic curves in $\mathbb{P}^2$. We have shown that $\mathcal{S}_{6,0}$ contains the semi-algebraic set of extreme forms in $P_{3,6}\backslash\Sigma_{3,6}$. Each rational sextic curve C in $\mathbb{P}^2$ is the image of a morphism $\mathbb{P}^1 \to \mathbb{P}^2$ defined by three binary forms of degree 6. Therefore, $\mathcal{S}_{6,0}$ is irreducible. To choose the three forms, we have $3 \cdot 7 = 21$ degrees of freedom. However, the image in $\mathbb{P}^2$ is invariant under the natural action of the 4-dimensional group $\mathrm{GL}(2, \mathbb{C})$ on the parametrization, and hence $\mathcal{S}_{6,0}$ has dimension $21 - 4 = 17$. The general member C has exactly 10 nodes. Moreover, that set of 10 nodes in $\mathbb{P}^2$ uniquely identifies the rational curve C.

The degree of $\mathcal{S}_{6,0}$ is the number of rational sextics passing through 17 given points in $\mathbb{P}^2$. This is one of the Gromov–Witten numbers of $\mathbb{P}^2$. For rational curves, these numbers were computed

https://doi.org/10.1112/S0010437X12000437 Published online by Cambridge University Press

by Kontsevich and Manin [KM94] using an explicit recursion formula equivalent to the WDVV equations. From their recursion, one gets degree$(\mathcal{S}_{6,0}) = 26\,312\,976$.

To complete the proof, it remains to show that the semi-algebraic set of extreme forms in $P_{3,6}\backslash\Sigma_{3,6}$ is Zariski dense in the Severi variety $\mathcal{S}_{6,0}$. We deduce this from [Rez07, Theorem 4.1 and § 5]. There, starting with a specific set Γ of 8 points in $\mathbb{P}^2$, an explicit one-parameter family of extreme non-negative sextics with 10 zeros, 8 of which come from Γ, was constructed using *Hilbert's method*. Furthermore, by Theorem 4.1, Hilbert's method can be applied to any 8-point configuration in the neighborhood of Γ. By a continuity argument, all 8-point configurations sufficiently close to Γ will also have a one-parameter family of extreme non-negative forms with 10 zeros. This identifies a semi-algebraic set of extreme non-negative forms having dimension $16 + 1 = 17$. We conclude that this set is Zariski dense in $\mathcal{S}_{6,0}$. □

Remark 8. Our analysis implies the following result concerning $\partial P_{3,6}\backslash\Sigma_{3,6}$. All exposed extreme rays are generated by sextics with ten *acnodes* (real nodes whose tangents are non-real complex conjugates), and all extreme rays are generated by limits of sextics with ten acnodes. This proves the second part of [Rez07, Conjecture 7.9]. Indeed, in the second paragraph of the above proof we saw that C has ten ordinary singularities. These cannot be cusps since the form F is non-negative; hence they have to be acnodes, or *round zeros*, as these are called in [Rez07].

Our next goal is to derive Theorem 3, the analogue of Theorem 2 for quartic surfaces in $\mathbb{P}^3$. The role of the Severi variety $\mathcal{S}_{6,0}$ is now played by the variety $\mathcal{QS}$ of quartic symmetroids, i.e. the surfaces whose defining polynomial equals

$$F(x_1, x_2, x_3, x_4) = \det(A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4) \tag{10}$$

where A_1, A_2, A_3 and A_4 are complex symmetric 4×4 matrices.

Lemma 9. *The variety $\mathcal{QS}$ is irreducible and has codimension 10 in $\mathbb{P}^{34}$.*

Proof. Each of the four symmetric matrices A_i has 10 free parameters. The formula (10) expresses the 35 coefficients of F as quartic polynomials in the 40 parameters, and hence defines a rational map $\mathbb{P}^{39} \dashrightarrow \mathbb{P}^{34}$. Our variety $\mathcal{QS}$ is the Zariski closure of the image of this map, so it is irreducible. To compute its dimension, we form the 35×40 Jacobian matrix of the parametrization. By evaluating at a generic point $(A_1, \ldots, A_4)$, we find that the Jacobian matrix has rank 25. Hence the dimension of the symmetroid variety $\mathcal{QS} \subset \mathbb{P}^{34}$ is 24. For a theoretical argument see [Jes16, p. 168, ch. IX.101]. □

