Appendix D
Vector Fields and Their Lie Bracket

D.1 Construction

In this appendix we recall the construction of the Lie algebra of vector fields.

D.1 Example The tangent bundle $\pi_M : TM \to M$ is a vector bundle (bundle trivialisations are given by the canonical charts $T\varphi$). A smooth section of the tangent bundle is called (smooth) *vector field* and we write shorter $\mathcal{V}(M) :=\Gamma(TM)$ for the vector space of all vector fields.

D.2 Example If $U \Subset E$ in a locally convex space, we have $TU = U \times E$ and $\pi_U : U \times E \to U$, $(u,e) \mapsto u$. Thus a vector field of U can be written as $X = (X_U, X_E): U \to U \times E$ and we must have $X_U = \mathrm{id}_U$. Hence a vector field on U is uniquely determined by the smooth map $X_E \in C^\infty(U,E)$.

D.3 If M is a manifold and (ϕ, U_ϕ) a manifold chart, then we have an analogue of X_E on U_ϕ for $X \in \mathcal{V}(M)$: Clearly $T\phi \circ X \circ \phi^{-1} = (\mathrm{id}_{V_\phi}, X_\phi)$ for the smooth map $X_\phi := d\phi \circ X \circ \phi^{-1} : V_\phi \to E$. We call X_ϕ the *local representative of* X or *the principal part of* X with respect to the chart ϕ.

For later use, consider a vector field $X \in \mathcal{V}(M)$ and a smooth function $f: M \to F$, where F is a locally convex space. Then we define a function $X.f \in C^\infty(M,F)$ via

$$X.f(m) := df \circ X(m) = \mathrm{pr}_2 \circ Tf \circ X(m). \qquad \text{(D.1)}$$

D.4 Similar to C.7 we topologise $\mathcal{V}(M)$: Pick an atlas $\mathcal{A}$ of M whose charts we denote by $\varphi: U_\varphi \to V_\varphi \subseteq E_\varphi$. Then we declare the topology to be the initial topology with respect to the map

$$\kappa: \mathcal{V}(M) \to \prod_{\varphi \in \mathcal{A}} C^\infty(V_\varphi, E_\varphi), \quad X \mapsto (X_\varphi)_{\varphi \in X},$$

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where the factors on the right-hand side carry the compact open C^∞-topology. In particular, this topology turns the vector fields into a locally convex space.

We will use the notion of integral curves and flows for vector fields, whence we recall the definition of these objects.

D.5 Let $X \in \mathcal{V}(M)$. We say a C^1-curve $c\colon [a,b] \to M$ is an *integral curve* for X if for every $t \in [a,b]$ the curve satisfies $\dot{c}(t) = X(c(t))$.

If M is a Banach manifold, it follows from the theory of ordinary differential equations, Lang (1999, IV), that for every $m \in M$ there exists an integral curve c_m of X on some open interval $J_m :=]-\varepsilon,\varepsilon[$ such that $c_m(0) = m$. Moreover, the *flow*

$$\mathrm{Fl}^X\colon \bigcup_{m\in M} \{m\} \times J_m \to M, \quad (m,t) \mapsto c_m(t)$$

defines a continuous map on some open subset of $M \times \mathbb{R}$. If M is modelled on a locally convex space, the existence of integral curves and flows is not automatic; see Appendix A.6.

D.6 Definition Let $f\colon M \to N$ be smooth. We call the vector fields $X \in \mathcal{V}(M), Y \in \mathcal{V}(N)$ *f-related* if $Y \circ f = Tf \circ X$.

D.7 Lemma *Let M be a manifold modelled on a locally convex space E with atlas $\mathcal{A}$. Let $(X_\phi)_{\phi\in\mathcal{A}}$ be a family of smooth maps $X_\phi\colon V_\phi \to E$ such that every pair X_ϕ, X_ψ is $\psi\circ\phi^{-1}$ related on $\phi(U_\psi \cap U_\phi)$. Then there is a unique vector field $X \in \mathcal{V}(M)$ whose local representatives coincide with the X_ϕ.*

Proof Define $X\colon M \to TM$, $p \mapsto T\phi^{-1}(\phi(p), X_\phi(\phi(p)))$ for $p \in U_\phi$. Since the maps X_ϕ, X_ψ are related by the change of charts on the overlap $U_\phi \cap U_\psi$, the mapping is well defined. By construction it is smooth and a vector field. □

D.8 For principal parts of vector fields X,Y on $U \Subset E$ write $X_E.Y_E(z) := dY_E \circ X(z) := dY_E(z; X_E(z))$. Define

$$[X,Y] := X.Y - Y.X. \qquad X,Y \in C^\infty(U,E).$$

We will see in the following that the bracket of principal parts of vector fields gives rise to a Lie bracket of vector fields.

