ON UNIONS OF TWO CONVEX SETS

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1. Introduction. Valentine **(3)** introduced the three-point convexity property P_3 : a set S in E_n satisfies P_3 if for each triple of points x, y, z in S at least one of the closed segments xy, yz, xz is in S. He proved, **(3** or **1)** that in the plane a closed connected set satisfying P_3 is the union of some three convex subsets. The problem of characterizing those sets that are the union of two convex subsets was suggested. Stamey and Marr **(2)** have provided an answer for compact subsets of the plane. We present here a generalization of property P_3 which characterizes closed sets in an arbitrary topological linear space which are the union of two convex subsets.

2. Preliminaries. Standard results concerning convex sets in linear topological spaces will be assumed. The reader may wish to refer to **(4).** We shall say that two points of a set S are *visible* (relative to S) if the closed line segment joining them lies entirely in S.

Definition. Let S be a closed set in a topological linear space. Then S has property P_0 if for each finite subset $x_1, \ldots, x_n$ of S, n odd, with the property that x_i and x_{i+1} are not visible $(i = 1, \ldots, n-1)$, it follows that x_1 and x_n are visible.

Remarks. 1. A set satisfying P_0 clearly has Valentine's property P_3.

2. A set S that is the union of two convex sets must have property P_0.

3. P_0 is equivalent to the property: for each finite subset $x_1, \ldots, x_n$, $n > 2$, of S such that x_i and x_{i+1} are not visible $(i = 1, \ldots, n-1)$, it follows that x_i and x_j are visible if i and j are both even or both odd.

Recall that the *convex kernel* of a set S is the (convex) set of all points of S that are visible with every point of S.

3. Characterization theorem.

THEOREM. *Let S be a closed non-convex set in a topological linear space. Then S is the union of two convex subsets if and only if S satisfies property P_0.*

Proof. By Remark 2 we need only consider the "if" statement.

Let K denote the convex kernel of S. Define $\mathfrak{P}$ to be the set of all pairs of

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sets (A, B) satisfying the following properties:

1. $A \cup B \subset S \sim K$.
2. Any two points of A (of B) are visible in S.
3. To each point of A (of B) there corresponds a point of B (of A) such that the pair is not visible in S.
4. Every point of S that is not visible with some point of A (of B) is in B (in A).

We show first that $\mathfrak{P}$ is non-empty. Let a be a point of $S \sim K$. Let $\mathfrak{C}_a$ be the set of all finite chains $\{x_1, x_2, \ldots, x_n\}$ consisting of points of S such that $x_1 = a$ and such that x_i and x_{i+1} are not visible for all i, $1 \leqslant i \leqslant n - 1$. Define

$$A = \{x \in S : x = x_i \text{ belongs to a chain in } \mathfrak{C}_a \text{ in which } i \text{ is odd}\},$$

$$B = \{x \in S : x = x_i \text{ belongs to a chain in } \mathfrak{C}_a \text{ in which } i \text{ is even}\}.$$

To show that $(A, B) \in \mathfrak{P}$ we verify 1 to 4.

Property 1 follows since clearly no point of a chain in $\mathfrak{C}_a$ can be in K.

To prove 2, let a_1 and a_2 be distinct points of A. Then there exist chains $\{a, x_2, x_3, \ldots, x_{2n}, a_1\}$ and $\{a, y_2, y_3, \ldots, y_{2m}, a_2\}$ in $\mathfrak{C}_a$. But then the chain $\{a_1, x_{2n}, \ldots, x_2, a, y_2, y_3, \ldots, y_{2m}, a_2\}$ has odd length and it follows from property P_0 that a_1 and a_2 are visible. The argument is similar for two points of B.

Property 3 is obvious since each point of A or B has a neighbour in some chain belonging to $\mathfrak{C}_a$. Finally, suppose that x is a point of S that is not visible with a point a_0 of A. Since a_0 is in A, there is some chain $\{a, x_2, \ldots, x_{2n}, a_0\}$ in $\mathfrak{C}_a$. But then $\{a, x_2, \ldots, x_{2n}, a_0, x\}$ is also in $\mathfrak{C}_a$ and hence x is in B.

A similar argument completes the verification of 4.

Now order the elements of $\mathfrak{P}$ in the following manner: If $(A_1, B_1) \in \mathfrak{P}$ and $(A_2, B_2) \in \mathfrak{P}$, then

$$(A_1, B_1) < (A_2, B_2) \Leftrightarrow A_1 \subset A_2 \text{ and } B_1 \subset B_2.$$

Let $\{(A_\alpha, B_\alpha)_{\alpha \in \Gamma}\}$ be a linearly ordered subset of $\mathfrak{P}$. Then it is easily verified that

$$\left(\bigcup_{\alpha \in \Gamma} A_\alpha, \bigcup_{\alpha \in \Gamma} B_\alpha\right) \in \mathfrak{P}.$$

Hence, we may assume that there is a maximal element (A_0, B_0) in $\mathfrak{P}$.