A general complex symmetroid S has 10 nodes, but not every quartic with 10 or more nodes in $\mathbb{P}^3$ is a symmetroid. To identify symmetroids, we employ the following lemma from Jessop's classical treatise [Jes16] on singular quartic surfaces. Let S be a 10-nodal quartic with a node at $p = (0:0:0:1)$. Its defining polynomial equals $F = fx_4^2 + 2gx_4 + h$ where $f, g, h \in \mathbb{C}[x_1, x_2, x_3]$ are homogeneous of degrees 2, 3 and 4, respectively. The projection of S from p is a double cover of the plane with coordinates x_1, x_2, x_3 ramified along the sextic curve C_p defined by $g^2 - fh$. The curve C_p has nodes exactly at the image of the nodes on S that are distinct from p. Since no three nodes on S are collinear, the curve C_p has a node for each node on S distinct from p, i.e. at least 9 nodes. The following result appears in [Jes16, ch. I.8].

Lemma 10. *Let S be a quartic surface with 10 or more nodes, and let p be one of these nodes. If the sextic ramification curve C_p is the union of two cubic curves, then the quartic surface S is a symmetroid.*

Proof of Theorem 3. Let $\mathcal{E}$ denote the semi-algebraic set of all non-negative extreme forms F in $P_{4,4}\backslash\Sigma_{4,4}$. Each $F\in\mathcal{E}$ satisfies $|\mathcal{V}_{\mathbb{R}}(F)|=10$, by Proposition 7 and the same argument as in the first paragraph of the proof of Theorem 2.

We shall prove that $\mathcal{E}$ is a subset of $\mathcal{QS}$. Let $F\in\mathcal{E}$ and let $S=\mathcal{V}_{\mathbb{C}}(F)$ be the corresponding complex surface. Then S is a real quartic with 10 real nodes and possibly some pairs of conjugate complex nodes. The 10 real nodes are isolated, so the Hessian at each of them is definite as a real quadratic form.

Our goal is to show that S is a symmetroid over $\mathbb{C}$. If $p\in\mathbb{P}^3_{\mathbb{R}}$ is one of the real nodes of F, then the ramification curve C_p is a real sextic curve with 9 real nodes at the image of the nodes distinct from p. Since the real nodes on S are the only real points, these 9 nodes are the only real points on C_p. Furthermore, the Hessian of each of these nodes is again definite, and the tangent lines of C_p are non-real complex conjugates.

Through any 9 of the real nodes of S there is a real quadratic surface. This quadric is unique; otherwise there would be a real quadric through all 10 real nodes and F would not be extreme. Let q be a real node on S distinct from p, and let A be a real quadratic form vanishing on all nodes on S except q. Consider the pencil of quartic forms

$$F_t=F+tA^2 \quad \text{for } t\in\mathbb{R}.$$

Suppose $p=(0:0:0:1)$. The polynomial A has the form ux_4+v, where $u,v\in\mathbb{R}[x_1,x_2,x_3]$ have degrees 1 and 2. The equation of F_t is then given by

$$F_t=(f+tu^2)x_4^2+2(g+tuv)x_4+h+tv^2.$$

Any surface $S_t=\{F_t=0\}$ has at least 9 real singular points, namely the nodes of S other than q. Since F is non-negative, F_t is non-negative for $t>0$ with zeros precisely at the 9 real nodes. On the other hand, F has an additional zero at q. Since A^2 is positive at q, the real surface $\{F_t=0\}$ must have a 2-dimensional component when $t<0$. Projecting from p, we get a pencil of ramification loci $C_p(t)$. In the above notation, this family of sextic curves is defined by the forms

$$G_t=fh-g^2+t(hu^2-2guv+fv^2)\in\mathbb{R}[x_1,x_2,x_3]_6.$$

The curves in this pencil have common nodes at eight real points $p_1,\dots,p_8$ in the plane $\mathbb{P}^2$, namely the images of the real nodes on S other than p and q.

