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THE COMPLEXITY OF THOMASON'S ALGORITHM FOR FINDING A SECOND HAMILTONIAN CYCLE

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Abstract

By Smith's theorem, if a cubic graph has a Hamiltonian cycle, then it has a second Hamiltonian cycle. Thomason ['Hamilton cycles and uniquely edge-colourable graphs', *Ann. Discrete Math.* **3** (1978), 259–268] gave a simple algorithm to find the second cycle. Thomassen [private communication] observed that if there exists a polynomially bounded algorithm for finding a second Hamiltonian cycle in a cubic cyclically 4-edge connected graph G, then there exists a polynomially bounded algorithm for finding a second Hamiltonian cycle in any cubic graph G. In this paper we present a class of cyclically 4-edge connected cubic bipartite graphs G_i with $16(i + 1)$ vertices such that Thomason's algorithm takes $12(2^i - 1) + 3$ steps to find a second Hamiltonian cycle in G_i.

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1. Introduction

It is well known that determining whether there is a Hamiltonian cycle in a cubic graph is an NP-complete problem [2]. Smith's theorem (see [5]) states that for any cubic graph and a given edge e, the number of Hamiltonian cycles through e is even. From Smith's theorem, if we find one Hamiltonian cycle then there must be another one. This leads to an interesting question: is finding the second Hamiltonian cycle still an NP-complete problem?

The first published proof of Smith's theorem was a beautiful but nonconstructive counting argument of Tutte [5]. Thomason [4] gave a simple constructive argument called the lollipop method to find a second Hamiltonian cycle.

Since Thomason's algorithm is the only known algorithm for finding a second Hamiltonian cycle, it is important to investigate its complexity. Krawczyk [3] presented a class of graphs on $8n + 2$ vertices, where $n \geq 1$, for which Thomason's algorithm requires at least 2^n steps to find a second Hamiltonian cycle. Later Cameron

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[1] proved a more general result showing that Thomason's algorithm is exponential on a family of cubic planar graphs.

A cyclic k-edge cut in a graph G is a k-edge cut $E' \subset E(G)$ such that at least two of the connected components in $G - E'$ contain cycles. A graph G is cyclically k-edge connected if and only if there is no cyclic k'-edge cut in G with $k' < k$.

As pointed out by Carsten Thomassen (private communication), if there exists a polynomially bounded algorithm for finding a second Hamiltonian cycle in a cubic cyclically 4-edge connected graph G, then there exists a polynomially bounded algorithm for finding a second Hamiltonian cycle in any cubic graph G. We will give a proof of this reduction theorem in Section 2.

Since the graphs in [1, 3] are not cyclically 4-edge connected, it is natural to ask for examples of cubic cyclically 4-edge connected graphs on which the complexity of Thomason's algorithm grows exponentially with the number of vertices. To this end, we prove the following theorem.

THEOREM 1.1. *For each $i \geq 0$, there exists a cyclically 4-edge connected cubic bipartite Hamiltonian graph G_i on $16(i + 1)$ vertices such that Thomason's algorithm takes $12(2^i - 1) + 3$ steps to find a second Hamiltonian cycle in G_i.*

2. The reduction to the cyclically 4-edge connected graph

THEOREM 2.1. *Suppose there exists a polynomially bounded algorithm A for the following problem: given a cubic cyclically 4-edge connected graph G possibly with multiple edges, an edge e in G and a Hamiltonian cycle C containing e, find a Hamiltonian cycle which is distinct from C and which contains e. Then there also exists a polynomially bounded algorithm B for the following more general problem: given a cubic graph G possibly with multiple edges, an edge e in G and a Hamiltonian cycle C containing e, find a Hamiltonian cycle which is distinct from C and which contains e.*

PROOF. Suppose the complexity of algorithm A for a cubic cyclically 4-edge connected graph G with n vertices is $O(n^k)$ where $k \geq 4$ is a fixed constant. We will show that algorithm B exists and for any cubic graph G with n vertices the complexity of B is still $O(n^k)$.