D.9 Lemma *Let $U \Subset E$, $V \Subset F$ be open in locally convex spaces and $f \in C^\infty(U,V)$, $X_1, X_2 \in C^\infty(U,E)$ and $Y_1, Y_2 \in C^\infty(V,F)$. Assume that X_i is f-related to Y_i for $i = 1,2$. Then $[X_1,X_2]$ is f-related to $[Y_1,Y_2]$.*

Proof Using the chain rule, (1.7) and relatedness we obtain $i{=}1,2 (x,v){\in}U{\times}E$.

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$$df(x, dX_i(x; v)) = dY_i(f(x), df(x; v)) - d^2 f(x; X_i(x), v), \tag{D.2}$$

We use this relation together with relatedness to obtain

$$\begin{aligned} df(x; [X_1, X_2](x)) &= df(x; dX_2(x, X_1(x))) - df(x; dX_1(x; X_2(x))) \\ &= dY_2(f(x); df(x; X_1(x))) - d^2 f(x; X_2(x), X_1(x)) \\ &\quad - dY_1(f(x); df(x; X_2(x))) + d^2 f(x; X_1(x), X_2(x)) \\ &= dY_2(f(x); Y_1(f(x))) - dY_1(f(x); Y_2(f(x))) \\ &= [Y_1, Y_2](f(x)), \end{aligned}$$

where the second-order terms cancel by Schwarz' theorem. □

Before we now establish the Lie algebra properties, let us recall a general definition useful for our purpose.

D.10 Definition Let $(A, \cdot)$ be an associative algebra. Then the linear mappings $L(A, A)$ form a Lie algebra under the commutator bracket $[\phi, \psi] := \phi \circ \psi - \psi \circ \phi$, Example 3.16 (where $\circ$ is the usual composition of linear maps). A mapping $\phi \in L(A, A)$ is called *derivation* of the algebra A if it satisfies the Leibniz rule

$$\phi(a \cdot b) = \phi(a) \cdot b + a \cdot \phi(b) \quad \text{for all } a, b \in A.$$

We denote by $\mathrm{der}(A)$ the *set of all derivations* of A and note that it forms a Lie subalgebra of $(L(A, A), [\cdot, \cdot])$. (As no topology is involved, this will, in general, not be a locally convex Lie algebra.)

For E a locally convex space, $U \Subset E$ and $X \in \mathcal{V}(U)$ define the *Lie derivative*

$$\mathcal{L}_X(f) := df \circ X = df(\mathrm{id}_U, X_E) \text{ for } f \in C^\infty(U, \mathbb{R}). \tag{D.3}$$

By definition $\mathcal{L}_X(f) = X.f$ in the special case that f is real valued. The reason for the new notation and name will become apparent from the following observations (see also Definition E.9): The pointwise multiplication turns $C^\infty(U, \mathbb{R})$ into an associative algebra. Then $\mathcal{L}_X$ is linear in f. Thus

$$\mathcal{L}_X(f \cdot g) = \mathcal{L}_X(f) \cdot g + f \cdot \mathcal{L}_X(g). \tag{D.4}$$

In other words, $\mathcal{L}_X$ is a derivation of the algebra $C^\infty(U, \mathbb{R})$.