Consider the vector space V of real sextic forms that are singular at $p_1,\dots,p_8$. Since each p_i imposes three linear conditions, we have $\dim V\geqslant 28-3\cdot 8=4$. We claim that $\dim V=4$. To see this, consider a general curve $C_p(t)$ with $t>0$. It has only eight real points, so as a complex curve it is irreducible and smooth outside the eight nodes. Hence the geometric genus of $C_p(t)$ is 2. Let X denote the blow-up of the plane in the points $p_1,\dots,p_8$, and denote by C the strict transform of $C_p(t)$ on X. By Riemann–Roch, $\dim H^0(\mathcal{O}_X(C)|_C)=3$ since $C^2=4$. Combined with the cohomology of the exact sequence

$$0\to\mathcal{O}_X\to\mathcal{O}_X(C)\to\mathcal{O}_X(C)|_C\to 0,$$

we conclude that $\dim V=\dim H^0(X,\mathcal{O}_X(C))\leqslant 4$, and hence $\dim V=4$.

The pencil $\mathbb{R}\{k_1,k_2\}$ of real cubic forms through the eight points $p_1,\dots,p_8$ determines a 3-dimensional subspace $U=\mathbb{R}\{k_1^2,k_1k_2,k_2^2\}$ of V, while the sextic forms G_t span a 2-dimensional subspace L of V. Since G_t has no real zeros except the nodes, when $t>0$ we see that L is not contained in U. Hence L and U intersect in a 1-dimensional subspace of V, so there is a unique value $t_0\in\mathbb{R}$ such that $G_{t_0}=K_1\cdot K_2$ where $K_1,K_2\in\mathbb{C}[x_1,x_2,x_3]_3$.

We now have two possibilities: either K_1 and K_2 are both real, or K_1 and K_2 are complex conjugates. We claim that the latter is the case.

Consider the intersection

$$\{G_t = 0\} \cap \{K_1 \cdot K_2 = 0\}.$$

This scheme is the union of a scheme Z_{nodes} of length 32 supported on the 8 nodes and a scheme Z_{res} of length $6 \cdot 6 - 4 \cdot 8 = 4$. The defining ideal of the intersection is $\langle fh - g^2, hu^2 - 2guv + fv^2 \rangle$ and contains, in particular, the square $(gu - fv)^2$. It follows that Z_{res} is supported in two non-real conjugate points contained in $\{f = g = hu^2 = 0\}$, if this subscheme is non-empty, or Z_{res} is residual to $\{f = g = 0\}$ and Z_{nodes} in $\{fh - g^2 = 0\} \cap \{(gu - 2fv)^2 = 0\}$. In both cases, each component of Z_{res} has even length. The intersection $\{K_i = 0\} \cap \{G_t = 0\}$ has length 18, so the subscheme $Z_i = \{K_i = 0\} \cap Z_{\text{res}}$ has length 2 for each $i = 1, 2$. Since the general curve $C_p(t) = \{G_t = 0\}$ does not contain the ninth intersection point of $\{K_1 = 0\}$ and $\{K_2 = 0\}$, Z_1 and Z_2 are distinct non-reduced subschemes of length 2. Furthermore, since the 8 nodes are the only real points of $C_p(t)$ for $t > 0$, the subschemes Z_1 and Z_2 are non-real complex conjugates.

If both K_i were real, then $Z_i = \{K_i = 0\} \cap Z_{\text{res}}$ would also be real, which is a contradiction. We conclude that the two cubics K_1 and K_2 are complex conjugates and that their only real points are the 9 common intersection points.

We now claim that $t_0 = 0$. Indeed, if $t_0 < 0$, then S_{t_0} has 2-dimensional real components and the real points in the ramification locus $C_p(t_0)$ would have dimension 1. If $t_0 > 0$, then S_{t_0} has only 9 real points and $C_p(t_0)$ has only 8 real points. Since $C_p(t_0) = \{K_1 \cdot K_2 = 0\}$ has 9 real zeros, it follows that $t_0 = 0$. Using Jessop's result, Lemma 10, we conclude that $F = F_0$ is a symmetroid.