Suppose that G is a cubic graph with n vertices and we have a Hamiltonian cycle C in G which contains an edge $e \in E(G)$. If G is cyclically 4-edge connected, then we just let $B = A$. Otherwise, we observe that G is 2-edge connected, since G has a Hamiltonian cycle. Consequently, if we consider the minimum edge cut in G, there are two cases:

(1) The minimum edge cut contains two edges.
(2) The minimum edge cut contains three edges.

Case (1). In this case we can find a 2-edge cut in $O(n^3)$ steps by choosing all pairs of edges and checking whether the deletion of these edges disconnects G.

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(Faster algorithms for solving this problem do exist, but we do not attempt to optimise the complexity here.) Let (x_1x_2, y_1y_2) be such a cut and let the part that does not contain the edge e in $G - x_1x_2 - y_1y_2$ be G_1. (If neither part contains e, let G_1 be an arbitrary part.) Suppose $x_1 \in G_1$, $y_1 \in G_1$ and $|V(G_1)| = n_1$. Note that $x_1 \neq y_1$, otherwise there will be a cut edge attached to x_1 since G is cubic. Now $G_1 + x_1y_1$ is a cubic graph which is smaller than G and there is a Hamiltonian cycle C_1 containing x_1y_1 in this graph (which arises from C). By the induction hypothesis we can use algorithm B in $O(n_1^k)$ steps to find another Hamiltonian cycle C_1' in $G_1 + x_1y_1$ that goes through x_1y_1. Now the cycle $C - (C_1 \cap G_1) + (C_1' \cap G_1)$ is a second Hamiltonian cycle in G which contains e, and we find it in $O(n_1^k) + O(n^3) = O(n^k)$ steps.

Case (2). In this case there must exist a cyclic 3-edge cut by the assumption that G is not cyclically 4-edge connected. We can find such a cut (e_1, e_2, e_3) in $O(n^4)$ steps by choosing all triples of edges and checking whether the deletion of these edges disconnects G and both connected components have cycles. Let G_1 and G_2 be the two connected components of $G - e_1 - e_2 - e_3$, let $G_1' = G/G_2$, $G_2' = G/G_1$ and let $n_1 = |V(G_1')|$, $n_2 = |V(G_2')|$. Then $n_1 + n_2 = n + 2$. For each G_i', we have a Hamiltonian cycle C_i which arises from C. Without loss of generality, we can assume that C contains e_1 and e_2, which means both C_1 and C_2 contain e_1 and e_2.

If the edge e is one of the edges of the cyclic 3-edge cut, say $e = e_1$, by the induction hypothesis we can use algorithm B to find another Hamiltonian cycle $C_1' \in G_1'$ which contains e_1 in $O(n_1^k)$ steps. If C_1' contains e_2, then C_1' together with C_2 forms a Hamiltonian cycle that differs from C and still contains e in G, and we find it in $O(n_1^k) + O(n^4) = O(n^k)$ steps. This allows us to assume that C_1' contains both e_1 and e_3. Again by the induction hypothesis, we can find another Hamiltonian cycle $C_2' \in G_2'$ by algorithm B in $O(n_2^k)$ steps which contains e_1. For the same reason, C_2' must contain both e_1 and e_3. Now C_1' together with C_2' forms a Hamiltonian cycle that differs from C and still contains e in G, and we find it in $O(n_1^k) + O(n_2^k) + O(n^4) = O(n^k)$ steps.

