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THE DECOMPOSITION OF ARTINIAN MODULES OVER HYPER-(CYCLIC OR FINITE) GROUPS

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If G is a hyperfinite locally soluble group and A an artinian $\mathbf{Z}G$-module then Zaĭcev proved that A has an f-decomposition. For G being a hyper-(cyclic or finite) locally soluble group, Z. Y. Duan has shown that any periodic artinian $\mathbf{Z}G$-module A has an f-decomposition. Here we prove that: if G is a hyper-(cyclic or finite) group, then any artinian $\mathbf{Z}G$-module A has an f-decomposition.

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In 1986, D. I. Zaĭcev [7] proved that if G is a hyperfinite locally soluble group, then any artinian $\mathbf{Z}G$-module A has an f-decomposition. Also in 1991, Z. Y. Duan [2] proved that any periodic artinian module over a hyper-(cyclic or finite) locally soluble group has an f-decomposition. In this paper, we consider artinian modules over hyper-(cyclic or finite) groups and generalize these two results. That is

Theorem. *If G is a hyper-(cyclic or finite) group, then any artinian $\mathbf{Z}G$-module A has an f-decomposition.*

Our proof of the above theorem depends on the results in [1] and is similar with that of Theorem 1 in [2]. First, we prove the following two lemmas which are necessary for the proof of the theorem.

Lemma 1. *Any finitely generated hyper-(cyclic or finite) group is a supersoluble-by-finite group.*

Proof. Let $1=G_0 \lhd G_1 \lhd \ldots \lhd G_\alpha = G$ be an ascending normal series of subgroups of a finitely generated group G in which each factor $G_{\beta+1}/G_\beta$ is cyclic or finite and furthermore we may assume that if $G_{\beta+1}/G_\beta$ is finite, then there is no subgroup K such that K is normal in G and $G_\beta < K < G_{\beta+1}$. Since $G/G_\beta = 1$ is clearly a supersoluble-by-finite group, we may assume that there exists $\beta \le \alpha$ such that G/G_β is supersoluble-by-finite but G/G_γ is not for all $\gamma<\beta$. We claim that $\beta=0$. For otherwise, if $\beta-1$ exists, then we consider the following two cases:

(i) $G_\beta/G_{\beta-1}$ is cyclic. Clearly $G/G_{\beta-1}$ is supersoluble-by-finite, a contradiction.
(ii) $G_\beta/G_{\beta-1}$ is finite. If $G_\beta/G_{\beta-1}\le C_{G/G_{\beta-1}}(G_\beta/G_{\beta-1})$, then $G_\beta/G_{\beta-1}$ is abelian. Since G/G_β is supersoluble-by-finite, there is an H such that G/H is finite and H/G_β is supersoluble. By $G_\beta/G_{\beta-1}$ being finite and abelian, $H/G_{\beta-1}$ is polycyclic and so is residually finite, therefore there is an N with H/N finite and $N\cap G_\beta=G_{\beta-1}$. Thus $N/G_{\beta-1}\cong NG_\beta/G_\beta$ is supersoluble and so $G/G_{\beta-1}$ is supersoluble-by-finite, a contradiction. Thus $(G_\beta/G_{\beta-1})\cap C_{G/G_{\beta-1}}(G_\beta/G_{\beta-1})=1$. Let $K/G_{\beta-1}=C_{G/G_{\beta-1}}(G_\beta/G_{\beta-1})$, then G/K is finite and $K\cap G_\beta=G_{\beta-1}$. Thus $K/G_{\beta-1}\cong KG_\beta/G_\beta$ is supersoluble-by-finite and so is $G/G_{\beta-1}$, a contradiction again. Thus $\beta-1$ does not exist, i.e. β is a limit ordinal. Since G/G_β is finitely generated, by [**5**, p. 403], G_β is finitely generated as a G-operator group. Let $G_\beta=\langle x_1,\dots,x_n\rangle^G$ and since $G_\beta=\bigcup_{\alpha>\beta}G_\gamma$ so there exist $\gamma_1\dots\gamma_n$ such that $x_i\in G_{\gamma_i}$. Let $\gamma_0<\beta$ such that $\gamma_i<\gamma_0$ for all $i=0,\dots,n$, then $x_i\in G_{\gamma_0}$ for all i. Since $G_{\gamma_0}\lhd G$, so $G_\beta=\langle x_1,\dots,x_n\rangle^G\le\langle G_{\gamma_0}\rangle^G=G_{\gamma_0}$. Thus $G/G_\beta=G/G_{\gamma_0}$, contrary to the hypothesis for β. Hence $\beta=0$ and then the result is proved.

The above lemma is necessary for removing the locally soluble condition and in fact, the following lemma is the beginning of the proof of the theorem.