We have shown that the semi-algebraic set $\mathcal{E}$ is contained in the symmetroid variety $\mathcal{QS}$. It remains to be proved that $\mathcal{E}$ is Zariski dense in $\mathcal{QS}$. To see this, we start with any particular extreme quartic. For instance, take the following extreme quartic due to Choi *et al.* [CLR80, proof of Proposition 4.13]:

$$F_b = \sum_{i,j} x_i^2 x_j^2 + b \sum_{i,j,k} x_i^2 x_j x_k + (4b^2 - 4b - 2) x_1 x_2 x_3 x_4 \quad \text{for } 1 < b < 2, \tag{11}$$

where the sums are taken over all distinct pairs and triples of indices. The complex surface defined by F_b has 10 nodes, namely the points in $\mathcal{V}_{\mathbb{R}}(F_b)$. Our proof above shows that F_b is a symmetroid. Since the Hessian of F_b is positive definite at each of the 10 real points, we can perturb these freely in a small open neighborhood inside the 24-dimensional variety of 10-tuples of real points that are nodes of a symmetroid. The real dimension here is equal to the complex dimension, since existence of a symmetroid with 10 prescribed nodes can be stated in terms of the number of linearly independent conditions that double vanishing on the 10 points imposes on quartics. Since double vanishing on real points enforces real conditions, the dimension is independent of the field. Each corresponding quartic is real, non-negative and extreme. This adaptation of 'Hilbert's method' constructs a semi-algebraic family of dimension 24 in $\mathcal{E}$. We conclude that $\mathcal{QS}$ is the Zariski closure of $\mathcal{E}$. □

Our proof raises the question of whether Lemma 10 can be turned into an algorithm. To be precise, given an extreme quartic, such as (11), what is a practical method for computing a complex symmetric determinantal representation (10)? We shall address this question in the second half of the next section.

5. Numerical algebraic geometry

We verified the results of Theorems 1 and 5 using the algorithms implemented in `Bertini` [BHSW]. In what follows, we shall explain our methodology and findings. An introduction to numerical algebraic geometry can be found in [SW05].

The main computational method used in `Bertini` is homotopy continuation. Given a polynomial system F with the same number of variables and equations, *basic homotopy continuation* computes a finite set $\mathcal{S}$ of complex roots of F which contains the set of isolated roots. By 'computes $\mathcal{S}$' we mean providing a numerical approximation of each point in $\mathcal{S}$ together with an algorithm for computing each point in $\mathcal{S}$ to arbitrary accuracy. The basic idea is to consider a parametrized family $\mathcal{F}$ of polynomial systems which contains F. One first computes the isolated roots of a sufficiently general member of $\mathcal{F}$, say G, and then tracks the solution paths starting with the isolated roots of G at $t=1$ of the homotopy

$$H(x,t)=F(x)(1-t)+tG(x).$$

The solution paths are tracked numerically using predictor–corrector methods. For enhanced numerical reliability, the adaptive step size and adaptive precision path-tracking method of [BHSW09] is used. The endpoints at $t=0$ of these paths can be computed to arbitrary accuracy by using endgames with the set of finite endpoints being the set $\mathcal{S}$. If F has finitely many roots, then $\mathcal{S}$ is the set of all roots of F. If the variety of F is not zero-dimensional, then the set of isolated roots of F is obtained from $\mathcal{S}$ by using the local dimension test of [BHPS09b].

Our computations to verify the degrees in Theorem 1 used only basic homotopy continuation. For the $\Sigma_{3,6}$ case, we computed the intersection of the set of rank-3 quadrics in cubics with a random line in the space $\mathbb{P}^{27}$ of ternary sextics. In particular, for random $p,q\in\mathbb{C}[x_0,x_1,x_2]_6$, we computed the complex values of s such that there exists $f,g,h\in\mathbb{C}[x_0,x_1,x_2]_3$ with

$$fh-g^2=p+sq.$$

We used the two degrees of freedom in the parametrization of a rank-3 quadric in cubics by taking the coefficients of x_0^3 in g and $x_0^2x_1$ in f to be zero, and we dehomogenized by taking the coefficient of x_0^3 in f to be 1. The resulting system $F=0$ consists of 26 quadratic and two linear equations in 28 variables. Since the solution set of $F=0$ is invariant under the action of negating g, we considered F as a member of the family $\mathcal{F}$ of all polynomial systems in 28 variables consisting of two linear and 26 quadratic polynomials which are invariant under this action. It is easy to verify that a general member of $\mathcal{F}$ has 2^{26} roots, which consist of 2^{25} orbits of order 2 under the action of negating g. We took the system G to be a dense linear product polynomial system [VC93] with random coefficients which respect this action. By tracking one path from each of the 2^{25} orbits, which took about 40 h using 80 processors, this yielded 166 400 points, corresponding to 83 200 distinct values of s.