So now we can assume that e is not in the cyclic 3-edge cut. Without loss of generality we assume that $e \in E(G_1)$. By the induction hypothesis we can use algorithm B to find a different Hamiltonian cycle in G_2 which contains edge e_1 in $O(n_2^k)$ steps. By the argument used above, this Hamiltonian cycle contains e_1 and e_3. Let this Hamiltonian cycle be C_{13}. Then again by the induction hypothesis and algorithm B we can find a Hamiltonian cycle in G_2 different from C_2 which contains the edge e_2 in $O(n_2^k)$ steps. Again, by the same argument as above, this Hamiltonian cycle contains e_2 and e_3. Let this Hamiltonian cycle be C_{23}. Recall that C_2 contains both e_1 and e_2. Let it be the Hamiltonian cycle C_{12}. Now by the induction hypothesis we can find a new Hamiltonian cycle C' in G_1 which contains e by algorithm B in $O(n_1^k)$ steps. Since C' is Hamiltonian, it must contain exactly two of the edges e_1, e_2 and e_3, say it contains e_i and e_j with $1 \leq i < j \leq 3$. Now C' together with C_{ij} forms a Hamiltonian cycle that differs from C and still contains e in G and we find it in $O(n_1^k) + O(n_2^k) + O(n_2^k) + O(n^4) = O(n^k)$ steps. This completes the proof. □

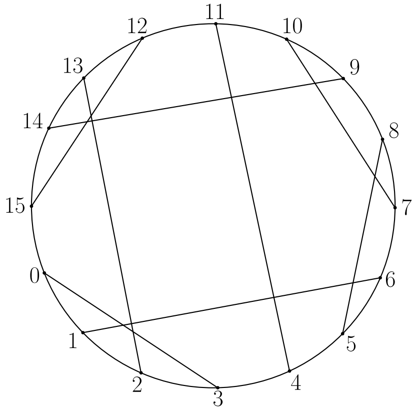

FIGURE 1. The graph G.

3. The construction and proof of Theorem 1.1

We start by showing how to construct the graph G_i. First take the graph G with 16 vertices and label the vertices as in Figure 1. This graph is cyclically 4-edge connected and bipartite and there is a Hamiltonian cycle $H_0 = 0, 1, \ldots, 15$. Apply the lollipop method to this Hamiltonian cycle with starting edge $(0, 1)$. The algorithm takes three steps to find the second Hamiltonian cycle in G, passing through the following three Hamiltonian paths (P_0^0 is the starting Hamiltonian cycle):

$$
\begin{aligned}
P_0^0 &= 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,\\
P_1^0 &= 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 14, 13,\\
P_2^0 &= 0, 1, 2, 13, 14, 15, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3,\\
P_3^0 &= 0, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 14, 13, 2, 1.
\end{aligned}
$$

Put $G_0 = G$. Take G_0 and a new copy of G. For the sake of convenience, we use roman font to represent the vertices from G_0 and underlined roman font to represent the vertices from the new copy of G. We delete the edges $(2, 3)$ and $(6, 7)$ from G_0 and delete the edges $(\underline{10}, \underline{11})$ and $(\underline{14}, \underline{15})$ from the new copy of G, and we make four new edges $(2, \underline{11}), (3, \underline{14}), (6, \underline{15}), (7, \underline{10})$. This is the graph G_1. There is a Hamiltonian cycle $H_1 = 0, 1, 2, \underline{11}, \underline{12}, \underline{13}, \underline{14}, 3, 4, 5, 6, \underline{15}, \underline{0}, \underline{1}, \ldots, \underline{9}, \underline{10}, 7, 8, \ldots, 15$ in this graph.

For every $i \geq 2$, we construct the graph G_i by taking G_{i-1} and a new copy of G, deleting the edges $(2, 3)$ and $(6, 7)$ from the last copy of G in G_{i-1} and deleting the edges $(\underline{10}, \underline{11})$ and $(\underline{14}, \underline{15})$ from the new copy of G, then making four new edges $(2, \underline{11}), (3, \underline{14}), (6, \underline{15}), (7, \underline{10})$. Now roman font denotes vertices from G_{i-1} and underlined roman font denotes vertices from the new copy of G. We can easily find a new Hamiltonian cycle H_i in G_i by replacing two edges of the Hamiltonian cycle H_{i-1} in G_{i-1} with two paths in the new copy of G. See Figure 2 for an example.