Next we show that $A_0 \cup B_0 = S \sim K$. Otherwise there is a point $y \in S \sim (K \cup A_0 \cup B_0)$. Define

$$A_* = A_0 \cup \{x \in S : x = x_i \text{ belongs to a chain in } \mathfrak{C}_y \text{ in which } i \text{ is odd}\},$$

$$B_* = B_0 \cup \{x \in S : x = x_i \text{ belongs to a chain in } \mathfrak{C}_y \text{ in which } i \text{ is even}\}.$$

Then $(A_*, B_*) \in \mathfrak{P}$ since 1, 3, and 4 are obviously satisfied and for 2 the only non-trivial case is $a_1 \in A_0$ and s_1 in a chain $\{y, x_2, \ldots, x_{2m}, s_1\}$ in $\mathfrak{C}_y$ (and the similar case with A_0 replaced by B_0). If a_1 and s_1 are not visible, then $s_1 \in B_0$. But then $x_{2m} \in A_0, x_{2m-1} \in B_0, \ldots, y \in B_0$ contrary to our assumption.

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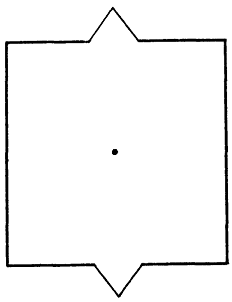
Since $y \notin K$, (A_*, B_*) is an element of $\mathfrak{P}$ strictly larger than (A_0, B_0) contradicting the maximality of (A_0, B_0). Hence, $A_0 \cup B_0 = S \sim K$.

The proof will be completed by showing that $\text{conv}(A_0 \cup K) \subset S$ and $\text{conv}(B_0 \cup K) \subset S$. Since the proofs are symmetric, we need to consider only the first statement. It is sufficient to show that if $\{c_1, \ldots, c_n\} \subset A_0 \cup K$ for arbitrary n, then every point of the relative interior of the simplex $T = \text{conv}\{c_1, \ldots, c_n\}$ is in S. Since this is obvious for $n \leqslant 2$, we shall use induction on n assuming the result for all $m < n$. If T contains a point of K, then the desired result is immediate. Hence, assume $\{c_1, \ldots, c_n\} \subset A_0$ and $T \cap K = \emptyset$. Suppose there exists a point x of the relative interior of T which is not in S. Then, since S is closed, $T \sim S$ is a non-empty relatively open subset of T.

We prove that $T \sim S$ is convex. By the induction assumption the proper faces of the simplex T are all subsets of S. Let x_1 and x_2 be distinct points of $T \sim S$ and suppose the line segment joining them contains a point s of S. The line through x_1 and x_2 must intersect the boundary of T in distinct points y_1 and y_2 on opposite sides of x_1 and x_2 respectively from s. If y_1 and y_2 are both in A_0 or both in B_0, it follows that x_1 is in S, contrary to assumption. Hence either y_1 and s or y_2 and s are both in the same set A_0 or B_0 and it follows that x_1 or x_2 is in S, which is again a contradiction.

Let P be the plane spanned by $\{c_1, c_2, x\}$. Then $P \cap (T \sim S)$ is a non-empty relatively open convex subset of $P \cap T$. By the Krein–Milman theorem, $\overline{P \cap (T \sim S)}$ is the convex hull of its set of extreme points. Clearly there must be at least three such extreme points, say e_1, e_2, and e_3, and each of these belongs to $A_0 \cup B_0$. There are two of these that belong to the same set, say e_1 and e_2 are in A_0. In the plane P, let l be the translate of the line determined by e_1 and e_2 that passes through x. Since x is in the relative interior of $\overline{P \cap (T \sim S)}$, there exist points s_1 and s_2 of S on l, on opposite sides of x and on the boundary of $T \sim S$. At least one of $\{s_1, s_2\}$ must be in A_0, say s_1. But then $\frac{1}{2}(e_1 + s_1)$ and $\frac{1}{2}(e_2 + s_1)$ are both in A_0 and hence in S. Since at least one of these must also be in $P \cap (T \sim S)$, we have the desired contradiction.

Consideration of the set indicated below, where the centre of symmetry has been removed, shows that the assumption that S be closed in the theorem is essential.



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The same example with the centre point inserted shows that $\mathfrak{P}$ may contain more than one element.

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