The $\Sigma_{4,4}$ case of Theorem 1 was solved similarly, and the number 38 475 was verified. We took advantage of the bi-homogeneous structure of the system

$$fg-hk=p+sq.$$

Numerical algebraic geometry can be used to compute all irreducible components of a complex algebraic variety. Here the methods combine the ability to compute isolated solutions with the use of random hyperplane sections. Each irreducible component V of $F=0$ is represented by a witness set, which is a triple $(F,\mathcal{L},W)$ where $\mathcal{L}$ is a system of $\dim V$ random linear polynomials and W is the finite set consisting of the points of intersection of V with $\mathcal{L}=0$.

Briefly, the basic approach to computing a witness set for the irreducible components of $F=0$ of dimension k is to first compute the isolated solutions W of $F=\mathcal{L}_k=0$ where $\mathcal{L}_k$ is a system of k random linear polynomials. The set W is then partitioned into subsets, each of which corresponds to the intersection of $\mathcal{L}_k=0$ with an irreducible component of $F=0$ of dimension k. The cascade [SV00] and regenerative cascade [HSW11] algorithms use a sequence of homotopies to compute the isolated solutions of $F=\mathcal{L}_k=0$ for all relevant values of k.

We applied these techniques to verify the results of Theorem 5 concerning our 10×10 Hankel matrices. Our computations combined the regenerative cascade algorithm with the *numerical rank-deficiency method* of [BHPS09a]. In short, if $A(x)$ is an $n\times N$ matrix with polynomial entries, consider the polynomial system

$$F_r = A(x)\cdot B\cdot \begin{bmatrix} I_{N-r} \\ \Xi \end{bmatrix}$$

where $B\in\mathbb{C}^{N\times N}$ is random, I_{N-r} is the $(N-r)\times(N-r)$ identity matrix, and Ξ is an $r\times(N-r)$ matrix of unknowns. One computes the irreducible components of $F_r=0$ whose general fiber under the projection $(x,\Xi)\mapsto x$ is zero-dimensional. Their images are the components of

$$\mathcal{S}_r(A)=\{x : \operatorname{rank} A(x)\leqslant r\}.$$

The degree of such degeneracy loci is then computed using the method of [HS10].

The results on degree and codimension in Theorem 5 were thus verified, with the workhorse being the regenerative cascade algorithm. For instance, we ran `Bertini` for 12 h on 80 processors to find that the variety of Hankel matrices (7) of rank 7 or less is indeed irreducible of dimension 21 and degree 2640.

We now shift gears and discuss the problem that arose at the end of § 4, namely, how to compute a symmetric determinantal representation (10) for a given extremal quartic $F\in\mathcal{E}\subset\partial P_{4,4}\backslash\Sigma_{4,4}$. As a concrete example let us consider the Choi–Lam–Reznick quartic in (11) with $b=3/2$. We found that $F_{3/2}=\det(M)/\gamma$, where $\gamma=-54874315598400(735\omega+2201)$, with $\omega=2\sqrt{-10}/7$, and M is the symmetric matrix with entries

$$\begin{aligned}
m_{11} &= (-11844\omega+8100)x_1+(3024\omega+13140)x_3,\\
m_{12} &= (7980\omega+14820)x_3,\\
m_{13} &= (19971\omega-17460)x_1+(4494\omega+9600)x_3,\\
m_{14} &= (-1596\omega-26790)x_3+(15561\omega-6840)x_4,\\
m_{22} &= (30324\omega-7220)x_2+(20216\omega+21660)x_3,\\
m_{23} &= (20216\omega+21660)x_2+(6384\omega+27740)x_3,\\
m_{24} &= (-20216\omega-21660)x_2-39710x_3+(7581\omega-21660)x_4,\\
m_{33} &= (-13230\omega+31860)x_1+39710x_2+(-28910\omega+29910)x_3,\\
m_{34} &= -39710x_2+(25004\omega-17100)x_3+((5187/2)\omega-1140)x_4,\\
m_{44} &= 39710x_2+(-20216\omega+37905)x_3+(-30324\omega+27075)x_4.
\end{aligned}\tag{12}$$