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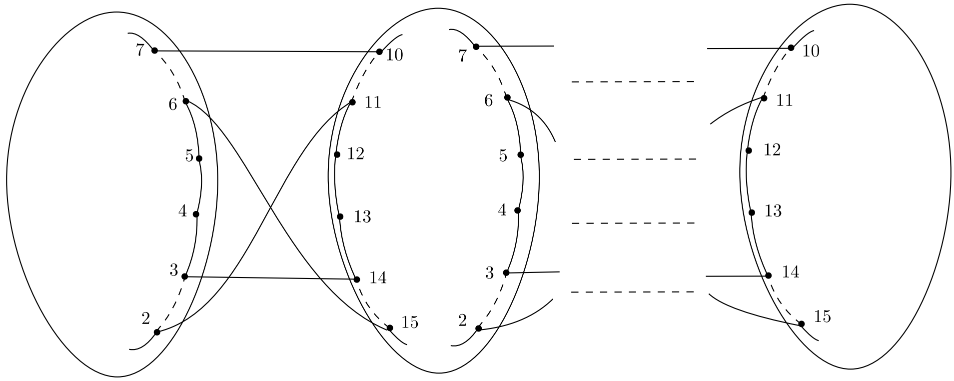

FIGURE 2. The construction of G_i.

Apply the lollipop method to the Hamiltonian cycle H_1 in G_1 with starting edge $(0, 1)$. The algorithm takes 15 steps to find the second Hamiltonian cycle in G_1, passing through the following 15 Hamiltonian paths (P^1_0 is the starting Hamiltonian cycle H_1):

$$P^1_0 = 0,1,2,\underline{11,12,13,14},3,4,5,6,\underline{15,0,1,2,3,4,5,6,7,8,9,10},7,8,9,10,11,12,13,14,15$$
$$P^1_1 = 0,1,2,\underline{11,12,13,14},3,4,5,6,\underline{15,0,1,2,3,4,5,6,7,8,9,10},7,8,9,10,11,12,15,14,13$$
$$P^1_2 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,9,8,7,6,5,4,3,2,1,0,15},6,5,4,3,\underline{14,13,12,11}$$
$$P^1_3 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,9,8,7,6,5,4,11,12,13,14},3,4,5,6,\underline{15,0,1,2,3}$$
$$P^1_4 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,9,8,7,6,5,4,11,12,13,14},3,4,5,6,\underline{15,0,3,2,1}$$
$$P^1_5 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,9,8,7,6,1,2,3,0,15},6,5,4,3,\underline{14,13,12,11,4,5}$$
$$P^1_6 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,9,8,5,4,11,12,13,14},3,4,5,6,\underline{15,0,3,2,1,6,7}$$
$$P^1_7 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,1,2,3,0,15},6,5,4,3,\underline{14,13,12,11,4,5,8,9}$$
$$P^1_8 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,1,2,3,0,15},6,5,4,3,\underline{14,9,8,5,4,11,12,13}$$
$$P^1_9 = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,1,2,13,12,11,4,5,8,9,14},3,4,5,6,\underline{15,0,3}$$
$$P^1_{10} = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,1,2,13,12,11,4,3,0,15},6,5,4,3,\underline{14,9,8,5}$$
$$P^1_{11} = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,5,8,9,14},3,4,5,6,\underline{15,0,3,4,11,12,13,2,1}$$
$$P^1_{12} = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,5,8,9,14},3,4,5,6,\underline{15,0,1,2,13,12,11,4,3}$$
$$P^1_{13} = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,5,8,9,14},3,4,5,6,\underline{15,0,1,2,3,4,11,12,13}$$
$$P^1_{14} = 0,1,2,13,14,15,12,11,10,9,8,7,\underline{10,7,6,5,8,9,14,13,12,11,4,3,2,1,0,15},6,5,4,3$$
$$P^1_{15} = 0,3,4,5,6,\underline{15,0,1,2,3,4,11,12,13,14,9,8,5,6,7,10},7,8,9,10,11,12,15,14,13,2,1$$

(The vertices in roman font are the vertices from G_0 and the vertices in underlined roman font are the vertices from the new copy of G.)