D.11 Lemma *Let $U \Subset E$ in a locally convex space.*

(a) $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$.
(b) *The map $\mathcal{L} \colon C^\infty(U, E) \to \mathrm{der}(C^\infty(U, R))$, $X \mapsto \mathcal{L}_X$ is linear and injective.*

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(c) *The map* $[\cdot,\cdot]\colon C^\infty(U,E)\times C^\infty(U,E)\to C^\infty(U,E)$, $(X,Y)\mapsto [X,Y] = X\,.\,Y - Y\,.\,X$ *turns the space* $C^\infty(U,E)$ *into a Lie algebra.*

Proof (a) From (1.7) we deduce

$$\mathcal{L}_X(\mathcal{L}_Y(f)) = d^2 f(x;Y(x),X(x)) + df(x;dY(x;X(x))).$$

Also using the formula for X and Y interchanged, we see that the second-order terms cancel by Schwarz' theorem and thus

$$[\mathcal{L}_X,\mathcal{L}_Y](f)(x) = \mathcal{L}_{[X,Y]}(f)(x).$$

(b) $\mathcal{L}_X$ is linear in X as $df(x;\cdot)$ is. Thus it suffices to prove that the kernel of $\mathcal{L}$ is trivial. Let $X\in C^\infty(U,E)$ be a map with $X(x)\neq 0$ for some $x\in U$. By the Hahn–Banach theorem, 1.7, we find $\lambda\in E'$ with $\lambda(X(x))\neq 0$. Then $\mathcal{L}_X(\lambda)(x) = d\lambda(x,X(x)) = \lambda(X(x))\neq 0$ and thus $\mathcal{L}_X\neq 0$.

(c) Clearly $[\cdot,\cdot]$ is bilinear, whence $(C^\infty(U,E),[\cdot,\cdot])$ is an algebra. Now $[X,X] = X.X - X.X = 0$. Recall that in the Jacobi identity, the entries of the iterated Lie bracket are cyclically permuted. We write shorter $\sum_{\text{cycl}}[X,[Y,Z]]$ for this and thus have to check that this expression vanishes for all $X,Y,Z\in C^\infty(U,E)$. However,

$$\mathcal{L}\left(\sum_{\text{cycl}}[X,[Y,Z]]\right) = \sum_{\text{cycl}}[\mathcal{L}_X,[\mathcal{L}_Y,\mathcal{L}_Z]] = 0,$$

where we have used linearity of $\mathcal{L}$, (a), (b) and the fact that the derivations form a Lie algebra. Since $\mathcal{L}$ is injective by (b), we see that the Jacobi identity holds. □

Finally, we show that the Lie bracket of vector fields is continuous if the space E is finite dimensional.

D.12 Lemma *Let E be a finite-dimensional space and $U \Subset E$. Then the Lie bracket*

$$[\cdot,\cdot]\colon C^\infty(U,E)\times C^\infty(U,E)\to C^\infty(U,E)$$

is continuous. Hence $(C^\infty(U,E),[\cdot,\cdot])$ *is a locally convex Lie algebra.*

Proof Note that $C^\infty(U,E)$ is a locally convex space with respect to the compact open C^∞-topology, Proposition 2.4. To establish continuity of the Lie bracket, we deduce from Lemma 2.10 that it suffices to establish continuity of the adjoint map

$$p\colon C^\infty(U,E)\times C^\infty(U,E)\times U\to E,\quad (X,Y,u)\mapsto dY(u;X(u)).$$

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Recall that the compact-open C^∞-topology is initial with respect to the mappings $d^k\colon C^\infty(U,E) \to C(U\times E^k,E)_{\text{c.o}}$, $f \mapsto d^k f$. Hence the map $d\colon C^\infty(U,E) \to C(U\times E,E)$ is continuous. We can thus write the adjoint map as a composition of continuous mappings (see Lemma B.10, which uses the that U is locally compact, i.e. E finite dimensional) $p(f,g) = \mathrm{ev}(d(f),u,\mathrm{ev}(g,u))$, whence the Lie bracket is continuous. □

D.13 Corollary *Let M be a finite-dimensional manifold. Then $(\mathcal{V}(M),[\cdot,\cdot])$ is a locally convex Lie algebra.*

Proof That the vector fields form a Lie algebra is checked in Exercise 3.2.3. Recall from D.4 that the vector fields were topologised as a subspace of a product of spaces of the form $C^\infty(U,E)$, where $U \Subset M$. By construction of the Lie bracket of two vector fields, the bracket is given by a local formula on chart domains U. Hence it suffices to establish continuity of the local formula on the spaces $C^\infty(U,E)$. This was exactly the content of Lemma D.12. □