A naive approach to obtaining such representations is to extend the numerical techniques for quartic curves in [PSV12, § 2]: after changing coordinates so that x_1^4 appears with coefficient 1 in F, one assumes that A_1 is the identity matrix, A_2 an unknown diagonal matrix, and A_3

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and A_4 arbitrary symmetric 4×4 matrices with unknown entries. The total number of unknowns is $4+10+10=24$, which matches the dimension of the symmetroid variety $\mathcal{QS}$. With this, the identity (10) translates into a system of 34 polynomial equations in 24 unknowns. Solving these equations directly using `Bertini` is currently not possible. Since the system is overdetermined, `Bertini` actually uses a random subsystem which has a total degree of $3^6 4^{15}$. The randomization destroys much of the underlying structure so that solving this system is currently infeasible.

In what follows, we outline a better algorithm based on the geometry of the problem. The input is a 10-nodal quartic surface $S=\{F=0\}$. After changing coordinates so that $p=(0:0:0:1)$ is one of the nodes, the quartic takes the form

$$F = fx_4^2 + 2gx_4 + h \quad \text{where } f, g, h \in \mathbb{R}[x_1, x_2, x_3].$$

The projection from p defines a double cover $\pi : S \to \mathbb{P}^2$, and the ramification locus is the sextic curve whose defining polynomial is $fh-g^2$ and which splits into a product of two complex conjugate cubic forms K_1 and K_2. The intersection of S with $\{K_1=0\}$, regarded as a cubic cone in $\mathbb{P}^3$, is supported on the branch locus of the double cover and therefore equals two times a curve C of degree 6. The curve C has a triple point at the vertex p, its arithmetic genus is 3, and it is arithmetically Cohen–Macaulay. By the Hilbert–Burch theorem, the ideal of C is generated by the 3×3 minors $g_1,\ldots,g_4$ of a 3×4 matrix whose entries are linear forms in $\mathbb{C}[x_1,x_2,x_3,x_4]_1$:

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \end{bmatrix}. \tag{13}$$

The rows of this matrix give three linear syzygies between the four cubics g_i. Furthermore, F itself is in the ideal generated by these cubics, so there is a linear relation $F=l_1g_1+\cdots+l_4g_4$. Hence the quartic F is equal, up to multiplication by a non-zero scalar in $\mathbb{C}$, to the determinant of

$$L = \begin{bmatrix} l_1 & -l_2 & l_3 & -l_4 \\ l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \end{bmatrix}.$$

To find a *symmetric* matrix M with the same property, we solve the linear system $PL=(PL)^T$ for some matrix $P\in \mathrm{GL}(4,\mathbb{C})$ and define $M=PL$.

A numerical version of the above algorithm was implemented and runs almost exactly as explained above, except that a basis for the ideal I_C of the genus-3 curve C is found by computing a large sample of points in the intersection $\{F=K_1=0\}$ and then computing a basis $g_1,\ldots,g_4$ for the 4-dimensional space of cubic forms vanishing on this set. Next, a basis for the 3-dimensional set of linear syzygies between these cubics is computed. This yields the matrix in (13) whose 3×3 minors are the four cubics g_i. For the quartic (11) with $b=3/2$, we used `Bertini` to compute 100 random points in this intersection and then used standard numerical linear algebra algorithms. In total, it took 30 s to compute a symmetric determinantal representation for $F_{3/2}$. To four digits, with $i=\sqrt{-1}$, the output we found is the symmetric

https://doi.org/10.1112/S0010437X12000437 Published online by Cambridge University Press