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We can see that after the second step of the algorithm (P_2^1) the last vertex of the Hamiltonian path is in the new copy of G and it comes back to G_0 after the 14th step. Consider the Hamiltonian paths where the last vertex is in G_0 (that is, P_1^1, P_{14}^1 and P_{15}^1). If we only focus on the vertices from G_0 in these three paths, then we can see that they are the same as the three paths we get when we apply the lollipop method to G_0 (that is, the part of P_1^1 in roman font is the same as P_1^0, the part of P_{14}^1 in roman font is the same as P_2^0 and the part of P_{15}^1 in roman font is the same as P_3^0). Thus these vertices appear in the same order as when we apply the lollipop method to G_0. The 12 extra Hamiltonian paths (from P_2^1 to P_{13}^1) are added in between these three Hamiltonian paths. We get these 12 extra Hamiltonian paths because, when we apply the lollipop method to G_0, after the second step the last vertex is 3 (the last number of P_2^0), but by our construction of G_1, the edge $(2, 3)$ disappears and it is replaced by two edges $(2, \underline{11}), (3, \underline{14})$, so the algorithm finds a new end for the Hamiltonian path in the new copy of G (the last vertex of P_2^1 in underlined roman font). This is the beginning of the 12 extra Hamiltonian paths.

Then we apply the lollipop method to graph G_2. The algorithm takes 39 steps to find the second Hamiltonian cycle in G_2. The 39 Hamiltonian paths are given in the Appendix (P_0^2 is the starting Hamiltonian cycle, the vertices in roman font are from first copy of G, the vertices in underlined roman font are from second copy of G and the vertices in bold italic font are from the third copy of G).

Consider the Hamiltonian paths where the last vertex is in the new copy of G. They appear in two groups, each containing 12 paths, namely $P_9^2, \ldots, P_{20}^2$ and $P_{25}^2, \ldots, P_{36}^2$. If we focus on the vertices that are in the last copy of G (the vertices in bold italic font) in these paths, we can see that these vertices appear in a reverse order. (The part of P_9^2 in bold italic font is the same as the part of P_{36}^2 in bold italic font, the part of P_{10}^2 in bold italic font is the same as the part of P_{35}^2 in bold italic font, and more generally, the part of P_i^2 in bold italic font is the same as the part of P_{45-i}^2 in bold italic font for $9 \le i \le 20$.) Also, if we compare the 12 extra paths when we apply the lollipop method in G_1 ($P_2^1, \ldots, P_{13}^1$) and the 24 extra paths when we apply the lollipop method in G_2 ($P_9^2, \ldots, P_{20}^2$ and $P_{25}^2, \ldots, P_{36}^2$), we can see that the part of P_9^2 in bold italic font is the same as the part of P_2^1 in underlined roman font, the part of P_{10}^2 in bold italic font is the same as the part of P_3^1 in underlined roman font, and more generally, the part of P_i^2 in bold italic font is the same as the part of P_{i-7}^1 in bold italic font for $9 \le i \le 20$. This means the vertices in bold italic font appear in the same order as the vertices in underlined roman font appear in G_1.

Next we focus on the paths where the last vertex is not in the new copy of G (namely $P_1^2, \ldots, P_8^2, P_{21}^2, \ldots, P_{24}^2, P_{37}^2, P_{38}^2, P_{39}^2$) and the vertices in the first or the second copy of G in these paths (in roman font and underlined roman font). We can see that they are the same as the paths we get when we apply the lollipop method to G_1 (the part of P_1^2 in roman and underlined roman font is the same as the part of P_1^1 in roman and underlined roman font, the part of P_2^2 in roman and underlined roman font is the same as the part of P_2^1 in roman and underlined roman font, ..., the part of P_{39}^2 in roman and underlined roman font is the same as the part of P^1_{15} in roman and underlined roman font).

This pattern repeats if we continue constructing G_i in this way. For each G_i, the lollipop method takes $12 \cdot 2^{i-1}$ more steps to find the second Hamiltonian cycle than it takes in G_{i-1}. This observation completes the proof of Theorem 1.1.