D.14 Remark In general, the Lie algebra of vector fields $\mathcal{V}(M)$ will not be a locally convex Lie algebra if M is an infinite-dimensional manifold. Indeed, it can be shown that Lemma D.12 becomes false beyond the realm of Banach spaces. To see this, let $U \Subset E$ be an open subset of a non-normable space. We consider the subalgebra

$$\begin{aligned}\mathcal{A} = \{X_{A,b} \in C^\infty(U,E) \mid\ &\text{for all } v\in E,\ X_{A,b}(v)\\ &= Av + b,\ \text{for } A\in L(E,E), b\in E\}\end{aligned}$$

of affine vector fields. By construction we can identify $\mathcal{A} \cong L(E,E)\times E$. Here the subspace topology induced by the compact-open C^∞-topology of $C^\infty(U,E)$ on $\mathcal{A}$ is the product topology, where E carries its natural locally convex topology and the space of continuous linear mappings $L(E,E)$ is endowed with the compact-open topology (i.e. the topology induced by the embedding $L(E,E) \subseteq C_{\text{c.o.}}(E,E)$). Indeed the latter fact is irrelevant for us; we are only interested in the fact that this topology turns $L(E,E)$ into a topological vector space. Now, the Lie bracket of $C^\infty(U,E)$ induces the Lie bracket

$$[X_{A,b},X_{C,d}](v) = (A\circ C - C\circ A)(v) + (A(d) - C(b))$$

on the affine vector fields (these facts are left as Exercise D.1.3). To see that this Lie bracket is, in general, not continuous, it suffices to note that the evaluation map $L(E,E)\times E \to E$, $(A,v) \mapsto A(v)$ is discontinuous. For this we pick $0\neq v\in E$ and consider the mapping

$$j\colon E' = L(E,\mathbb{R}) \to L(E,E),\quad \lambda \mapsto (x\mapsto \lambda(x)\cdot v). \tag{D.5}$$

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If we endow the dual space E' with the compact-open topology (again the subspace topology of $E' \subseteq C_{\text{c.o.}}(E,\mathbb{R})$) then E' becomes a topological vector space and j continuous. However, Proposition A.19 shows that the evaluation map $E' \times E \to \mathbb{R}$ is discontinuous for every topological vector space E which is not normable. As j and scalar multiplication in E are continuous, this implies that the evaluation of $L(E,E)$ must be discontinuous if E is not normable. We deduce that the Lie bracket on $C^\infty(U,E)$ must be discontinuous if E is not normable.[1]

Exercises

D.1.1 Show that the construction of the topology for $\mathcal{V}(M)$ in D.4 is just a special case of C.7.

D.1.2 Let A be an associative algebra. Show that the set of derivations $\operatorname{der}(A)$ (see Definition D.10) forms a Lie subalgebra of $(L(A,A),[\cdot,\cdot])$, where the bracket is given by the commutator bracket $[f,g] = f \circ g - g \circ f$ of linear maps.

D.1.3 We provide the missing details in Remark D.14. To this end let $U \Subset E$ in a locally convex space and endow $C^\infty(U,E)$ with the compact-open topology (i.e. the topology induced by the embedding $L(E,E) \subseteq C_{\text{c.o.}}(E,E)$). We consider the affine vector fields

$$\mathcal{A} = \{X_{A,b} \in C^\infty(U,E) \mid \text{for all } v \in E,\ X_{A,b}(v) = Av + b, \\ \text{for } A \in L(E,E),\ b \in E\}$$

and identify $\mathcal{A} = L(E,E) \times E$ (where $L(E,E)$ denote continuous linear maps). Show that:

(a) The subspace topology on $\mathcal{A}$ is the product topology of the compact-open topology on $L(E,E)$ and the locally convex topology of E.

(b) The Lie bracket on $C^\infty(U,E)$ induces the Lie bracket

$$[X_{A,b}, X_{C,d}] = (A \circ C - C \circ A) + (A(d) - C(b)) \text{ on } \mathcal{A}.$$

(c) If we endow the dual space E' with the compact-open topology (i.e. the subspace topology of $E' \subseteq C_{\text{c.o.}}(E,\mathbb{R})$), then E' is a topological vector space and the map j from (D.5) becomes continuous.

[1] Even stronger, one can show that the evaluation must be discontinuous on $L(E,E)$ with the compact-open topology for all infinite-dimensional spaces E; see Neeb (2006, Remark I.5.3) for an exposition.

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