matrix M with entries

$$\begin{aligned}
m_{11} &= (15.5378 + 5.6547i)x_1 - (20.4008 - 5.8116i)x_2 \\
&\quad - (23.1956 + 16.9236i)x_3 + (12.4987 + 26.8206i)x_4, \\
m_{12} &= (18.3458 - 5.8125i)x_1 - (14.0867 - 25.1505i)x_2 \\
&\quad - (35.0029 - 5.2948i)x_3 + (36.1417 + 15.7167i)x_4, \\
m_{13} &= (11.6232 + 5.6624i)x_1 - (15.6076 - 5.9393i)x_2 \\
&\quad - (17.3057 + 12.3685i)x_3 + (11.0079 + 22.8305i)x_4, \\
m_{14} &= (25.7222 + 1.2098i)x_1 - (27.4233 - 22.3864i)x_2 \\
&\quad - (45.8046 + 14.1068i)x_3 + (35.3836 + 37.8454i)x_4, \\
m_{22} &= (12.6315 - 18.4638i)x_1 + (9.6932 + 37.6953i)x_2 \\
&\quad - (26.0269 - 34.9909i)x_3 + (49.9098 - 16.2993i)x_4, \\
m_{23} &= (14.6285 - 3.0705i)x_1 - (9.5983 - 20.6203i)x_2 \\
&\quad - (25.8489 - 4.2265i)x_3 + (31.1616 + 13.0794i)x_4, \\
m_{24} &= (24.1544 - 17.3589i)x_1 - (5.2755 - 47.6528i)x_2 \\
&\quad - (52.3363 - 27.5281i)x_3 + (68.7313 + 6.4353i)x_4, \\
m_{33} &= (8.5030 + 5.3275i)x_1 - (11.9127 - 5.6822i)x_2 \\
&\quad - (12.9473 + 8.9555i)x_3 + (9.6646 + 19.4288i)x_4, \\
m_{34} &= (19.6130 + 2.9165i)x_1 - (20.0754 - 19.3371i)x_2 \\
&\quad - (34.2042 + 9.9911i)x_3 + (30.7454 + 32.0581i)x_4, \\
m_{44} &= (37.6831 - 10.7034i)x_1 - (27.3051 - 52.2852i)x_2 \\
&\quad - (80.4558 - 2.6947i)x_3 + (79.5452 + 43.7001i)x_4.
\end{aligned} \tag{14}$$

The symbolic solution (12) and the numerical solution (14) are in the same equivalence class of symmetric matrix representations. In fact, we close with the result that the output of the algorithm is essentially unique, independent of the choice of node p and cubic form K_i.

Proposition 11. *For any 10-nodal symmetroid $F \in \mathcal{QS}$, the representation (10) is unique up to the natural action of* $\mathrm{GL}(4, \mathbb{C})$ *via $A_i \mapsto UA_iU^T$ for $i = 1, 2, 3, 4$.*

Proof. Let $M = \sum x_i A_i$ be a symmetric matrix such that $F = \det(M)$. Any three of the four rows of M determine a curve C by taking 3×3 minors. This gives a 4-dimensional linear system L_M of curves of arithmetic genus 3 and degree 6 on S. The doubling of any curve in L_M is the complete intersection of S and a cubic surface defined by a 3×3 symmetric submatrix of M. Conversely, the linear system determines the matrix M up to a change of basis.

Each curve in L_M passes through all the nodes of S, and these are the common zeros of the curves in L_M. If $\tilde{S}$ is the smooth K3 surface obtained by resolving the nodes, then, by Riemann–Roch, L_M defines a *complete* linear system on $\tilde{S}$. Since $\mathrm{Pic}(\tilde{S})$ is torsion-free, we see that L_M is uniquely determined as the linear system of degree-6 curves on S passing through all nodes and whose doubling form a complete intersection. Therefore the equivalence class of the symmetric matrix representation is also unique. □

ACKNOWLEDGEMENTS

This project was started at the Institut Mittag-Leffler, Djursholm, Sweden, whose support and hospitality was enjoyed by all authors. We thank Paul Larsen, Giorgio Ottaviani, Rahul Pandharipande and Ulf Persson for helpful discussions and comments. G.B., J.H. and B.S. were also supported by the US National Science Foundation.

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