Acknowledgement

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Appendix. The 39 Hamiltonian paths in G_2

P^2_0 = 0,1,2,11, 12, 13, 14,3,4,5,6,15, 0, 1, 2,***11,12,13,14,***3, 4, 5, 6,
15,0,1,2,3,4,5,6,7,8,9,10,7, 8, 9, 10,7,8,9,10,11,12,13,14,15

P^2_1 = 0,1,2,11, 12, 13, 14,3,4,5,6,15, 0, 1, 2,***11,12,13,14,***3, 4, 5, 6,
15,0,1,2,3,4,5,6,7,8,9,10,7, 8, 9, 10,7,8,9,10,11,12,15,14,13

P^2_2 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 9, 8, 7,***10,9,8,7,6,5,4,***
3,2,1,0,15,6, 5, 4, 3,***14,13,12,11,***2, 1, 0, 15,6,5,4,3,14, 13, 12, 11

P^2_3 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 9, 8, 7,***10,9,8,7,6,5,4,***
3,2,1,0,15,6, 5, 4, 11, 12, 13, 14,3,4,5,6,15, 0, 1, 2,***11,12,13,14,***3

P^2_4 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 9, 8, 7,***10,9,8,7,6,5,4,***
3,2,1,0,15,6, 5, 4, 11, 12, 13, 14,3,4,5,6,15, 0, 3,***14,13,12,11,***2, 1

P^2_5 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 9, 8, 7,***10,9,8,7,6,5,4,***
3,2,1,0,15,6, 1, 2,***11,12,13,14,***3, 0, 15,6,5,4,3,14, 13, 12, 11, 4, 5

P^2_6 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 9, 8, 5, 4, 11, 12, 13, 14,3,4,
5,6,15, 0, 3,***14,13,12,11,***2, 1, 6,***15,0,1,2,3,4,5,6,7,8,9,10,***7

P^2_7 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,3,2,***
1,0,15,6, 1, 2,***11,12,13,14,***3, 0, 15,6,5,4,3,14, 13, 12, 11, 4, 5, 8, 9

P^2_8 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,3,2,***
1,0,15,6, 1, 2,***11,12,13,14,***3, 0, 15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13

P^2_9 = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,3,2,***
1,0,15,6, 1, 2, 13, 12, 11, 4, 5, 8, 9, 14,3,4,5,6,15, 0, 3,***14,13,12,11***

P^2_{10} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,11,12,***
13,14,3, 0, 15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13, 2, 1, 6,***15,0,1,2,3***

P^2_{11} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,11,12,***
13,14,3, 0, 15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13, 2, 1, 6,***15,0,3,2,1***

P^2_{12} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,1,2,3,0,***
15,6, 1, 2, 13, 12, 11, 4, 5, 8, 9, 14,3,4,5,6,15, 0, 3,***14,13,12,11,4,5***

P^2_{13} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,5,4,11,12,13,14,***
3, 0, 15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13, 2, 1, 6,***15,0,3,2,1,6,7***

P^2_{14} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,3,0,15,***6, 1, 2, 13, 12, 11, 4, 5, 8, 9, 14,3,4,5,6,15, 0, 3,***14,13,12,11,4,5,8,9***

P^2_{15} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,3,0,15,***6,
1, 2, 13, 12, 11, 4, 5, 8, 9, 14,3,4,5,6,15, 0, 3,***14,9,8,5,4,11,12,13***

P^2_{16} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,13,12,11,4,***
5,8,9,14,3, 0, 15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13, 2, 1, 6,***15,0,3***

P^2_{17} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,13,12,11,4,***
3,0,15,6, 1, 2, 13, 12, 11, 4, 5, 8, 9, 14,3,4,5,6,15, 0, 3,***14,9,8,5***

P^2_{18} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,***3, 0,
15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13, 2, 1, 6,***15,0,3,4,11,12,13,2,1***

P^2_{19} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,***3, 0,
15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13, 2, 1, 6,***15,0,1,2,13,12,11,4,3***

P^2_{20} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,***3, 0,
15,6,5,4,3,14, 9, 8, 5, 4, 11, 12, 13, 2, 1, 6,***15,0,1,2,3,4,11,12,13***