Lemma 2. *Let G be a hyper-(cyclic or finite) group, B an artinian $\mathbf{Z}G$-module and A a ZG-submodule of B such that all irreducible $\mathbf{Z}G$-factors of A are infinite. If B/A is finite, Then $B=A\oplus B_1$ for some $\mathbf{Z}G$-submodule B_1 of B.*

Proof. Since A is artinian, it is possible to choose a $\mathbf{Z}G$-submodule B_1 of B such that $B=A+B_1$ and for each $U\le B$ with $B=A+U$, the intersection $A\cap U$ and $A\cap B_1$ are equal. We prove that $A\cap B_1=0$.

Suppose $A_1=A\cap B_1\neq 0$ and assume that G acts faithfully on B_1. Since $B/A\cong_{ZG}B_1/A_1$, then B_1/A_1 is finite. Let $G_1=C_G(B_1/A_1)$, then G_1 contains a normal subgroup K of G which is either finite or cyclic. Let $H=G_1\cap C_G(K)$, then G/H is finite. Let $1\neq x\in K$, then $H\le C_G(x)\le G$. Consider B_1, A_1 as $\mathbf{Z}H$-modules. By Lemma 3 in [**3**], $B_1(x-1)$ and $C_{B_1}(x)$ are $\mathbf{Z}H$-submodules of B_1 and $B_1(x-1)\cong_{ZH}B_1/C_{B_1}(x)$. Since $A_1<A$ and all irreducible $\mathbf{Z}G$-factors of A_1 are infinite, then so are all irreducible $\mathbf{Z}H$-factors of A_1. Since $x\in C_G(B_1/A_1)$, then $B_1(x-1)<A_1$ and so $B_1(x-1)=(B_1(x-1))^f$. On the other hand, define φ: $B_1/A_1\to B_1(x-1)/A_1(x-1)$ such that $\varphi(b+A_1)=b(x-1)+A_1(x-1)$, where $b\in B_1$. Clearly φ is a homomorphism from B_1/A_1 to $B_1(x-1)/A_1(x-1)$. Since B_1/A_1 is finite, then $B_1(x-1)/A_1(x-1)$ is finite, contrary to $B_1(x-1)=(B_1(x-1))^f$. So $B_1(x-1)=A_1(x-1)$, and then $B_1=A_1+C_{B_1}(x)$. Since $B=A+B_1=A_1+C_{B_1}(x)+A=A+C_{B_1}(x)$, then $A\cap C_{B_1}(x)<A\cap B_1$ which is contrary to the choice of B_1. So $A\cap B_1=0$ and the lemma is proved.

Now we can prove the theorem completely.

Theorem. *If G is a hyper-(cyclic or finite) group, then any artinian $\mathbf{Z}G$-module A has an f-decomposition.*

Proof. Suppose that A does not have an f-decomposition, then we can find a submodule A_1 not having an f-decomposition such that every proper submodule of A_1 does have an f-decomposition. We may assume that A satisfies this condition, i.e., $A=A_1$, and further assume that G acts faithfully on A. It follows that A is not a sum of proper submodules and so A has a unique maximal submodule M containing every proper submodule of A. For each $a\in A-M$, $\langle a\rangle^G=A$. If G were finite, then the abelian group A would be finitely generated and since A is artinian, then A is finite, a contradiction. So G is infinite.

Let $M=M^f\oplus M^{\bar f}$ be the f-decomposition of M. We consider the following two cases A/M is finite or infinite. If A/M is finite, then $M^{\bar f}\neq 0$. For otherwise, for any irreducible $\mathbf{Z}G$-factor C/D of A, if $C\le M$, then C/D is finite since $M=M^f$; if $C>M$, then $C=A$ and so $D=M$ since M is the unique maximal submodule of A and C/D is irreducible, contrary to $A\neq A^f$. Consider A/M^f. By Lemma 2, $A/M^f=M/M^f\oplus B/M^f$ for some $\mathbf{Z}G$-submodule B of A, then $A=M+B<M$ since M contains all proper $\mathbf{Z}G$-submodules of A, a contradiction. Thus A/M is infinite. Also $M^f\neq 0$, for otherwise we can get a contradiction using the same method as above. Considering A/M^f, we may assume that $M^f=0$ and so $M=M^{\bar f}$.