P^2_{21} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,13,12,***
11,4,3,2,1,0,15,6, 1, 2, 13, 12, 11, 4, 5, 8, 9, 14,3,4,5,6,15, 0, 3

P^2_{22} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,13,12,***
11,4,3,2,1,0,15,6, 1, 2, 13, 12, 11, 4, 3, 0, 15,6,5,4,3,14, 9, 8, 5

P^2_{23} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,13,12,***
11,4,3,2,1,0,15,6, 5, 8, 9, 14,3,4,5,6,15, 0, 3, 4, 11, 12, 13, 2, 1

P^2_{24} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,13,12,***
11,4,3,2,1,0,15,6, 5, 8, 9, 14,3,4,5,6,15, 0, 1, 2, 13, 12, 11, 4, 3

P^2_{25} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,***3, 4,
11, 12, 13, 2, 1, 0, 15,6,5,4,3,14, 9, 8, 5, 6,***15,0,1,2,3,4,11,12,13***

P^2_{26} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,***3, 4,
11, 12, 13, 2, 1, 0, 15,6,5,4,3,14, 9, 8, 5, 6,***15,0,1,2,13,12,11,4,3***

P^2_{27} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,5,8,9,14,***3, 4,
11, 12, 13, 2, 1, 0, 15,6,5,4,3,14, 9, 8, 5, 6,***15,0,3,4,11,12,13,2,1***

P^2_{28} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,13,12,11,4,***
3,0,15,6, 5, 8, 9, 14,3,4,5,6,15, 0, 1, 2, 13, 12, 11, 4, 3,***14,9,8,5***

P^2_{29} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,13,12,11,4,***
5,8,9,14,3, 4, 11, 12, 13, 2, 1, 0, 15,6,5,4,3,14, 9, 8, 5, 6,***15,0,3***

P^2_{30} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,3,0,15,***6,
5, 8, 9, 14,3,4,5,6,15, 0, 1, 2, 13, 12, 11, 4, 3,***14,9,8,5,4,11,12,13***

P^2_{31} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,7,6,1,2,3,0,15,***6,
5, 8, 9, 14,3,4,5,6,15, 0, 1, 2, 13, 12, 11, 4, 3,***14,13,12,11,4,5,8,9***

P^2_{32} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,5,4,11,12,13,14,***
3, 4, 11, 12, 13, 2, 1, 0, 15,6,5,4,3,14, 9, 8, 5, 6,***15,0,3,2,1,6,7***

P^2_{33} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,1,2,3,0,***
15,6, 5, 8, 9, 14,3,4,5,6,15, 0, 1, 2, 13, 12, 11, 4, 3,***14,13,12,11,4,5***

P^2_{34} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,11,12,***
13,14,3, 4, 11, 12, 13, 2, 1, 0, 15,6,5,4,3,14, 9, 8, 5, 6,***15,0,3,2,1***

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P^2_{35} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,11,12,***
13,14,3, 4, 11, 12, 13, 2, 1, 0, 15,6,5,4,3,14, 9, 8, 5, 6,***15,0,1,2,3***

P^2_{36} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,3,2,***
1,0,15,6, 5, 8, 9, 14,3,4,5,6,15, 0, 1, 2, 13, 12, 11, 4, 3,***14,13,12,11***

P^2_{37} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,3,2,***
1,0,15,6, 5, 8, 9, 14,3,4,5,6,15, 0, 1, 2,***11,12,13,14,***3, 4, 11, 12, 13

P^2_{38} = 0,1,2,13,14,15,12,11,10,9,8,7,10, 7,***10,9,8,7,6,5,4,3,2,***
1,0,15,6, 5, 8, 9, 14, 13, 12, 11, 4, 3,***14,13,12,11,***2, 1, 0, 15,6,5,4,3

P^2_{39} = 0,3,4,5,6,15, 0, 1, 2,***11,12,13,14,***3, 4, 11, 12, 13, 14, 9, 8, 5, 6,
15,0,1,2,3,4,5,6,7,8,9,10,7, 10,7,8,9,10,11,12,15,14,13,2,1

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