Let D be a finite $\mathbf{Z}G$-submodule of M and $H=C_G(D)$, then G/H is finite, H contains a nontrivial infinite cyclic or finite subgroup K normal in G. Let $G_1=C_G(K)$, then G/G_1 is finite. By Corollary 2.2.2 in [**1**], A/M contains an infinite irreducible $\mathbf{Z}G$-submodule A_1^*/M and also by Lemma 2.2.4 in [**1**], $A/M=Dr_{s\in S}(A_1^*/M)s$, where S is a subset of transversal T to G_1 in G. Thus A/M is a direct sum of finitely many irreducible $\mathbf{Z}G_1$-submodules. Let $A/M=A_1^*/M\oplus A_2^*/M\oplus\ldots\oplus A_n^*/M$. Let $x\in K$, $x\neq 1$, then $G_1=C_G(K)\le C_G(x)\le G$ and so $A_i^*(x-1)$ and $C_{A_i^*}(x)$ are both $\mathbf{Z}G$-submodules of A_i^* for each i. Now we can choose some $i\in\{1,\ldots,n\}$ such that x acts nontrivially on A_i^*. For otherwise, if x acts trivially on A_i^* for each i, then x acts trivially on A, contrary to G acting trivially on A. For A_i^*, it is clear that: (a) A_i^* does not have an $f(\mathbf{Z}G_1)$-decomposition and $C_{A_i^*}(x)\neq A_i^*$; (b) M is the unique maximal $\mathbf{Z}G_1$-submodule of A_i^* with $M=M^f$; (c) A_i^*/M is infinite and for each $a\in A_i^*-M$, $A_i^*=\langle a\rangle^{G_1}$. Now we consider the following two cases:

(1) If $A_i^*(x-1)<A_i^*$, then $A_i^*(x-1)<M$. For φ: $a+M\to a(x-1)+M(x-1)$ $(a\in A_i^*)$, we have $A_i^*/M\overset{\varphi}{\cong}_{\mathbf{Z}G_1}A_i^*(x-1)/M(x-1)$ and $\operatorname{Ker}\varphi=0$ or A_i^*/M. If $\operatorname{Ker}\varphi=0$, then $A_i^*(x-1)/M(x-1)$ is an infinite irreducible $\mathbf{Z}G_1$-factor of M, a contradiction. So $\operatorname{Ker}\varphi=A_i^*/M$. That is $A_i^*(x-1)=M(x-1)$ and then $A_i^*=M+C_{A_i^*}(x)$. Since $C_{A_i^*}(x)\neq A_i^*$, then $C_{A_i^*}(x)<M$ and $A_i^*=M+C_{A_i^*}(x)=M<A_i^*$, a contradiction again.

(2) If $A_i^*(x-1)=A_i^*$, so for $a\in A_i^*-M$, there exists $a_0\in A_i^*$ such that $a=a_0(x-1)$ and $A_i^*=\langle a\rangle^{G_1}$. Choose a finitely generated subgroup L^* of G_1 such that $a_0\in\langle a\rangle^{L^*}$, $D\le\langle a\rangle^L$. Let $L=\langle L^*,x\rangle$ and $A_1=\langle a\rangle^L$, then A_1 is a finitely generated $\mathbf{Z}L$-module and L is a finitely generated hyper-(cyclic or finite) group. By Lemma 1, L is a supersoluble-by-finite group. Thus A_1 has a $\mathbf{Z}L$-submodule B_1 of finite index such that $D\cap B_1\neq D$ by the residual finiteness of finitely generated abelian-by-polycyclic groups [**4**]. Consider

the finite $\mathbf{Z}L$-module A_1/B_1. Since $G_1 = C_G(K) \leq C_G(x)$ and so $[G_1, x] = 1$, we have $[L^*, x] = 1$ and then $x \in Z(L)$ since $L = \langle L^*, x \rangle$. Thus A_1/B_1 can be viewed as a $\mathbf{Z}\langle x \rangle$-module. Then, by [6], we can get $A_1/B_1 = B/B_1 \oplus C/B_1$, where the $\mathbf{Z}\langle x \rangle$-submodule B/B_1 has a $\mathbf{Z}\langle x \rangle$-composition series in which each $\mathbf{Z}\langle x \rangle$-factor is $\langle x \rangle$-trivial and the $\mathbf{Z}\langle x \rangle$-submodule C/B_1 has no nonzero $\mathbf{Z}\langle x \rangle$-factors which are $\langle x \rangle$-trivial. Since $(D + B_1)/B_1$ is an x-trivial $\mathbf{Z}\langle x \rangle$-submodule of A_1/B_1, so $B/B_1 \neq 0$. Thus A_1/C is a nonzero finite $\mathbf{Z}\langle x \rangle$-module and $A_1/C \cong_{Z\langle x \rangle} (A_1/B_1)/(C/B_1) \cong_{Z\langle x \rangle} B/B_1$ shows that A_1/C has a finite $\mathbf{Z}\langle x \rangle$-composition series in which each $\mathbf{Z}\langle x \rangle$-factor is $\langle x \rangle$-trivial. Hence $(A_1(x-1) + C)/C = \bar{A}_1(x-1) < \bar{A}_1$, where $\bar{A}_1 = A_1/C$. But on the other hand, since $A_1(x-1)$ is a $\mathbf{Z}L$-module and $a_0 \in A_1$, so

$$A_1 = \langle a \rangle^L = \langle a_0(x-1) \rangle^L = (\langle a_0 \rangle (x-1))^L \leq (A_1(x-1))^L = A_1(x-1)$$

a contradiction. This finishes the proof of the theorem.

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